Large time asymptotics for a continuous coagulation-fragmentation model with degenerate size-dependent diffusion

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Abstract

We study a continuous coagulation-fragmentation model with constant kernels for reacting polymers (see [AB]). The polymers are set to diffuse within a smooth bounded one-dimensional domain with no-flux boundary conditions. In particular, we consider size-dependent diffusion coefficients, which may degenerate for small and large cluster-sizes. We prove that the entropy-entropy dissipation method applies directly in this inhomogeneous setting. We first show the necessary basic a priori estimates in dimension one, and secondly we show faster-than-polynomial convergence towards global equilibria for diffusion coefficients which vanish not faster than linearly for large sizes. This extends the previous results of [CDF], which assumes that the diffusion coefficients are bounded below.

Key words: Continuous coagulation-fragmentation with spatial diffusion degenerate in size, faster than polynomial equilibration rates, entropy-entropy dissipation method,

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1 Introduction

We analyse the spatially inhomogeneous version of a size-continuous model for reacting polymers or clusters of aggregates:

$$\partial_t f - a(y)\, \partial_{xx} f = Q(f, f)\,. \tag{1.1}$$

Here, $f := f(t, x, y)$ is the concentration of polymers/clusters with length/size $y \geq 0$ at time $t \geq 0$ and point $x \in [0, 1]$. These polymers/clusters diffuse in the environment. Equation (1.1) is to be considered with homogeneous Neumann boundary condition

$$\partial_x f(t, 0, y) = \partial_x f(t, 1, y) = 0, \tag{1.2}$$

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so that there is no polymer flux through the physical boundary.

The diffusion coefficient $a := a(y)$ is supposed to be bounded above and below on compact intervals $y \in [\delta, \delta^{-1}]$ for all $0 < \delta < 1$, but can possibly degenerate for small and large sizes (that is, tend to ∞ at $y = 0$ and tend to 0 at $y \to \infty$). More precisely, we shall assume that there exist $a_* > 0$, $\delta \mapsto a^*(\delta) > 0$ such that :

$$y \mapsto a(y) \quad \text{measurable satisfies} \quad \begin{cases} a(y) \leq a^*(\delta)\,, & \forall y \in [\delta, \delta^{-1}] \\ 0 < \frac{a_*}{1+y} \leq a(y)\,, & \forall y \in [0, \infty). \end{cases} \tag{1.3}$$

The reaction term $Q(f,f)$ of (1.1) models chemical degradation (break-up or fragmentation) and polymerisation (coalescence or coagulation) of polymers/clusters. The full collision operator sums a gain- and a loss-term from both coagulation and fragmentation :

$$\begin{aligned} Q(f,f) &= Q^+(f,f) - Q^-(f,f) \\ &= \int_0^y f(t,x,y-y')f(t,x,y')\,dy' + 2\int_y^\infty f(t,x,y')\,dy' \\ &\quad - 2f(t,x,y)\int_0^\infty f(t,x,y')\,dy' - y\,f(t,x,y)\,. \end{aligned} \tag{1.4}$$

These four terms model the following phenomena: Coagulation of clusters of size $y' \leq y$ and $y - y'$ results into clusters of size y, break-up of clusters of size y' larger than y creates clusters of size y, coagulation of clusters of size y with clusters of size y' produces a loss, as does, finally, break-up of clusters of size y.

This kind of models finds its application not only in polymers and cluster aggregation in aerosols [S16, S17, AB, Al, Dr] but also in cell physiology [PS], population dynamics [Ok] and astrophysics [Sa]. Here, fragmentation and coagulation kernels are all set to constants as in the original Aizenman-Bak model [AB]. This will be of paramount importance in the basic a-priori estimates as well as in the use of the entropy-entropy dissipation method.

A fundamental conservation-of-mass law follows from the collision invariance $\int_0^\infty y\,Q(f,f)\,dy = 0$, entailing that the total number of monomers (or mass of polymers)

$$N(t,x) := \int_0^\infty y\,f(t,x,y)\,dy$$

(assumed initially to be positive) is formally conserved for times $t \geq 0$:

$$\int_0^1 N(t,x)\,dx = \int_0^1\int_0^\infty y\,f(t,x,y)\,dy\,dx = \int_0^1 N(0,x)\,dx := N_\infty > 0. \tag{1.5}$$

Another macroscopic quantity of interest is the number density of polymers,

$$M(t,x) := \int_0^\infty f(t,x,y)\,dy,$$

that together with the total number of monomers $N(t,x)$ (formally) satisfies the (non-closed) reaction-diffusion system

$$\partial_t N - \partial_{xx}\left(\int_0^\infty y a(y)\, f(t,x,y)\, dy\right) = 0\,, \tag{1.6}$$

$$\partial_t M - \partial_{xx}\left(\int_0^\infty a(y)\, f(t,x,y)\, dy\right) = N - M^2\,. \tag{1.7}$$

The definition of the full collision operator has to be understood in the weak sense. Integrating by parts the gain term of the fragmentation operator, we obtain for any smooth function $\varphi := \varphi(y)$ and function $f := f(y)$ such that the integrals exist:

$$\begin{aligned}\int_0^\infty Q(f,f)(y)\,\varphi(y)\,dy = &\int_0^\infty\!\!\int_0^\infty [\varphi(y'') - \varphi(y) - \varphi(y')]\, f(y)f(y')\,dy\,dy' \\ &+ 2\int_0^\infty \Phi(y)\, f(y)\,dy - \int_0^\infty y\,\varphi(y)\, f(y)\,dy\,,\end{aligned} \tag{1.8}$$

where the function Φ denotes the primitive of φ ($\partial_y \Phi = \varphi$) with $\Phi(0) = 0$ and $y'' = y + y'$.

Let us consider the (free-energy) entropy functional associated to any positive density $f := f(y)$ as

$$H(f) = \int_0^\infty (f \ln f - f)\; dy\,,$$

and the relative entropy $H(f|g) = H(f) - H(g)$ of two states f and g (not necessarily with the same L^1_y-norm). Then, the entropy (integrated w.r.t. x) formally dissipates for solutions of eq. (1.1) as

$$\begin{aligned}\frac{d}{dt}\int_0^1 H(f)\,dx = &-\int_0^1\!\!\int_0^\infty a(y)\frac{|\partial_x f|^2}{f}\,dy\,dx \\ &-\int_0^1\!\!\int_0^\infty\!\!\int_0^\infty (f'' - ff')\ln\left(\frac{f''}{ff'}\right) dy\,dy' dx \;:= -D(f)\end{aligned} \tag{1.9}$$

with $f' := f(t,x,y')$ and $f'' := f(t,x,y'')$.

In the present paper, we shall rather use a weaker dissipation inequality (see (3.1) below), which is obtained by using a remarkable inequality proven in [AB, Propositions 4.2 and 4.3]. It reads (for functions of y only) as (Cf. [CDF]):

$$\int_0^\infty\!\!\int_0^\infty (f'' - ff')\ln\left(\frac{f''}{ff'}\right) dy\,dy' \geq M\,H(f|f_N) + 2(M - \sqrt{N})^2\,. \tag{1.10}$$

Herein, $f_N := f_N(y)$ denotes a distinguished, exponential-in-size distribution with the very moments $M = \sqrt{N}$ and N :

$$f_N(y) = e^{-\frac{y}{\sqrt{N}}}\,.$$

These distributions f_N appear as analog to the so-called intermediate or local equilibria in the study of inhomogeneous kinetic equation (e.g. [DV01, CCG, FNS, DV05, FMS, NS]). Finally, the conservation of mass (1.5) identifies (at least formally) the global equilibrium f_∞ with constant moments $M_\infty^2 = N = N_\infty$:

$$f_\infty = e^{-\frac{y}{\sqrt{N_\infty}}} . \tag{1.11}$$

The analogy to intermediate equilibria carries over to the following additivity of relative entropies

$$H(f|f_\infty) = H(f|f_N) + H(f_N|f_\infty) . \tag{1.12}$$

It is worthy to point out that even if f_N and f_∞ do not have the same L^1_y-norm, its global relative entropy $\int_0^1 H(f_N|f_\infty)\, dx = 2(\sqrt{\int_0^1 N\, dx} - \int_0^1 \sqrt{N}\, dx) \geq 0$ is a nonnegative quantity, as easily checked via Jensens's inequality.

Global existence and uniqueness of classical solutions to equations of the form (1.1)-(1.4) has been studied in [Am, AW] but with restrictions for the coagulation and fragmentation kernel which do not enable to cope with the Aizenman-Bak model (1.4). For the initial boundary-value problem to (1.1)-(1.2), global existence of weak solutions was then proven in [LM02-1], assuming only the first condition of (1.3), and for much more general coagulation and fragmentation kernels including the Aizenman-Bak model (1.4).

In [LM02-1], it is also proven that f_∞ attracts all global weak solutions in $L^1([0,1] \times (0,\infty))$ of (1.1)-(1.2) but no time decay rate is obtained. This result is the analog to convergence results along subsequences for the classical Boltzmann equation in [De].

In the present paper, we are able to obtain explicit rates and constants for the decay to equilibrium for degenerate diffusion coefficients as stated in assumption (1.3). As a trade-off we are not able to recover exponential decay. Nevertheless we shall show decay faster than any polynomial, and in fact of the type $\exp(-(\ln t)^\beta)$ for all $\beta < 2$. Up to our knowledge, this is the first result of explicit convergence for inhomogeneous coagulation-fragmentation models with degenerate diffusion, and the first example in which the entropy/entropy dissipation method leads to a convergence with such a strange rate.

Our key Lemma 3.1 in section 3 establishes a functional inequality between entropy and entropy dissipation provided we have lower and upper bounds on the moment M, an upper bound on higher moments, and assumption (1.3). While Lemma 3.1 holds in all space dimensions, it is in the one-dimensional case that we are able to apply this functional inequality to solutions of (1.1)-(1.4), in which the entropy dissipation entails sufficiently strong a-priori estimates. The decay to equilibrium with a rate given above follows finally via a suitable Gronwall argument (see section 4). Our main Theorem reads as :

Theorem 1.1 *Consider a diffusion coefficient satisfying* (1.3). *Let us also assume that* $f_0 \neq 0$ *(i.e.* f_0 *is not constant zero, for which the theorem holds* *trivially by setting* $f_\infty = 0$ *for* $N_\infty = 0$*) is a nonnegative initial datum such that* $(1 + y + \ln f_0)f_0 \in L^1((0,1) \times (0,\infty))$.

Then, the global weak solutions $f(t,x,y) \in L^\infty_{loc}(\mathbb{R}_+; L^1((0,1)\times(0,\infty)))$ *of* (1.1)–(1.4) *satisfying the entropy/entropy dissipation estimate*

$$\int_0^1 H(f(t,x,\cdot))\,dx + \int_0^t D(f(s,\cdot,\cdot))\,ds \le \int_0^1 H(f_0(x,\cdot))\,dx$$

decay to the global equilibrium state (1.11) *with the following rate: For all* $\beta < 2$*, there exists* $C_\beta > 0$ *(which can be explicitly bounded above w.r.t.* f_0*,* a_* *and* a^**) such that (for all* $t > 0$*)*

$$\int_0^1 H(f(t,\cdot)|f_\infty)\,dx \le C_\beta\, e^{-(\ln t)^\beta}, \tag{1.13}$$

and:

$$\|f(t,\cdot,\cdot) - f_\infty\|_{L^1_{x,y}} \le C_\beta\, e^{-(\ln t)^\beta}, \tag{1.14}$$

where f_∞ *is defined by* (1.11) *[and* $N_\infty > 0$ *is determined by the conservation of mass* (1.5)*].*

Remark 1.1 *The second part of* (1.3) *[that is, the at most linear degeneracy of the diffusion coefficient for large sizes] is unavoidable in our method. To illustrate why this is so, let us calculate the evolution of second order moments* $M_2(f)(t) := \int_0^1\int_0^\infty y^2 f(t,x,y)\,dydx$ *using the weak formulation* (1.8)*. We find (dropping* t *and* x *for notational convenience)*

$$\frac{d}{dt}M_2(f) = \int_0^1\int_0^\infty\int_0^\infty 2\,y\,y'\,f(y)\,f(y')\,dydy'dx - \int_0^1\int_0^\infty \frac{y^3}{3}f(y)\,dydx\,,$$

and, using Young's inequality $\frac{y}{a(y)}\frac{y'}{a(y')} \le \frac12\left(\frac{y^2}{a^2(y)} + \frac{(y')^2}{a^2(y')}\right)$,

$$\frac{d}{dt}M_2(f) \le \int_0^1 \left[\int_0^\infty a(y')f(y')\,dy'\right]\left[\int_0^\infty \frac{2y^2}{a(y)}f(y)\,dy\right]dx - \int_0^1\int_0^\infty \frac{y^3}{3}f(y)\,dydx.$$

Then, we notice that the Fisher information being bounded as in (1.9) *"almost" implies that* $\int_0^\infty a(y')f(y')\,dy' \in L^\infty_{t,x}$ *(this is not quite true, Cf. lemma 2.2 for a more precise statement). Interpolating* $2y^2/a(y)$ *between* y^3 *and* y*, and using a Gronwall argument leads to a global bound on* $M_2(f)$*, and the appearance of a third order moment. It is such bounds on moments (Cf. Lemma 3.1) which yield an explicit decay towards equilibrium.*

But of course, such an interpolation holds only when $a(y) \ge a_*(1+y)^{-\delta}$ *with* $\delta < 1$*, which is a slightly degraded version of assumption* (1.3)*. We think therefore that it is not possible to significantly relax the condition on the diffusion coefficient for large* y *with our method.*

Remark 1.2 *On the opposite, it is possible to relax the condition on the diffusion coefficient $a := a(y)$ for small y to allow an unbounded yet integrable inverse, i.e. $\int_0^1 a^{-1}(y)\,dy < \infty$, if we assume initial data $f_0 := f_0(x,y) \in L^\infty_{x,y}$. For such initial data a multiplier technique (see [CP][CDF1, Lemma 3.2]) shows the propagation of the L^∞ bound for all times and Lemma 2.2 can be suitably modified.*

Remark 1.3 *We cannot expect the explicit decay rate given in eq. (1.13), (1.14) to be optimal, since it is a consequence of estimates on moments (more precisely, the dependence w.r.t. p of the bounds on the moment of order p) which are probably not optimal themselves, as well as a consequence of the entropy-entropy dissipation estimate, which is certainly not optimal in the steps 2 and 3. We nevertheless suspect that the convergence towards equilibrium might not be exponential, the degeneracy when $y \to \infty$ of the diffusion rate playing here the same role as the degeneracy when $v \to \infty$ in soft potentials for the Boltzmann equation (Cf. [TV]).*

It is further possible to interpolate the faster-than-polynomial decay in a "weak" norm like L^1 with polynomially growing bounds in "strong" norms like (weighted) $L^1_y(H^1_x)$ in order to get faster-than-polynomial decay in a "medium" norm like $L^1_y(L^\infty_x)$. Thus, the decay toward equilibrium can be extended to these stronger norms. This idea is used in the proof of the following proposition (Cf. the end of section 4):

Proposition 1.1 *Under the assumptions of Theorem* 1.1*, for all $q \geq 0$ and $\beta < 2$, there are (explicitly computable) constants $C_{\beta,q}$ such that*

$$\int_0^\infty (1+y)^q \, \|f(t,\cdot,y) - f_\infty(y)\|_{L^\infty_x} \, dy \leq C_{\beta,q} \, e^{-(\ln t)^\beta}\,, \tag{1.15}$$

for all $t \geq t_* > 0$.

A bootstrap argument in the spirit of the proof of Proposition 1.1 can even allow to replace the L^∞_x norm by any Sobolev norm in (1.15).

Note that on one hand no extra assumption on the initial datum is needed for this Proposition, since the regularity w.r.t. x is created thanks to the diffusive character of eq. (1.1). On the other hand, one definitely needs extra assumptions on the initial datum (typically $\int_0^\infty (1+y)\|f_0(t,\cdot,y)\|_{L^\infty_x}\,dy < +\infty$) if one wishes to use Proposition 1.1 (or more simply estimate (4.6)) in order to prove a result of uniqueness for weak solutions of eq. (1.1). We shall come back to the issue of uniqueness with general initial data before stating lemma 2.2.

Explicit rates of decay for coagulation-fragmentation models without diffusion have been obtained in [AB] and, for the Becker-Döring model, in [JN]. Explicit rates of decay for reversible reaction-diffusion models (corresponding to a finite number of possible size for polymers) have been obtained in

[DF05, DF06, DF]. Non-constructive exponential rates via a contradiction argument was shown for general drift-diffusion-reaction systems in [Grö, GGH]. In [DF07], the case of a degenerate diffusion in reaction-diffusion models is studied.

The first result of explicit rate of decay for inhomogeneous coagulation-fragmentation models was proven in [CDF], under the physically unrealistic assumption that the diffusion is bounded below and above. The present paper is devoted to the removal of this assumption.

The method of proof makes use of the entropy–entropy dissipation method (Cf. [Des] for a general introduction to this method in the context of kinetic equations). It is in particular reminiscent of works in which "slowly growing a priori bounds" appear, such as [TV] and [DF06].

Moreover, one uses here bounds on moments in which one keeps track of the constants (w.r.t. the order of the moment) so that some summability of those bounds can be recovered in the end. This idea is already present in papers such as [BGP]. The strange functions of time recovered in (1.13) is directly related to the summability mentioned above. (Cf. the end of the proof of theorem 1.1).

2 A-priori Estimates

We begin with a-priori estimates used in the proof of Theorem 1.1. In the lemmas and propositions of this section, f, M or N always refer to a global weak solution of (1.1)–(1.2) satisfying the assumptions of Theorem 1.1.

Lemma 2.1 *There exists $\mathcal{M}_0^* > 0$ (depending only on the initial datum) such that*

$$\sup_{t\geq 0} \int_0^1 M(t,x)\,dx \leq \mathcal{M}_0^*. \tag{2.1}$$

Proof.- The proof is a consequence of integrating (1.7) and Jensen's inequality:

$$\frac{d}{dt}\int_0^1 M(t,x)\,dx \leq N_\infty - \left(\int_0^1 M(t,x)\,dx\right)^2.$$

Cf. [CDF, Lemma 4] for more details. □

We now turn to a control of $M(t,\cdot)$ in L^∞_x. Note that the estimate obtained in Lemma 2.2 below is a significant step towards a proof of uniqueness of weak solutions to eq. (1.1) with general initial datum of finite mass and entropy. One would in fact need the same kind of estimate for N in order to get such a result (see e.g. [LM04]).

Lemma 2.2 *The number density of polymers $M := M(t,x)$ lies in $[L^2 + L^\infty](0,\infty; L^\infty_x(0,1))$. More precisely, there exist $m_\infty > 0$ and an $L^1 \cap L^2(0,\infty)$-function $m_2(t)$ such that (for a.e. $t \geq 0$)*

$$\|M(t,\cdot)\|_{L^\infty_x} \leq m_\infty + m_2(t).$$

Proof.- We integrate

$$f(t,x,y) - f(t,\tilde{x},y) = 2\int_{\tilde{x}}^{x} \sqrt{f}(t,\xi,y)\,\partial_x\sqrt{f}(t,\xi,y)\,d\xi$$

with respect to $\tilde{x} \in (0,1)$ and $y \in \mathbb{R}_+$, and get

$$\int_0^\infty \left| f(t,x,y) - \int_0^1 f(t,\tilde{x},y)\,d\tilde{x} \right| dy \le 2\int_0^\infty\int_0^1 |\sqrt{f}(t,\xi,y)|\,|\partial_x\sqrt{f}(t,\xi,y)|\,d\xi dy$$

$$\le 2\left[\int_0^\infty\int_0^1 a(y)^{-1}\,f(t,x,y)\,dxdy\right]^{\frac{1}{2}} \left[\int_0^\infty\int_0^1 a(y)\,|\partial_x\sqrt{f}(t,x,y)|^2\,dxdy\right]^{\frac{1}{2}}$$

$$\le \left[a_*^{-1}\int_0^1 (M(t,x)+N(t,x))\,dx\right]^{\frac{1}{2}} (D(f(t)))^{1/2},$$

using assumption (1.3). Then,

$$M(t,x) \le \int_0^1 M(t,\tilde{x})\,d\tilde{x} + a_*^{-1/2}\,(\mathcal{M}_0^* + N_\infty)^{1/2}\,D(f(t))^{1/2}.$$

The first term in this estimate belongs to L_t^∞ thanks to Lemma 2.1 and the second one to L_t^2. Note that the L^2 function can be split as a sum of an L^1 and an L^∞ function. This completes the proof of Lemma 2.2. □

Lemma 2.3 *There exists a constant $\mathcal{M}_{0*} > 0$ such that (for all $t \ge 0$), one has*

$$\int_0^1 M(t,x)\,dx \ge \mathcal{M}_{0*}. \tag{2.2}$$

Proof.- A Gronwall type proof exploiting the estimate

$$\frac{d}{dt}\int_0^1 M(t,x)\,dx \ge N_\infty - (m_\infty + m_2(t))\int_0^1 M(t,x)\,dx,$$

can be found in [CDF, Lemma 6]. □

Next, we show the (uniform for time $t \ge t_* > 0$) control of all moments with respect to size y of the solutions. Let us define the moment of order $p > 1$ by

$$M_p(f)(t) := \int_0^1\int_0^\infty y^p\,f(t,x,y)\,dy\,dx$$

for all $t \ge 0$. The proof uses the fact that the fragmentation term (in terms of creation of moments) somehow dominates the non-linear coagulation term. This property was already used for homogeneous coagulation-fragmentation models (with so-called strong fragmentation) in the works of [BC, Cos], see also [Des, MW] for related kinetic models. Here, we used crucially Lemma 2.2 in order to adapt this idea to the spatially inhomogeneous case. The proof keeps record of the dependence upon p of the uniform bound of the moment of order p: this will enable us in the proof of Theorem 1.1 to conclude to the "exponential of the square of the logarithm" convergence rate via a summation argument.

Note also that the bounds which are obtained in Lemma 2.4 below ensure that the mass is conserved along the solutions of eq. (1.1) [that is, no gelation occurs], provided that some moments initially exist. We refer to [CnDF1], [CnDF2] and the references therein for a much broader approach to this problem (though in the "discrete in y" setting).

Lemma 2.4 *For any $p > 1$ and $t_* > 0$, one has*

$$M_p(f)(t) \le (2^{2p}\, C)^p =: \mathcal{M}_p^*\,, \qquad \textit{for } t \ge t_* > 0, \tag{2.3}$$

for a constant $C = C(t_*, f_0)$ depending only on the initial datum and t_*.

Proof.- We prove Lemma 2.4 in three steps . We denote $\mu_2 = \|m_2\|_{L^2}$ (where m_2 is defined in Lemma 2.2)
Step 1.- As in [CDF, Lemma 7], the evolution of the moment of order $p > 1$ is governed by

$$\frac{d}{dt} M_p(f)(t) \le (2^p - 2)\, M_p(f)(t)\ [m_\infty + m_2(t)] - \frac{p-1}{p+1} M_{p+1}(f)(t). \tag{2.4}$$

Trivial interpolation of the $(p+1)$-order moment with the moment of order one implies thanks to Young's inequality and the conservation of mass (1.5) that

$$-\frac{p-1}{p+1} M_{p+1}(f)(t) \le \frac{\epsilon^{-p}}{p+1} N_\infty - \frac{p}{p+1}\epsilon^{-1}\, M_p(f)(t)$$

for all $\epsilon > 0$. Thus

$$\frac{d}{dt} M_p(f)(t) \le \left[(2^p-2)(m_\infty + m_2(t)) - \frac{p}{p+1}\epsilon^{-1}\right] M_p(f)(t) + \frac{\epsilon^{-p}}{p+1} N_\infty\,.$$

Moreover, according to Duhamel's formula and using $\int_{t_*}^t m_2\, ds \le \mu_2 \sqrt{t - t_*}$, we estimate for all $p > 1$

$$M_p(f)(t) \le M_p(f)(t_*)\, e^{\theta(t-t_*)} + \frac{\epsilon^{-p}}{p+1} N_\infty \int_{t_*}^t e^{\theta(t-s)}\, ds\,, \tag{2.5}$$

where θ is the function defined by

$$\theta(r) = \left[(2^p-2) m_\infty - \frac{1}{2}\epsilon^{-1}\right] r + (2^p - 2)\mu_2 \sqrt{r}\,.$$

Choosing then $\epsilon^{-1} = 2^{2p}\, m_\infty$, it is easy to verify that θ is bounded above. More precisely,

$$\theta(r) = (2^p - 2)\mu_2 \left[\sqrt{r} - \frac{r}{\sqrt{r_p}}\right], \quad \text{where } \sqrt{r_p} := \frac{2^p - 2}{2^{2p-1} - 2^p + 2}\frac{\mu_2}{m_\infty} > 0\,,$$

and $0 < \theta(r) \le \theta(r_p/4) < \frac{\mu_2^2}{2m_\infty}$ for $r \in (0, r_p)$, $\theta(r_p) = 0$, $\theta(r) < 0$ for $r \in (r_p, \infty)$. Then, we obtain the estimate

$$\theta(r) \le \mathbb{I}_{\{r \le 2r_p\}} \frac{\mu_2^2}{2m_\infty} - \mathbb{I}_{\{r > 2r_p\}} (2^p - 2)\mu_2 (1 - \frac{1}{\sqrt{2}}) \frac{r}{\sqrt{r_p}} \,, \tag{2.6}$$

since $\sqrt{r} - \frac{r}{\sqrt{r_p}} \le -(1 - \frac{1}{\sqrt{2}})\frac{r}{\sqrt{r_p}}$ for $r > 2r_p$. Finally, we end up with

$$\begin{aligned} M_p(f)(t) &\le M_p(f)(t_*) \left[e^{\frac{\mu_2^2}{2m_\infty}} + e^{-(2^p-2)\mu_2(1-\frac{1}{\sqrt{2}})2\sqrt{r_p}} \right] \\ &\quad + (2^{2p}\, m_\infty)^p \, N_\infty \left[e^{\frac{\mu_2^2}{2m_\infty}} 2r_p + \int_0^t e^{-(2^p-2)\mu_2(1-\frac{1}{\sqrt{2}})\frac{t-s}{\sqrt{r_p}}} \, ds \right] \\ &\le M_p(f)(t_*) \left[e^{\frac{\mu_2^2}{2m_\infty}} + 1 \right] + (2^{2p}\, m_\infty)^p \, N_\infty \left[e^{\frac{\mu_2^2}{2m_\infty}} 2r_p + 6\, r_p \frac{m_\infty}{\mu_2^2} \right] \\ &\le \; C \left(M_p(f)(t_*) + (2^{2p}\, m_\infty)^p \right) . \end{aligned} \tag{2.7}$$

for a constant $C = C(N_\infty, m_\infty, \mu_2)$ and for all $t \ge t_* > 0$.

Step 2.- Next, we construct a sequence $\{t_k\}$ for $k = 2, 3, 4, \dots$ as follows : Let us fix $t_* > 0$, set $t_2 = \frac{t_*}{2}$, and assume that $M_2(f)(t_*/2) < \infty$. Let $0 < \lambda < 1$ be some parameter (in fact, we shall take $\lambda := \inf\{1, (\frac{m_\infty}{\mu_2})^2 \frac{t_*}{2}\}$). By (2.7), we have for all $t \in [t_k, t_k + \lambda 2r_k]$ that

$$M_k(f)(t) \le \left(M_k(f)(t_k) + (2^{2k}\, m_\infty)^k \, 2r_k \, N_\infty \right) \left[e^{\frac{\mu_2^2}{2m_\infty}} + 1 + 3\,\frac{m_\infty}{\mu_2^2} \right] .$$

Reinserting into (2.4) and integrating over $[t_k, t_k + \lambda 2r_k]$ yields

$$\begin{aligned} \frac{k-1}{k+1} \int_{t_k}^{t_k + \lambda 2r_k} M_{k+1}(f)(t)\, dt &\le M_k(f)(t_k) + (2^k - 2) \left(\lambda 2r_k\, m_\infty + \mu_2 \sqrt{\lambda 2r_k} \right) \\ &\quad \times \left(M_k(f)(t_k) + (2^{2k}\, m_\infty)^k \, 2r_k \, N_\infty \right) \left[e^{\frac{\mu_2^2}{2m_\infty}} + 1 + 3\,\frac{m_\infty}{\mu_2^2} \right] . \end{aligned}$$

Then, dividing by the interval length $\lambda 2r_k$ and recalling that $\sqrt{r_k} < 2^{1-k}\frac{\mu_2}{m_\infty}$ we obtain

$$\int_{t_k}^{t_k + \lambda 2r_k} \frac{M_{k+1}(f)(t)}{\lambda 2r_k} \, dt \le C\, 2^{2k}\, M_k(f)(t_k) + (2^{2k}\, C)^k \,,$$

where $C = C(N_\infty, m_\infty, \mu_2, \lambda)$ depends only on N_∞, m_∞, μ_2, and λ. Thus, there exists a time $t_{k+1} \in [t_k, t_k + \lambda 2r_k]$ such that

$$M_{k+1}(f)(t_{k+1}) \le C\, 2^{2k}\, M_k(f)(t_k) + (2^{2k}\, C)^k \,.$$

Moreover, by iteration in k (and with a larger constant C)

$$M_k(f)(t_k) < (2^{2(k-1)}\, C)^{(k-1)} \quad \Rightarrow \quad M_{k+1}(f)(t_{k+1}) \le (2^{2k}\, C)^k \,.$$

Finally, considering that

$$t_\infty = \lim_{k\to\infty} t_k \le t_2 + \sum_{k=2}^{\infty} \lambda 2 r_k \le t_2 + 2\lambda(\frac{\mu_2}{m_\infty})^2 \sum_{k=2}^{\infty} 2^{2-2k} = t_2 + \lambda(\frac{\mu_2}{m_\infty})^2 \frac{2}{3}\,,$$

we choose $\lambda < (t_* - t_2)\frac{3}{2}(\frac{m_\infty}{\mu_2})^2$ to ensure $t_\infty \le t_*$ and hence

$$M_k(f)(t_*) \le (2^{2k}\, C)^k \qquad \text{for all } k = 2,3,4,\dots\,,$$

so that

$$\forall t \ge t_*, \qquad M_k(f)(t) \le (2^{2k}\, C)^k \qquad \text{for all } k = 2,3,4,\dots\,,$$

where $C = C(N_\infty, m_\infty, \mu_2, t_*)$ depends only on N_∞, m_∞, μ_2, and t_*.

Step 3.- It remains to show that for given nontrivial initial data such that $y\, f_0 \in L^1_{x,y}$, there exists a time $t_0 \le \frac{t_*}{4}$ such that $M_p(f)(t_0) < \infty$ for some $p > 1$ and, further, a time $t_1 \le \frac{t_*}{2}$ such that $M_2(f)(t_1) < \infty$.

We start with the following observation [MW, Appendix A]: For a nonnegative integrable function $g(y) \neq 0$ on $(0,\infty)$, there exists a concave function $\Phi(y)$, depending on g, smoothly increasing from $\Phi(0) > 0$ to $\Phi(\infty) = \infty$ such that

$$\int_0^\infty \Phi(y)\, g(y)\, dy < \infty.$$

Moreover, the function Φ can be constructed to satisfy

$$\Phi(y) - \Phi(y') \ge C \frac{y - y'}{y \ln^2(e+y)} \tag{2.8}$$

for $0 < y' < y$ with C not depending on g. We refer to [MW, Appendix A] for all the details of this "by-now standard" construction.

To show now that $M_p(f)(t_0) < \infty$ for a $p > 1$ and a time $t_0 \le \frac{t_*}{4}$, we take functions $\Phi(x,y)$ constructed for nontrivial $y\, f_0(x,y) \in L^1_y(0,\infty)$ a.e. $x \in (0,1)$ [here x is only a parameter] and calculate - similar to Step 1 - the moment

$$M_{1,\Phi}(f)(t) = \int_0^1 \int_0^\infty y\, \Phi(x,y) f(x,y)\, dy\, dx.$$

For the fragmentation part, we use (2.8) for $0 < y' < y$ and estimate

$$\begin{aligned} 2\int_0^y y'(\Phi(y') - \Phi(y))\, dy'\, f(y) &\le -\frac{C}{\ln^2(e+y) y} \int_0^y y'(y - y')\, dy'\, f(y) \\ &= \frac{C}{\ln^2(e+y)} \frac{y^2}{6}\, f(y) \le -C_\delta\, y^{2-\delta}\, f(y)\,, \end{aligned}$$

for all $\delta > 0$ and a positive constant C_δ [the (t,x)-dependence has been dropped for notational convenience]. Hence, by estimating the coagulation part similar to [CDF, Lemma 7], making use of the concavity of Φ, we obtain

$$\frac{d}{dt} M_{1,\Phi}(f)(t) \le 3(m_\infty + m_2(t)) M_{1,\Phi}(f)(t) - C_\delta M_{2-\delta}(f)(t)\,,$$

and boundedness of the moment $M_{1,\Phi}$ follows by interpolation as well as the finiteness of $M_{2-\delta}(f)(t_0)$ (for some $t_0 \in (0, t_*/4)$) analogously to Step 2. Writing a differential inequality for $M_{3/2}(f)$ yields then the existence of some $t_1 \in (t_0, t_*/2)$ such that $M_2(f)(t_1) < \infty$. This yields the assumption at the beginning of step 2 and concludes the proof. $\square$

Next, we show that M is bounded below uniformly (with respect to t and x) for all $t \ge t_* > 0$.

Proposition 2.1 *Let $t_* > 0$ be given. Then, there is a strictly positive constant $\mathcal{M}_*$ (depending on t_*, a_* and $a^*(\delta)$ as in assumption (1.3), $m_\infty, \mu_1 := \|m_2\|_{L^1}$ and the initial datum) such that for all $t \ge t_* > 0$,*

$$M(t,x) \ge \mathcal{M}_*.$$

Proof.- We write the equation satisfied by f in this way :

$$\partial_t f - a(y)\, \partial_{xx} f = g_1 - y\, f - \|M(t,\cdot)\|_{L^\infty_x}\, f,$$

where g_1 is nonnegative. Then

$$(\partial_t - a(y)\, \partial_{xx})\left(f\, e^{ty + \int_0^t \|M(s,\cdot)\|_{L^\infty_x}\, ds} \right) = g_2,$$

where g_2 is nonnegative.

Now, we recall that the solution $h := h(t,x)$ of the heat equation

$$\partial_t h - a\, \partial_{xx} h = G,$$

with homogeneous Neumann boundary condition on the interval $(0,1)$, where $a > 0$ is a constant and $G := G(t,x) \in L^1$, is given by the formula

$$\begin{aligned} h(t,x) = {} & \frac{1}{2\sqrt{\pi}} \int_{-1}^{1} \tilde{h}(0,z) \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{a\,t}}\, e^{-\frac{(2k+x-z)^2}{4a\,t}}\, dz \\ & + \frac{1}{2\sqrt{\pi}} \int_0^t \int_{-1}^{1} \tilde{G}(s,z) \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{a\,(t-s)}}\, e^{-\frac{(2k+x-z)^2}{4a\,(t-s)}}\, dz ds, \end{aligned}$$

with $\tilde{h}$ and $\tilde{G}$ denoting the functions h and G "evenly mirrored around 0 in the x variable", that is

$$\tilde{h}(t,x) = \begin{cases} h(t,x) & x \in [0,1], \\ h(t,-x) & x \in [-1,0[, \end{cases}$$

Therefore, for all $t_1, t \ge 0$, and $x \in (0,1)$, $y \in \mathbb{R}_+$,

$$\begin{aligned} & f(t_1+t,x,y)\, e^{(t_1+t)\, y + \int_0^{t_1+t} \|M(s,\cdot)\|_{L^\infty_x}\, ds} \\ & \ge \frac{1}{2\sqrt{\pi}} \int_{-1}^{1} \tilde{f}(t_1,z,y) \frac{1}{\sqrt{a(y)\,t}}\, e^{-\frac{(x-z)^2}{4a(y)\,t}}\, e^{t_1\, y + \int_0^{t_1} \|M(s,\cdot)\|_{L^\infty_x}\, ds}\, dz\,, \end{aligned}$$

where we have neglected g_2 (and all the terms corresponding to $k \neq 0$ in the sum) as nonnegative. Considering $t \in [t_*, 2t_*]$ for some $t_* > 0$, recalling that $\|M(s,\cdot)\|_{L^\infty_x} \leq m_\infty + m_2(s)$ with $\int_0^\infty m_2(s)\,ds \leq \mu_1 < \infty$ by Lemma 2.2, and since $|x-z| < 2$ we have for all $\delta \leq y \leq 1/\delta$:

$$\begin{aligned} f(t_1+t,x,y) &\geq \frac{1}{2\sqrt{\pi}} \int_{-1}^{1} \tilde{f}(t_1,z,y)\, \frac{1}{\sqrt{a(y)\,t}}\, e^{-\frac{1}{a(y)\,t}}\, e^{-t\,y - \int_{t_1}^{t_1+t} \|M(s,\cdot)\|_{L^\infty_x}\,ds}\, dz \\ &\geq \frac{1}{\sqrt{2\pi\, a^*(\delta)\, t_*}} \int_0^1 f(t_1,z,y)\, e^{-\frac{1+y}{a_*\,t_*}}\, e^{-2t_*\,y - 2t_*\,m_\infty - \mu_1}\, dz \\ &\geq C \int_0^1 f(t_1,z,y)\, e^{-(2t_* + \frac{1}{a_*\,t_*})\,y}\, dz, \end{aligned}$$

where $C > 0$ depends on the constants $a_*, a^*(\delta)$, m_∞, μ_1 and $t_* > 0$. Using Lemma 2.3, for all K large enough, and $0 < \delta < 1$,

$$\begin{aligned} M(t_1+t,x) &\geq C\, e^{-(2t_* + \frac{1}{a_*\,t_*})\frac{1}{\delta}} \int_0^1 \int_\delta^{1/\delta} f(t_1,z,y)\, dydz \\ &\geq C\, e^{-(2t_* + \frac{1}{a_*\,t_*})\frac{1}{\delta}} \left(\mathcal{M}_{0*} - \delta\, N_\infty - K\,\delta - \int H(f)\,dx/\ln K \right), \end{aligned}$$

where we have used that $\int_{\frac{1}{\delta}}^\infty f(y)\,dy \leq \delta N$ and $\int_0^\delta f(y)\,dy \leq K\,\delta + \int_0^\delta f\,\frac{\ln f}{\ln K}\,dx$. Choosing δ and K, we get that $M(t_1+t,x) \geq \mathcal{M}_*$. Moreover, since $\mathcal{M}_* = \mathcal{M}_*(a_*, a^*, m_\infty, \mu_1, H(f_0), t_*)$ does not depend on t_1, we get Proposition 2.1. $\square$

3 Entropy-entropy dissipation estimate

For the subsequent large-time analysis, we shall study the relative entropy with respect to the global equilibrium, which dissipates according to (1.9) and (1.10) as

$$\frac{d}{dt} \int_\Omega H(f|f_\infty)\,dx \leq -D_1(f), \tag{3.1}$$

$$D_1(f) = \int_\Omega \int_0^\infty a(y) \frac{|\nabla_x f|^2}{f}\,dy\,dx + \int_\Omega \left[M\,H(f|f_N) + 2(M - \sqrt{N})^2 \right]\,dx \tag{3.2}$$

We introduce a lemma enabling to estimate the entropy of f by means of its entropy dissipation. This is a functional estimate, that is, the function f in this lemma does not depend on t and does not necessarily have something to do with the solution of our equation. Moreover, since this lemma is true in any dimension of space, we replace the interval $[0,1]$ by any bounded measurable subset Ω of $\mathbb{R}^N$ of measure 1. Then, for all quantity S, we denote by $\bar{S}$ its average w.r.t. $x \in \Omega$, that is $\bar{S} = \int_\Omega S(x)\,dx$.

Lemma 3.1 *Assume* (1.3)*, in particular* $0 < \frac{a_*}{1+y} \le a(y)$ *for all* $y \in [0,\infty)$*. Let* $f := f(x,y) \ge 0$ *be a measurable function from* $\Omega \times [0,+\infty[$ *to* $\mathbb{R}$ *with moments satisfying* $0 < \mathcal{M}_* \le M(x) := \int_0^\infty f(x,y)\,dy \le \|M\|_{L^\infty_x}$ *and* $0 < N_\infty := \int_\Omega \int_0^\infty y\, f(x,y)\,dydx = \overline{N}$*. Let* $p > 1$ *and assume that the moment of order* $2p$ *is finite, i.e.* $\int_\Omega \int_0^\infty y^{2p} f(x,y)\,dxdy = \mathcal{M}_{2p} < +\infty$*. Then, the following entropy-entropy dissipation estimate holds for all* $A \ge 1$ *:*

$$D_1(f) \ge \frac{C}{A\,\|M\|_{L^\infty_x}} \int_\Omega H(f|f_\infty)\,dx - C\,\frac{\mathcal{M}_{2p}}{A^{2p+1}}\,, \tag{3.3}$$

with a constant $C = C(\mathcal{M}_*, N_\infty, a_*, P(\Omega))$ *depending as specified only on* $\mathcal{M}_*$*,* N_∞*,* a_**, and the Poincaré constant* $P(\Omega)$*.*

Proof.- Step 1.- We start with the right-hand side of (3.3) by using the additivity (1.12) and calculating

$$\int_\Omega H(f|f_\infty)\,dx = \int_\Omega H(f|f_N)\,dx + 2\left(\sqrt{\overline{N}} - \overline{\sqrt{N}}\right), \tag{3.4}$$

where we recall that $\bar{S} := \int_\Omega S(x)\,dx$ (with $|\Omega| = 1$).

Step 2.- The second term of (3.4) – which measures how far N is from being constant – is bounded as :

$$\sqrt{\overline{N}} - \overline{\sqrt{N}} \le \frac{2}{\sqrt{N_\infty}}\left[\|M - \sqrt{N}\|^2_{L^2_x} + \|M - \overline{M}\|^2_{L^2_x}\right]. \tag{3.5}$$

Indeed, since $\sqrt{N} - \overline{\sqrt{N}}$ is orthogonal to $\overline{\sqrt{N}} - \overline{M}$ in L^2_x, and, thus, $\|\sqrt{N} - \overline{M} + \overline{M} - \overline{\sqrt{N}}\|^2_{L^2_x} = \|\sqrt{N} - \overline{M}\|^2_{L^2_x} + \|\overline{M} - \overline{\sqrt{N}}\|^2_{L^2_x}$, we have

$$\sqrt{\overline{N}} - \overline{\sqrt{N}} \le \frac{\overline{N} - \overline{\sqrt{N}}^2}{\sqrt{\overline{N}}} \le \frac{1}{\sqrt{N_\infty}}\|\sqrt{N} - \overline{\sqrt{N}}\|^2_{L^2_x} \le \frac{1}{\sqrt{N_\infty}}\|\sqrt{N} - \overline{M}\|^2_{L^2_x}\,,$$

and further, we obtain (3.5) by expanding $\|\sqrt{N} - M\|^2_{L^2_x}$ and by using Young's inequality

$$\frac{1}{2}\|\sqrt{N} - \overline{M}\|^2_{L^2_x} - \|M - \overline{M}\|^2_{L^2_x} \le \|\sqrt{N} - \overline{M} - M + \overline{M}\|^2_{L^2_x}\,.$$

Thus, we obtain (using $0 < \mathcal{M}_* < M$)

$$\int_\Omega H(f|f_\infty)\,dx \le \max\left\{\mathcal{M}_*^{-1}, 2N_\infty^{-\frac{1}{2}}\right\}\left[\int_\Omega M H(f|f_N)\,dx + 2\|M - \sqrt{N}\|^2_{L^2_x}\right] + \frac{4}{\sqrt{N_\infty}}\|M - \overline{M}\|^2_{L^2_x}\,. \tag{3.6}$$

Step 3.- For a cut-off size $A > 0$, we introduce the finite size density integral $M_A(t,x) := \int_0^A f(t,x,y)\,dy$ and its complement $M_A^c(t,x) := \int_A^\infty f(t,x,y)\,dy$ and proceed to estimate the last term in (3.6) in the following way :

$$\begin{aligned}\|M-\overline{M}\|_{L^2_x}^2 &= \int_\Omega \left(M_A - \overline{M_A} + M_A^c - \overline{M_A^c}\right)^2 dx \\ &\le 2\|M_A-\overline{M_A}\|_{L^2_x}^2 + \frac{4}{A^{2p}}\int_\Omega \left(\int_0^\infty y^p f(t,x,y)\,dy\right)^2 dx \\ &\le 2\|M_A-\overline{M_A}\|_{L^2_x}^2 + \frac{4}{A^{2p}}\|M\|_{L^\infty_x}\mathcal{M}_{2p}\,, \end{aligned} \tag{3.7}$$

for any $p > 1$, where the last term has been estimated thanks to Cauchy-Schwarz inequality.

Step 4.- Next, the variance of M_A, i.e. the first term on the right-hand side of (3.7) is controlled by the first, "Fisher"-type term of (3.1). Denoting by $P(\Omega)$ the constant of Poincaré's inequality, we estimate using Cauchy-Schwartz inequality and assumption (1.3), for $A \ge 1$:

$$\begin{aligned}\|M_A-\overline{M_A}\|_{L^2_x}^2 &\le P(\Omega)\int_\Omega \left|\nabla_x \int_0^A f\,dy\right|^2 dx \\ &\le P(\Omega)\int_\Omega \left|\int_0^A a(y)\frac{|\nabla_x f|^2}{f}dy\right|\left|\int_0^A \frac{f}{a(y)}dy\right| dx \\ &\le P(\Omega)\,\frac{1+A}{a_*}\,\|M\|_{L^\infty_x}\int_\Omega\int_0^\infty a(y)\frac{|\nabla_x f|^2}{f}\,dydx \\ &\le P(\Omega)\,\frac{2\,A}{a_*}\,\|M\|_{L^\infty_x}\int_\Omega\int_0^\infty a(y)\frac{|\nabla_x f|^2}{f}\,dydx\,, \end{aligned} \tag{3.8}$$

since we have $\frac{1}{a(y)} \le \frac{1+y}{a_*} \le \frac{1+A}{a_*}$ for $y \in [0,A]$.

Step 5.- Finally, combining (3.6) with (3.7) and (3.8), and further with (3.2), we have (still for $A \ge 1$)

$$\begin{aligned}\int_\Omega H(f|f_\infty)\,dx \le{}& C(\mathcal{M}_*, N_\infty, P(\Omega), a_*)\|M\|_{L^\infty_x}\,A\,D_1(f) + C(\mathcal{M}_*, N_\infty)\,D_1(f) \\ &+ C(N_\infty)\,\mathcal{M}_{2p}\|M\|_{L^\infty_x}\,A^{-2p}, \end{aligned}$$

which yields the proof of Lemma 3.1, since $\|M\|_{L^\infty_x}\,A \ge \mathcal{M}_*$. □

4 Proof of Theorem 1.1

With Proposition 2.1 and Lemmas 2.2 and 2.4 providing the moment bounds required by the entropy-entropy dissipation Lemma 3.1 in the one dimensional case $\Omega = (0,1)$, we turn now to the

Proof.- [Theorem 1.1] We divide the proof into two steps: First, we show polynomial convergence rates, and, secondly, we prove faster than polynomial rates via a summation argument.

Step 1.- We denote by C_1, C_2, etc., various constants which only depend on f_0, a_* and a^*. According to Lemmas 2.4 and 3.1, for any $A > 1$

$$\begin{aligned}\frac{d}{dt}\int_0^1 H(f|f_\infty)\,dx &\le -D_1(f)\\ &\le -\frac{C_1}{\|M\|_{L^\infty_x}}\frac{1}{A}\int_0^1 H(f|f_\infty)\,dx + C_2\,C_3^p\,\frac{2^{8p^2}}{A^{2p+1}}\,, \qquad (4.1)\end{aligned}$$

where $\|M\|_{L^\infty_x}(t) \le m_\infty + m_2(t)$ by Lemma 2.2. By choosing $A = A(t) = \max\{2, A^*(t)\}$ where $A^*(t)$ is defined by

$$C_2\,C_3^p\frac{2^{8p^2}}{(A^*)^{2p+1}} = \frac{1}{2}\frac{C_1}{\|M\|_{L^\infty_x}}\frac{1}{A^*}\int_0^1 H(f|f_\infty)\,dx\,,$$

it follows that

$$C_2\,C_3^p\frac{2^{8p^2}}{A^{2p+1}} \le \frac{1}{2}\frac{C_1}{\|M\|_{L^\infty_x}}\frac{1}{A}\int_0^1 H(f|f_\infty)\,dx\,.$$

Hence, denoting $C_4 = C_1/(2\,C_2)$, we see that

$$\frac{1}{A} \le C_3^{-1/2}\left(\frac{C_4\int_0^1 H(f|f_\infty)\,dx}{\|M\|_{L^\infty_x}\,2^{8p^2}}\right)^{\frac{1}{2p}}\,,$$

and we obtain

$$\frac{d}{dt}\int_0^1 H(f|f_\infty)\,dx \le -\frac{1}{2}\frac{C_1\,C_3^{-1/2}}{\|M\|_{L^\infty_x}}\left(\frac{C_4\int_0^1 H(f|f_\infty)\,dx}{\|M\|_{L^\infty_x}\,2^{8p^2}}\right)^{\frac{1}{2p}}\int_0^1 H(f|f_\infty)\,dx,$$

which integrates (for $t > t_* > 0$) like

$$\begin{aligned}\left(\int_0^1 H(f(t)|f_\infty)\,dx\right)^{-\frac{1}{2p}} - \left(\int_0^1 H(f(t_*)|f_\infty)\,dx\right)^{-\frac{1}{2p}} &\ge \frac{C_5}{p\,2^{4p}}\int_{t_*}^t \frac{ds}{\|M\|_{L^\infty_x}^{1+\frac{1}{2p}}}\\ \ge \frac{C_5}{p\,2^{4p}}\frac{1}{(1+m_\infty)^{1+\frac{1}{2p}}}\int_{t_*}^t \mathbb{I}_{\{m_2(s)<1\}}\,ds &\ge \frac{C_5}{p\,2^{4p}}\frac{t-t_*-\mu_1}{(1+m_\infty)^{1+\frac{1}{2p}}}\end{aligned}$$

(since $|\{m_2 \ge 1\}| \le \mu_1$ where $\mu_1 = \|m_2\|_{L^1}$ as defined in Lemma 2.2). Using the fact that $\int_0^1 H(f(t_*)|f_\infty)\,dx \ge 0$ and denoting $C_8 = ((1+m_\infty)/C_5)^2$, $C_7 = t_* + \mu_1$, we get

$$\int_0^1 H(f(t)|f_\infty)\,dx \le C_8^p\,p^{2p}\,2^{8p^2}\,(t-C_7)^{-2p}.$$

Step 2.- We may now sum over all $2 \leq p \in \mathbb{N}$

$$\left[\sum_{p\geq 2} \frac{(t-C_7)^{2p}}{(2C_8)^p\, p^{2p}\, 2^{8p^2}}\right] \int_0^1 H(f(t)|f_\infty)\, dx \leq 1,$$

i.e.

$$\int_0^1 H(f(t)|f_\infty)\, dx \leq L(t-C_7),$$

where (for all $1 < \alpha < 2$)

$$\begin{aligned} L^{-1}(t) &= \sum_{q\geq 1,\ \text{even}} \frac{t^q}{(C_{10}\, q)^q\, 2^{2q^2}} = \sum_{q\geq 1,\ \text{even}} t^q\, e^{-2q^2 \ln 2 - q\ln(q\, C_{10})} \\ &\geq C(\alpha) \sum_{q\geq 1,\ \text{even}} t^q\, e^{-\alpha\left(\frac{q}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} + \left(\frac{q+2}{\alpha}\right)^2}, \end{aligned}$$

since

$$e^{2q^2 \ln 2 + q\ln(q\, C_{10}) + \left(\frac{q+2}{\alpha}\right)^2} = O(e^{\alpha\left(\frac{q}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}).$$

Note that our assumption on α implies that $\frac{\alpha}{\alpha-1} > 2$.

Finally, choosing p (even) such that $t \in [\exp((p/\alpha)^{\frac{1}{\alpha-1}}), \exp(((p+2)/\alpha)^{\frac{1}{\alpha-1}})]$ and applying to the following elementary computation on these intervals :

$$\frac{e^{(\ln t)^{[2(\alpha-1)]}}}{t^p} \leq e^{\left(\frac{p+2}{\alpha}\right)^2 - p\left(\frac{p}{\alpha}\right)^{\frac{1}{\alpha-1}}} = e^{\left(\frac{p+2}{\alpha}\right)^2 - \alpha\left(\frac{p}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}},$$

we have

$$L^{-1}(t) \geq C(\alpha)\, e^{[\ln^{2(\alpha-1)}(t)]}$$

for all t large enough. and $1 < \alpha < 2$. This ends the proof of (1.13). The proof of the L^1-decay estimate (1.14) follows from the Csiszar-Kullback inequality. □

In the proof of Proposition 1.1, we use explicit L^r bounds ($r \geq 1$) for the 1D heat equation. Similar bounds were already established in [DF06]. Here we prove an improved version allowing, in particular, unbounded diffusion coefficients. As these bounds will be used pointwise in y, we will suppress for notational convenience the dependence on y.

Lemma 4.1 *Let u denote the solution of the 1D heat equation ($t > 0, x \in [0,1]$, and (constant) diffusivity a) with homogeneous Neumann boundary condition, i.e.*

$$\partial_t u - a\, \partial_{xx} u = g\,, \qquad \partial_x u(t,0) = \partial_x u(t,1) = 0\,, \tag{4.2}$$

and assume for the initial data $u(0,x) = u_0(x)$ and the source term $g(t,x)$ that

$$u_0 \in L^p([0,1])\,, \qquad g \in L^p([0,+\infty) \times [0,1])\,.$$

Then, for the exponents $r, p \geq 1$ and $q \in [1,3)$ satisfying $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ and for all $T > 0$, the norm $\|u\|_{L^r([0,T]\times[0,1])}$ grows at most polynomially in T like

$$\begin{aligned}\|u\|_{L^r([0,T]\times[0,1])} \leq &\left[C\,T^{\frac{1}{r}} + C\,a^{-\frac{1}{3r}}T^{\frac{2}{3r}}\right]\|u_0\|^{\frac{1}{r}}_{L^p[0,1]}\\ &+\left[C\,T^{\frac{1}{q}} + C\,a^{-\frac{1}{3}}T^{\frac{3-q}{3q}}\right]\|g\|_{L^p([0,T]\times[0,1])}\,,\end{aligned} \tag{4.3}$$

for various constants C (depending only on r, p).

Proof.- As in [DF06], the proof uses Fourier series, which simplify when (4.2) is mirrored evenly around $x = 0$, i.e. when the functions are extended like

$$\tilde{u}(t,x) = \begin{cases} u(t,x) & x \in [0,1],\\ u(t,-x) & x \in [-1,0],\end{cases}$$

and when $\tilde{g}$ and $\tilde{u_0}$ are defined analogously. Then, expanding the solution of the heat equation in a Fourier series in terms of the eigenfunctions of the Laplacian with periodic boundary conditions on $\tilde{x} \in [-1,1]$, we proceed as in [DF06] using Poisson's summation formula and Young's inequality for convolutions to obtain (with $L^r_{T,\tilde{x}}$ and $L^r_{\tilde{x}}$ as short-cuts for $L^r([0,T]\times[-1,1])$ and $L^r([-1,1])$, respectively)

$$\|\tilde{u}\|_{L^r_{T,\tilde{x}}} \leq \frac{1}{2\sqrt{\pi}}\left(\int_0^T \|S\|_{L^q_{\tilde{x}}}\,dt\right)^{\frac{1}{r}}\|\tilde{u_0}\|_{L^p_{\tilde{x}}} + \frac{1}{2\sqrt{\pi}}\|S\|_{L^q_{T,\tilde{x}}}\|\tilde{g}\|_{L^p_{T,\tilde{x}}}\,, \tag{4.4}$$

where $S(t,x) := \frac{1}{\sqrt{at}}\left[e^{-\frac{x^2}{4at}} + 2\sum_{n=1}^{\infty} e^{-\frac{(2n+x)^2}{4at}}\right] \leq \frac{2}{\sqrt{at}}\sum_{n=0}^{\infty} e^{-\frac{(2n+x)^2}{4at}}$.

In order to estimate $\|S\|_{L^q_{\tilde{x}}}$, we estimate first

$$\begin{aligned}\left[\sum_{n=0}^{\infty} e^{-\frac{(2n+x)^2}{4at}}\right]^2 &= \sum_{n=0}^{\infty} e^{-\frac{(2n+x)^2}{2at}} + 2\sum_{n=1}^{\infty}\sum_{m=0}^{n-1} e^{-\frac{(2n+x)^2}{4at}}\, e^{-\frac{(2m+x)^2}{4at}}\\ &\leq \sum_{n=0}^{\infty} e^{-\frac{(2n+x)^2}{2at}} + 2\sum_{n=1}^{\infty} n\, e^{-\frac{(2n+x)^2}{4at}}\,.\end{aligned}$$

Similarly, we see that

$$\left[\sum_{n=0}^{\infty} e^{-\frac{(2n+x)^2}{4at}}\right]^3 \leq \sum_{n=0}^{\infty} e^{-\frac{3(2n+x)^2}{4at}} + 4\sum_{n=1}^{\infty} n\, e^{-\frac{(2n+x)^2}{4at}} + 4\sum_{n=1}^{\infty} n^2\, e^{-\frac{(2n+x)^2}{4at}}\,,$$

so that

$$\begin{aligned}\int_{-1}^{1}\left[\sum_{n=0}^{\infty} e^{-\frac{(2n+x)^2}{4at}}\right]^3 dx &\leq \int_{-1}^{\infty} e^{-\frac{3y^2}{4at}}\,dy + 4\int_1^{\infty} y\, e^{-\frac{y^2}{4at}}\,dy + 4\int_1^{\infty} y^2\, e^{-\frac{y^2}{4at}}\,dy\\ &\leq C\,(at)^{\frac{1}{2}} + C\,(at) + C\,(at)^{\frac{3}{2}}\,,\end{aligned}$$

for various constants C. Altogether, we see that $\|S\|^3_{L^3_{\tilde{x}}} = O(\frac{1}{at})$ and thus $\|S\|_{L^3_{T,\tilde{x}}}$ is not necessarily integrable at $t=0$. Nevertheless, using Hölders inequality for $q<3$

$$\int_{-1}^{1} \Big(\sum_{n=0}^{\infty} \frac{2}{\sqrt{at}}\, e^{-\frac{(2n+x)^2}{4at}}\Big)^q dx \leq 2^q \Big[\int_{-1}^{1} \Big(\sum_{n=0}^{\infty} \frac{1}{\sqrt{at}}\, e^{-\frac{(2n+x)^2}{4at}}\Big)^3 dx\Big]^{\frac{q}{3}} \Big[\int_{-1}^{1} 1\, dx\Big]^{\frac{3-q}{3}}$$
$$\leq \Big[C\,(at)^{-1} + C\,(at)^{-\frac{1}{2}} + C\Big]^{\frac{q}{3}} ,$$

for various constants C, which can be chosen independent of q. Finally, we find

$$\|S\|_{L^q_{\tilde{x}}} \leq C + C\,(at)^{-\frac{1}{3}} ,$$

Lemma 4.1 is then obtained after integration w.r.t time. ☐

Finally, we show Proposition 1.1. We denote by $C[T]$ any constant of the form $C(t)\,(1+T)^s$, where $s\in\mathbb{R}$ and $C(t)$ is bounded on any interval $[t_*,+\infty)$ with $t_*>0$.

Proof of Proposition 1.1.- We observe for the gain term Q^+ that using the bounds (2.3) and (2.1) and for all $q\geq 0$,

$$\int_0^T\!\!\int_0^1\!\!\int_0^\infty (1+y)^q\, Q^+(f,f)\, dy\, dx\, dt \leq \int_0^T\!\!\int_0^1\!\!\int_0^\infty \frac{(1+y)^{q+1}}{q+1}\, f(t,x,y)\, dy\, dx\, dt$$
$$+ \int_0^T\!\!\int_0^1\!\!\int_0^\infty\!\!\int_0^\infty (1+y+z)^q\, f(t,x,y)\, f(t,x,z)\, dz\, dy\, dx\, dt$$
$$\leq C_q\,(\mathcal{M}_0^* + \mathcal{M}_{q+1}^*)\, T + C_q \int_0^T \|M(t,\cdot)\|_{L^\infty_x} (\mathcal{M}_0^* + \mathcal{M}_q^*)\, dt \leq C_q\, C[T]\,.$$

Using lemma 4.1 pointwise in y, we have

$$\|f(\cdot,\cdot,y)\|_{L^r([t_*,T]\times[0,1))} \leq \Big[C\,T^{\frac{1}{r}} + C\,a(y)^{-\frac{1}{3r}} T^{\frac{2}{3r}}\Big]\, \|f(t_*,\cdot,y)\|^{\frac{1}{r}}_{L^p_x}$$
$$+ \Big[C\,T^{\frac{1}{q}} + C\,a(y)^{-\frac{1}{3}} T^{\frac{3-q}{3q}}\Big]\, \|Q^+(f,f)(\cdot,\cdot,y)\|_{L^p([0,T]\times[0,1))}.$$

As a consequence for any $\varepsilon>0$ and $t_*>0$,

$$\int_0^\infty (1+y)^q\, \|f(\cdot,\cdot,y)\|_{L^{3-\varepsilon}([t_*,T]\times[0,1))}\, dy \leq C[T]. \tag{4.5}$$

Then, for all $r \in [2,3)$

$$
\begin{aligned}
&\int_0^\infty (1+y)^q \, \|Q^+(f,f)(\cdot,\cdot,y)\|_{L^{r/2}([t_*,T]\times[0,1])}\, dy \\
&\qquad \le \int_0^\infty \frac{(1+y)^{q+1}}{q+1} \, \|f(\cdot,\cdot,y)\|_{L^r([t_*,T]\times[0,1])}\, dy \\
&\qquad\quad + \int_0^\infty (1+y)^q \, \| \int_0^\infty f(\cdot,\cdot,y')\, f(\cdot,\cdot,y-y')\, dy' \|_{L^{r/2}([t_*,T]\times[0,1])}\, dy \\
&\qquad \le C_q\, C[T] + \int_0^\infty\!\!\int_0^\infty (1+y+z)^q \|f(\cdot,\cdot,y)\, f(\cdot,\cdot,z)\|_{L^{r/2}([t_*,T]\times[0,1])}\, dydz \\
&\qquad \le C_q\, C[T] + \left(C_q \int_0^\infty (1+y)^q \, \|f(\cdot,\cdot,y)\|_{L^r([t_*,T]\times[0,1])} dy \right)^2 \le C_q\, C[T].
\end{aligned}
$$

Using again the properties of the heat kernel (still described in [DF06]), we see that for any $s \in [1,\infty)$, $q \ge 0$ and $t_* > 0$,

$$
\int_0^\infty (1+y)^q \, \|f(\cdot,\cdot,y)\|_{L^s([t_*,T]\times[0,1])} dy \le C[T].
$$

It is finally possible to repeat this argument from (4.5) with $\varepsilon = 0$, and get

$$
\int_0^\infty (1+y)^q \, \|f(\cdot,\cdot,y)\|_{L^\infty([t_*,T]\times[0,1])} dy \le C[T]. \tag{4.6}
$$

The above argument can now be used with $r = 4$ and shows that

$$
\int_0^\infty (1+y)^q \, \|Q^+(f,f)(\cdot,\cdot,y)\|_{L^2([t_*,T]\times[0,1])}\, dy \le C[T].
$$

Next, we use, as in the proof of lemma 4.1, a convolution formula for solutions of the heat equation mirrored around $\tilde{x} = 0$. Taking the derivative in x, we find pointwise in y with $S(t,\tilde{x}) := \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{a(y)t}} e^{-\frac{(2k+\tilde{x})^2}{4a(y)t}}$ for $\tilde{x} \in [-1,1]$

$$
\tilde{f}_x(t,\tilde{x}) = \frac{1}{2\sqrt{\pi}}\, \tilde{f}_0 *_{\tilde{x}} S_x + \frac{1}{2\sqrt{\pi}}\, Q(\tilde{f}) *_{t,\tilde{x}} S_x\, ,
$$

where $S_x = \partial_x S$ and $*_x$ and $*_{t,x}$ denote the convolution in space and space/time, respectively. We then estimate

$$
\begin{aligned}
\|S_x\|_{L^1_x} &\le \frac{1}{\sqrt{a(y)\, t}} \int_{-1}^1 \left| \sum_{k=-\infty}^{\infty} e^{-\frac{(2k+\tilde{x})^2}{4a(y)t}} \left(-\frac{2k+\tilde{x}}{2a(y)t} \right) \right| dx \\
&\le \frac{1}{\sqrt{a(y)\, t}} \int_{-1}^1 \left[e^{-\frac{\tilde{x}^2}{4a(y)t}} \frac{|\tilde{x}|}{2a(y)t} + 2 \sum_{k=1}^{\infty} e^{-\frac{(2k+\tilde{x})^2}{4a(y)t}} \left(\frac{2k+\tilde{x}}{2a(y)t} \right) \right] dx \\
&= \frac{2}{\sqrt{a(y)\, t}} \left(\int_0^1 e^{-\frac{z^2}{4a(y)t}} \frac{z}{2a(y)t}\, dz + \int_1^\infty e^{-\frac{z^2}{4a(y)t}} \frac{z}{2a(y)t}\, dz \right) \\
&\le \frac{8}{\sqrt{a(y)\, t}}\, ,
\end{aligned}
$$

to obtain with Young's inequality

$$
\begin{aligned}
\|\tilde{f}_x(t)\|_{L^2_{\tilde{x}}} &\leq \frac{1}{2\sqrt{\pi}}\,\|f_0\|_{L^2_{\tilde{x}}}\|S_x\|_{L^1_{\tilde{x}}} + \frac{1}{2\sqrt{\pi}}\,\|Q(\tilde{f})\|_{L^2_{t,\tilde{x}}}\|S_x\|_{L^1_{t,\tilde{x}}} \\
&\leq \frac{C}{\sqrt{a(y)\,t}}\,\|f_0\|_{L^2_{\tilde{x}}} + \frac{C\sqrt{t}}{\sqrt{a(y)}}\|Q(\tilde{f})\|_{L^2_{t,\tilde{x}}}\,,
\end{aligned}
$$

for constants C. Hence, considering for instance $T \geq 2t_*$, we have that

$$
\int_0^\infty (1+y)^q\,\|f(T,\cdot,y)\|_{H^1_x}\,dy \leq C[T]\,,
$$

where $C[T]$ depends on t_*. Then, using a Gagliardo-Nirenberg type interpolation and Theorem 1.1, we obtain

$$
\begin{aligned}
\int_0^\infty (1+y)^q\,\|f(T,\cdot,y) - f_\infty(y)\|_{L^\infty_x}\,dy &\leq \int_0^\infty \left[(1+y)^q\,\|f(T,\cdot,y) - f_\infty(y)\|^{\frac{3}{4}}_{H^1_x}\right] \\
&\qquad \times \left[\|f(T,\cdot,y) - f_\infty(y)\|^{\frac{1}{4}}_{L^1_x}\right] dy \\
\leq \left[\int_0^\infty (1+y)^{\frac{4q}{3}}\,\|f(T,\cdot,y) - f_\infty(y)\|_{H^1_x} dy\right]^{\frac{3}{4}} &\left[\int_0^\infty \|f(T,\cdot,y) - f_\infty(y)\|_{L^1_x} dy\right]^{\frac{1}{4}} \\
&\leq C[T]^{\frac{3}{4}}\,\exp(-\ln^\beta(T)) \leq C\exp(-\ln^{\beta'}(T)),
\end{aligned}
$$

for $2 > \beta > \beta'$, which concludes the proof of Proposition 1.1. □

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