CONVERGENCE TO EQUILIBRIUM IN LARGE TIME FOR BOLTZMANN AND B.G.K. EQUATIONS

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Abstract

We study in this paper the equilibrium states of a gas in a box with specular reflexion conditions. We prove that the renormalized solutions of the Boltzmann equation converge towards those states in the large time asymptotics.

1 Introduction

Rarefied gas dynamics is usually described by the Boltzmann equation:

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \tag{1}$$

where $f(t, x, v)$ is the density of particles which at time t and point x move with velocity v, and Q is a quadratic collision term described in [Ce], [Ch, Co] or [Tr, Mu].

A large amount of information on the behavior of the gas is contained in the simpler model of B.G.K.,

$$\partial_t f + v \cdot \nabla_x f = M_f - f, \tag{2}$$

where M_f is the Maxwellian having the same moments as f (Cf. [Bh, Gr, Kr]).

From the physical point of view, the density of particles is assumed to converge to an equilibrium represented by a Maxwellian function of the velocity v when the time t becomes large. This Maxwellian is assumed to be global (independant of x) when x lies in a bounded domain with suitable boundary conditions or in a periodic box. In that case, the walls are said to "thermalize" the gas (Cf. [Ar 1]). The reader can find a survey on the problems of convergence towards equilibrium in [De 1].

The goal of this work is to give some mathematical results on these topics. Note that this problem has already been studied by L. Arkeryd in the case of the full Boltzmann equation in a periodic box (Cf. [Ar 1]). Note also that the more complicated case when electromagnetic self consistant forces are taken into account is now treated in [De, Do].

We shall systematically use the results on existence of solutions to the Boltzmann equation stated by R.J. Di Perna and P–L. Lions in [DP, L] and their extension when f is assumed to satisfy various boundary conditions obtained by K. Hamdache (Cf. [Ha]).

In section 2, we establish mathematically the convergence of f to a Maxwellian satisfying the free transport equation when x varies in a bounded domain.

A complete study of the Maxwellians satisfying this condition is given in section 3.

Then, section 4 is devoted to using this description together with some classical boundary conditions in order to establish the thermalizing effect of the walls.

Finally, section 5 is devoted to the study of the strong convergence in the case of the B.G.K. equation.

2 Convergence to equilibrium

This section is devoted to the study of the long time behavior of the Boltzmann and B.G.K equations when x lies in a bounded domain. Note that L. Arkeryd has already proved in [Ar 1] the following result, using non–standard techniques:

If $f(t,x,v)$ is a renormalized solution of the Boltzmann equation with x varying in a periodic box, then for every sequences t_n going to infinity, there exist a subsequence t_{n_k} and a global time–independant Maxwellian m such that $f_{n_k}(t,x,v) = f(t_{n_k}+t,x,v)$ converges weakly in $L^1([0,T]\times \mathbb{R}^N/\mathbb{Z}^N \times \mathbb{R}^N)$ to m for every $T>0$.

Let Ω be a bounded regular and connected open set of $\mathbb{R}^N$. We denote by $n(x)$ its outward normal at point x and consider the Boltzmann equation,

$$\partial_t f + v\cdot\nabla_x f = Q(f,f), \tag{3}$$

where Q is a quadratic collision kernel acting only on the variable v,

$$Q(f,f)(v) = \int_{v_1\in\mathbb{R}^N}\int_{\omega\in S^{N-1}} \{\, f(v')f(v_1') - f(v)f(v_1)\}\, B(v,v_1,\omega)\, d\omega dv_1, \tag{4}$$

with

$$v' = \quad v + ((v_1 - v)\cdot\omega)\,\omega, \tag{5}$$
$$v_1' = \quad v_1 - ((v_1 - v)\cdot\omega)\,\omega, \tag{6}$$

and $B(v,v_1,\omega)$ is a cross section. We make the

Assumption 1:

1. *The cross section B depends only on $|v-v_1|$ and $|(v-v_1)\cdot\omega|$,*
2. *The cross section B is strictly positive a.e.,*
3. *There exists $C>0$ such that*

$$B(v,v_1,\omega) \le C(1+|v|+|v_1|). \tag{7}$$

Note that assumption 1 is satisfied by the hard–spheres and hard potentials models with the hypothesis of angular cut–off of Grad (Cf. [Ce], [Ch, Co], [Gr] or [Tr, Mu]). Note also that the third part of assumption 1 can probably be relaxed, since the existence of renormalized solutions is proved under a somewhat weaker assumption (Cf. [DP, L]). However, we will keep here this assumption for the sake of simplicity.

The time variable t is in $[0, +\infty[$, the position x is in Ω and the velocity v is in $I\!R^N$.

We add to eq. (1) one of the two following linear boundary conditions: For all x lying in $\partial\Omega$ and all v such that $v \cdot n(x) \leq 0$,

1. Reverse reflexion holds:
$$f(t, x, v) = f(t, x, -v). \tag{8}$$

2. Specular reflexion holds:
$$f(t, x, v) = f(t, x, v - 2(v \cdot n(x))\, n(x)). \tag{9}$$

Finally, we make on the initial datum f_0 the

Assumption 2:

1. *The initial datum f_0 is nonnegative,*
2. *The initial datum f_0 satisfies the following natural bounds:*
$$\int_\Omega \int_{I\!R^N} f_0(x, v)(\, 1 + |v|^2 + |\log f_0(x, v)|\,)\, dvdx < +\infty. \tag{10}$$

According to [DP, L] and [Ha], under assumptions 1 and 2, equations (1), (8) or (1), (9) admit a nonnegative renormalized solution $f(t, x, v) \in C([0, +\infty[; L^1(\Omega \times I\!R^N))$, and the boundary condition holds for the trace of f on $[0, +\infty[\times \partial\Omega \times I\!R^N$. The main result of this section is the following:

Theorem 1: *Let $f(t, x, v)$ be a renormalized solution of equations (1), (8) or (1), (9) under assumptions 1 and 2. Then, for every sequences t_n going to infinity, there exist a subsequence t_{n_k} and a local time–dependant Maxwellian $m(t, x, v)$ such that $f_{n_k}(t, x, v) = f(t_{n_k} + t, x, v)$ converges weakly in $L^1([0, T] \times \Omega \times I\!R^N)$ to $m(t, x, v)$ for every $T > 0$. Moreover, this Maxwellian satisfies the free transport equation,*
$$\partial_t m + v \cdot \nabla_x m = 0, \tag{11}$$

and the boundary condition (8) or (9).

Proof of theorem 1: Note that the proof given by L. Arkeryd in [Ar 1] holds in this case (the assumption of x being in a periodic box is not used in this part of the proof). However, we give here a different proof, without any use of non–standard analysis.

We shall consider the solution f of theorem 1. According to [DP, L], [Ha] and using assumption 2, the conservations of mass and energy and Boltzmann's H–theorem ensure that

$$\sup_{t\in[0,+\infty[} \int_{x\in\Omega}\int_{v\in I\!R^N} f(t,x,v)\,(1+|v|^2+|\log f(t,x,v)|)\,dvdx$$

$$+\int_0^{+\infty}\int_{x\in\Omega}\int_{v\in I\!R^N}\int_{v_1\in I\!R^N}\int_{\omega\in S^{N-1}} \{\, f(t,x,v')f(t,x,v_1') - f(t,x,v)f(t,x,v_1)\,\}$$
$$\{\,\log(f(t,x,v')f(t,x,v_1')) - \log(f(t,x,v)f(t,x,v_1))\,\}$$
$$B(v,v_1,\omega)\,d\omega dv_1 dvdxdt < +\infty. \tag{12}$$

Therefore, $f_n(t,x,v) = f(t_n+t,x,v)$ is weakly compact in $L^1([0,T]\times\Omega\times I\!R^N)$ for all sequences t_n of nonnegative numbers and all $T>0$.

Now consider a sequence of nonnegative numbers t_n going to infinity. The weak compactness of $f_n(t,x,v)$ ensures the existence of a subsequence t_{n_k} and a function m in $L^1([0,T]\times\Omega\times I\!R^N)$ such that the functions f_{n_k} converge to m weakly in $L^1([0,T]\times\Omega\times I\!R^N)$ for all $T>0$. In order to prove that m is a Maxwellian, we use estimate (12) in the following way:

$$\int_{t_{n_k}}^{t_{n_k}+T}\int_{x\in\Omega}\int_{v\in I\!R^N}\int_{v_1\in I\!R^N}\int_{\omega\in S^{N-1}} \{\, f(t,x,v')f(t,x,v_1') - f(t,x,v)f(t,x,v_1)\,\}$$
$$\{\,\log(f(t,x,v')f(t,x,v_1')) - \log(f(t,x,v)f(t,x,v_1))\,\}$$
$$B(v,v_1,\omega)\,d\omega dv_1 dvdxdt \underset{k\to+\infty}{\to} 0, \tag{13}$$

and thus

$$\int_0^T\int_{x\in\Omega}\int_{v\in I\!R^N}\int_{v_1\in I\!R^N}\int_{\omega\in S^{N-1}} \{f_{n_k}(t,x,v')f_{n_k}(t,x,v_1')$$
$$-\,f_{n_k}(t,x,v)f_{n_k}(t,x,v_1)\}\,\{\,\log(f_{n_k}(t,x,v')f_{n_k}(t,x,v_1'))$$
$$-\,\log(f_{n_k}(t,x,v)f_{n_k}(t,x,v_1))\,\}\,B(v,v_1,\omega)\,d\omega dv_1 dvdxdt \underset{k\to+\infty}{\to} 0. \tag{14}$$

But according to the proof of [DP, L] and [Ha], for all smooth nonnegative functions ϕ, ψ with compact support,

$$\int_{v\in I\!R^N}\int_{v_1\in I\!R^N}\int_{\omega\in S^{N-1}} f_{n_k}(t,x,v')\, f_{n_k}(t,x,v')\, \phi(v)\, \psi(v_1)\, B(v,v_1,\omega)\, d\omega dv_1 dv$$

tends to

$$\int_{v\in I\!R^N}\int_{v_1\in I\!R^N}\int_{\omega\in S^{N-1}} m(t,x,v')\, m(t,x,v_1')\, \phi(v)\, \psi(v_1)\, B(v,v_1,\omega)\, d\omega dv_1 dv \tag{15}$$

in measure on $[0,T]\times\Omega$, and

$$\int_{v\in I\!R^N}\int_{v_1\in I\!R^N}\int_{\omega\in S^{N-1}} f_{n_k}(t,x,v)\, f_{n_k}(t,x,v_1)\, \phi(v)\, \psi(v_1)\, B(v,v_1,\omega)\, d\omega dv_1 dv$$

tends to

$$\int_{v\in I\!R^N}\int_{v_1\in I\!R^N}\int_{\omega\in S^{N-1}} m(t,x,v)\, m(t,x,v_1)\, \phi(v)\, \psi(v_1)\, B(v,v_1,\omega)\, d\omega dv_1 dv \tag{16}$$

in the same sense.

It is possible to extract a subsequence $f_{n_{k_p}}$ such that the convergence holds a.e in $[0,T]\times\Omega$ in (14), (15) and (16) for a dense and enumerable set in $C(I\!R^N)$ of nonnegative smooth functions ϕ and ψ. But $P(x,y) = (x-y)(\log x - \log y)$ is a nonnegative convex function from $I\!R_+\times I\!R_+$ to $I\!R_+$. Therefore, using the convexity of P, we get for a.e. (t,x,v,v_1,ω),

$$P(\, m(t,x,v')m(t,x,v_1'),\; m(t,x,v)m(t,x,v_1)\,)\, \phi(v)\, \psi(v_1)\, B(v,v_1,\omega) = 0. \tag{17}$$

Then, the nonnegativity of P and the strict positivity of B ensure that

$$m(t,x,v')\; m(t,x,v_1') = m(t,x,v)\; m(t,x,v_1) \tag{18}$$

for a.e (t,x,v,v_1,ω). According to [Tr, Mu], the function m is a Maxwellian. Moreover, following the proof of [DP, L] and [Ha] of weak stability, m is a renormalized solution of (1) and satisfies the boundary conditions (8) or (9). Therefore, $Q(m,m)=0$, and m satisfies the free transport equation

$$\partial_t m + v\cdot\nabla_x m = 0. \tag{19}$$

In a second step, we concentrate on the B.G.K. equation (2). B. Perthame proved in [Pe] that this equation admits a nonnegative solution when x varies in the whole space $I\!R^N$ and under natural assumptions on the initial datum f_0. This result was extended by E. Ringeissen in [Ri] in the case of a bounded domain with boundary conditions (8) or (9). More precisely, he proved that if the initial datum f_0 satisfies assumption 2, there exists a solution f to (2), (8) or (2), (9) in $C([0,+\infty[; L^1(\Omega \times I\!R^N))$, and the boundary condition holds for the trace of f on $\partial\Omega \times I\!R^N$. We prove the following result:

Theorem 2: *Let f be a solution of eq. (2) with boundary conditions (8) or (9) and initial datum f_0 satisfying assumption 2. We shall also assume that f satisfies the following property:*

$$\sup_{t\in[0,+\infty[} \int_{x\in\Omega} \int_{v\in I\!R^N} f(t,x,v)\,|v|^3\, dvdx < +\infty. \tag{20}$$

Then, for every sequence t_n going to infinity, there exists a subsequence t_{n_k} and a local time–dependant Maxwellian $m(t,x,v)$ such that $f_{n_k}(t,x,v) = f(t+t_{n_k},x,v)$ converges weakly in $L^1([0,T]\times\Omega\times I\!R^N)$ to $m(t,x,v)$ for every $T>0$. Moreover, m satisfies the free transport equation (11) and the boundary conditions (8) or (9).

Proof of theorem 2: According to [DP, L] and [Bh, Gr, Kr], and using (10) and (20), the conservation of mass and Boltzmann's H–theorem ensure that

$$\sup_{t\in[0,+\infty[} \int_{x\in\Omega} \int_{v\in I\!R^N} f(t,x,v)\,(1+|v|^3+|\log f(t,x,v)|)\, dvdx$$
$$+\int_0^{+\infty} \int_{x\in\Omega} \int_{v\in I\!R^N} \{\, M_f(t,x,v) - f(t,x,v)\,\}$$
$$\{\,\log M_f(t,x,v) - \log f(t,x,v)\,\}\, dvdxdt < +\infty. \tag{21}$$

Therefore, $f_n(t,x,v) = f(t_n+t,x,v)$ is weakly compact in $L^1([0,T]\times\Omega\times I\!R^N)$ for all sequences t_n and all $T>0$.

Now consider a sequence of nonnegative numbers t_n going to infinity, and a given $T>0$. The weak compactness of $f_n(t,x,v)$ ensures the existence of a subsequence t_{n_k} and a function m in $L^1([0,T]\times\Omega\times I\!R^N)$ such that the functions f_{n_k} converge to m weakly in $L^1([0,T]\times\Omega\times I\!R^N)$. In order to prove that m is a Maxwellian, we use estimate (21) in the following way:

$$\int_{t_{n_k}}^{t_{n_k}+T} \int_{x\in\Omega} \int_{v\in I\!R^N} \{\, M_f(t,x,v) - f(t,x,v)\,\}$$

$$\{\, \log M_f(t,x,v) - \log f(t,x,v)\,\}\, dvdxdt \underset{k\to+\infty}{\longrightarrow} 0, \tag{22}$$

and therefore,

$$\int_0^T \int_{x\in\Omega} \int_{v\in I\!R^N} \{\, M_{f_{n_k}}(t,x,v) - f_{n_k}(t,x,v)\,\}$$
$$\{\, \log M_{f_{n_k}}(t,x,v) - \log f_{n_k}(t,x,v)\,\}\, dvdxdt \underset{k\to+\infty}{\longrightarrow} 0. \tag{23}$$

Following the proof of [Ri], we prove that $M_{f_{n_k}}$ converges weakly (and even strongly) in L^1 to M_m because estimate (20) implies that

$$\int_0^T \int_{x\in\Omega} \int_{v\in I\!R^N} f_n(t,x,v)|v|^3\, dvdxdt \tag{24}$$

is uniformly bounded.

Now estimate (23) and the convexity of P ensure that for all smooth and nonnegative functions ϕ with compact support,

$$P(M_m(t,x,v), m(t,x,v))\, \phi(t,x,v) = 0. \tag{25}$$

Then, the nonnegativity of P implies that for a.e. $(t,x,v) \in [0,T]\times\Omega\times I\!R^N$,

$$M_m(t,x,v) = m(t,x,v). \tag{26}$$

Finally, m is a Maxwellian function of v. Following the proof of existence of [Ri], we can see that m satisfies eq. (11) and the boundary conditions (8) or (9).

3 Maxwellian solutions of the free transport equation

Theorem 1 ensures that when the time becomes large, the renormalized solutions of the Boltzmann equation (in a bounded domain with boundary conditions (8) or (9)) behave as Maxwellians satisfying the free transport equation (11).

Therefore, this section will be devoted to the complete description of such Maxwellians. We begin by the

Lemma 1: *Let $m(t,x,v) \in L^1([0,T]\times\Omega\times I\!R^N)$ be a nonnegative Maxwellian function of v satisfying the free transport equation (11). Moreover, we suppose that for all $t\in[0,T]$, $m(t,\cdot,\cdot)$ is not identically equal to 0. Then, m is strictly positive.*

Proof of lemma 1: In this lemma and the following, we do not take into account the negligeable sets on which the equations do not hold. The interested reader can find a detailed proof of lemma 1 in [De 2].

Since $m \in L^1$ is a nonnegative Maxwellian function of v, we can write

$$m(t,x,v) = \frac{\rho(t,x)}{(2\pi\Theta(t,x))^{N/2}}\, exp\Big\{ -\frac{|v-u(t,x)|^2}{2\Theta(t,x)} \Big\}, \tag{27}$$

where $0 < \Theta < +\infty$ and $\rho \geq 0$.

We begin by the case when Ω is convex. We consider $t \in [0,T]$. There exists $x_0 \in \Omega$ and $v_0 \in I\!\!R^N$ such that $m(t,x_0,v_0) > 0$. Therefore, $\rho(t,x_0) > 0$, and according to eq. (11), for any s, v such that $t+s \in [0,T]$ and $x_0+vs \in \Omega$, $\rho(t+s, x_0+vs) > 0$. Finally, f is strictly positive on $[0,T] \times \Omega \times I\!\!R^N$.

If Ω is not convex, note however that between two points $x, y \in \Omega$, one can find $x_1, ..., x_K \in \Omega$ such that $[x,x_1],[x_1,x_2],...,[x_K,y] \subset \Omega$. Then, by induction, one can find t_i such that $\rho(t_i,x_i) > 0$. According to the previous part of the proof, ρ is strictly positive in a convex neighbourhood of y for any time $t \in [0,T]$. Therefore, lemma 1 is proved.

According to lemma 1, we can consider the logarithm of any m satisfyng its hypothesis. Since m is a Maxwellian, one can find measurable $a(t,x)$, $b(t,x)$ and $c(t,x)$ such that

$$\log m(t,x,v) = a(t,x) + b(t,x)\cdot v + c(t,x)|v|^2. \tag{28}$$

Moreover, $\log m$ still satisfies eq. (11) in the following sense,

For any $t \in [0,T], x \in \Omega, v \in I\!\!R^3$, and s such that $s+t \in [0,T]$ and $[x, x+vs] \subset \Omega$,

$$\log m(t,x,v) = \log m(t+s, x+vs, v). \tag{29}$$

Then, we prove the following lemma,

Lemma 2: *Let h be a measurable function such that one can find $a(t,x), b(t,x)$ and $c(t,x)$ satisfying*

$$h(t,x,v) = a(t,x) + b(t,x)\cdot v + c(t,x)|v|^2, \tag{30}$$

and such that for any $t \in [0,T], x \in \Omega, v \in I\!\!R^3$, and s satisfying $s+t \in [0,T]$ and $[x, x+vs] \subset \Omega$,

$$h(t,x,v) = h(t+s, x+vs, v). \tag{31}$$

Then, a, b, c and h are smooth.

Proof of lemma 2: If $(t,x) \in [0,T] \times \overline{\Omega}$, we fix $t_1, ..., t_{N+2} \in [0,T]$ and $x_1, ..., x_{N+2} \in \Omega$ such that for all i, $t \neq t_i$, and

$$\begin{vmatrix} 1 & \frac{x_1^1 - x^1}{t_1 - t} & \frac{x_1^2 - x^2}{t_1 - t} & ... & |\frac{x_1 - x}{t_1 - t}|^2 \\ 1 & \frac{x_2^1 - x^1}{t_2 - t} & \frac{x_2^2 - x^2}{t_2 - t} & ... & |\frac{x_2 - x}{t_2 - t}|^2 \\ . & . & . & & . \\ . & . & . & & . \\ 1 & \frac{x_{N+2}^1 - x^1}{t_{N+2} - t} & \frac{x_{N+2}^2 - x_2}{t_{N+2} - t} & ... & |\frac{x_{N+2} - x}{t_{N+2} - t}|^2 \end{vmatrix} \neq \quad 0. \tag{32}$$

Then, this determinant is also different from 0 in a small convex neighbourhood V of (t,x).

Since for all i,

$$h(t, x, \frac{x_i - x}{t_i - t}) = h(t_i, x_i, \frac{x_i - x}{t_i - t}), \tag{33}$$

we get for all i,

$$a(t,x) \,+\, b(t,x) \cdot \frac{x_i - x}{t_i - t} \,+\, c(t,x) \, |\frac{x_i - x}{t_i - t}|^2$$

$$= a(t_i, x_i) \,+\, b(t_i, x_i) \cdot \frac{x_i - x}{t_i - t} \,+\, c(t_i, x_i) \, |\frac{x_i - x}{t_i - t}|^2. \tag{34}$$

Solving this linear system, we get the boundedness of a, b and c on V. Using then the compactness of $[0,T] \times \overline{\Omega}$, we can see that $h \in L^\infty([0,T] \times \Omega \times I\!R^N)$.

Therefore, h satisfies eq. (11) in the sense of distributions, and we can apply the averaging lemmas of [Go, L, Pe, Se]. But a, b and c can be obtained as moments of h, therefore a, b, c and h are smooth.

The main result of this section is the following:

Theorem 3: *Let $m(t,x,v) \in L^1([0,T] \times \Omega \times I\!R^N)$ be a nonnegative Maxwellian function of v satisfying the free transport equation (11). Moreover, we suppose that for all $t \in [0,T]$, $m(t, \cdot, \cdot)$ is not identically equal to 0. Then, m can be written under the form*

$$m(t,x,v) = \exp\{\, D_0 + C_1 \cdot \{x - vt\} + c_3 |x - vt|^2$$

$$+ C_0 \cdot v + c_2 \{x - vt\} \cdot v + c_1 |v|^2 + \Lambda_0(x) \cdot v \,\}, \tag{35}$$

where $D_0, c_1, c_2, c_3 \in I\!R$, $C_0, C_1 \in I\!R^N$, *and* Λ_0 *is a skew–symmetric tensor.*

Remark: Note that m is clearly a Maxwellian function of v and that it depends only upon $x - vt$ and v. Therefore, these functions give an exhaustive representation of the (non identically equal to 0) Maxwellian solutions of the Boltzmann, B.G.K. and free transport equations.

Proof of theorem 3: We note first that because of lemmas 1 and 2, $\log m$ exists and is smooth.

Therefore, we can substitute in eq. (11) identity (28) and write down the result under the form of a polynomial of order 3 in the variable v that can be identified to 0. We get

$$\partial_t a = 0, \tag{36}$$

$$\partial_t b + \nabla_x a = 0, \tag{37}$$

$$\partial_t c + \nabla_x b : k \otimes k = 0, \tag{38}$$

$$\nabla_x c = 0, \tag{39}$$

for all vectors k such that $|k| = 1$.

eq. (39) ensures that c depends only on t, therefore we can find c_0 such that

$$c(t,x) = c_0(t). \tag{40}$$

To deal with equation (38), we need the following result:

Lemma 3: *Let* ϕ *be a map from* Ω *to* $I\!R^N$ *such that* $\nabla\phi$ *is always skew–symmetric. Then,* ϕ *is an affine map and its linear associated map is skew–symmetric.*

Proof of lemma 3: For all i, j in $[1, N]$, we can write

$$\frac{\partial \phi_i}{\partial x_j} = -\frac{\partial \phi_j}{\partial x_i}. \tag{41}$$

Now let i, j, k be three distinct numbers in $[1, N]$, we compute

$$\frac{\partial^2 \phi_k}{\partial x_i \partial x_j} = -\frac{\partial^2 \phi_j}{\partial x_i \partial x_k} = \frac{\partial^2 \phi_i}{\partial x_j \partial x_k} = -\frac{\partial^2 \phi_k}{\partial x_j \partial x_i}, \tag{42}$$

and therefore

$$\frac{\partial^2 \phi_k}{\partial x_i \partial x_j} = 0. \tag{43}$$

For all distinct numbers i, j, eq. (43) provides the existence of functions ϕ_i^j depending only on x_j, such that

$$\phi_i(x_1, ..., x_n) = \phi_i^1(x_1) + ... + \phi_i^n(x_n), \tag{44}$$

and ϕ_i does not depend on x_i because of eq. (41). But eq. (41) applied to identity (44) ensures the existence of constants λ_i^j and C_i^j such that

$$\phi_i^j(x_j) = \lambda_i^j x_j + C_i^j, \tag{45}$$

and

$$\lambda_i^j = -\lambda_j^i, \tag{46}$$

which ends the proof of the lemma.

Coming back to eq. (38), we apply lemma 3 to the function W defined by:

$$W(t,x) = b(t,x) - c_0'(t)\, x\ . \tag{47}$$

We obtain a linear, skew–symmetric, time–dependant map $\Lambda(t)$, from $I\!R^N$ to $I\!R^N$, and a time–dependant vector $C(t)$ such that

$$W(t,x) = \Lambda(t)(x) + C(t). \tag{48}$$

Finally,

$$b(t,x) = \Lambda(t)(x) - c_0'(t)\, x + C(t). \tag{49}$$

Then, eq. (37) becomes

$$\Lambda'(t)(x) - c_0''(t)\, x + C'(t) + \nabla_x a = 0. \tag{50}$$

According to eq. (50), the quantity

$$\Phi(t,x) = \Lambda'(t)(x) - c_0''(t)\, x + C'(t) \tag{51}$$

must be a gradient, and therefore

$$\Lambda'(t) = 0. \tag{52}$$

Then, we can find a constant skew–symmetric tensor Λ_0 such that

$$\Lambda(t) = \Lambda_0. \tag{53}$$

Solving eq. (50), we get a function $D(t)$ such that

$$a(t,x) = c_0''(t)\,\frac{x^2}{2} - C'(t)\cdot x + D(t). \tag{54}$$

According to eq. (36), we have moreover

$$c_0'''(t) = 0, \quad C''(t) = 0, \quad D'(t) = 0. \tag{55}$$

Finally, we can find $D_0, c_1, c_2, c_3 \in I\!R$ and $C_0, C_1 \in I\!R^N$ such that

$$c_0(t) = c_1 - c_2\,t + c_3\,t^2, \tag{56}$$

$$C(t) = C_0 - C_1\,t, \tag{57}$$

and

$$D(t) = D_0. \tag{58}$$

Then, we get

$$a(t,x) = c_3\,|x|^2 + C_1\,x + D_0, \tag{59}$$

$$b(t,x) = \Lambda_0(x) + (c_2 - 2\,c_3\,t)\,x + (C_0 - C_1\,t), \tag{60}$$

and

$$c(t,x) = c_1 - c_2\,t + c_3\,t^2. \tag{61}$$

Therefore, theorem 3 is proved.

4 The effect of the boundary conditions on the equilibrium

In section 2, we proved the convergence of the solutions of the Boltzmann equation to Maxwellians satisfying the free transport equation. In section 3, we described explicitly the form of these functions. The goal of this section is to use this description and the boundary conditions in order to determine the possible states of equilibrium.

4.1 States of equilibrium in a bounded domain with reverse reflexion at the wall

This subsection is devoted to the description of the states of equilibrium in a bounded domain Ω when the function f satisfies the boundary condition (8), which models the reverse reflexion of all particles at the wall.

Its main result is the following:

Theorem 4 *let Ω be a bounded and regular (C^2) open set of $\mathbb{R}^N$, and let f be a renormalized solution of the Boltzmann equation (1) with boundary condition (8) under assumptions 1 and 2 (or of the B.G.K. equation (2) with the same boundary condition and under assumption 2 and (20)). Then, for every sequences t_n going to infinity, there exist a subsequence t_{n_k} and a global, time–independant Maxwellian $m(v)$ with zero bulk velocity:*

$$m(t,x,v) = r_0 \exp -\nu\, v^2 \tag{62}$$

with $r_0 \geq 0$ and $\nu > 0$, such that $f_{n_k}(t,x,v) = f(t_{n_k}+t,x,v)$ converges weakly in $L^1([0,T]\times\Omega\times\mathbb{R}^N)$ to $m(v)$ for any $T>0$.

Remark: This result proves the thermalizing effect of the wall with such a boundary condition. The constants r_0 and ν could be determined by the initial datum f_0 if both mass and energy were conserved for the renormalized solution of the Boltzmann equation (only the mass is known to be conserved). Note however that this is the case for the B.G.K. equation under assumption (20).

Proof of theorem 4: Because of the conservation of mass, Theorem 1 and 3 (or 2 and 3) imply the convergence of f_{n_k} to a Maxwellian m of the form (35) satisfying the boundary condition (8) if $f_0 \neq 0$ (The trivial case $f=0$ must be treated separately).

For a Maxwellian of this type, the boundary condition exactly means that for all v in $\mathbb{R}^N$, t in $[0,T]$, x in $\partial\Omega$,

$$-C_1\cdot v\,t - 2\,c_3\,(x\cdot v)\,t + C_0\,v + c_2\,v\cdot x + \Lambda_0(x)\cdot v = 0. \tag{63}$$

Therefore, for all x in $\partial\Omega$,

$$C_1 + 2\,c_3\,x = 0, \tag{64}$$

$$C_0 + c_2\,x + \Lambda_0(x) = 0. \tag{65}$$

But $\partial\Omega$ is not a single point, therefore $C_1 = 0, c_3 = 0$. Moreover, $\partial\Omega$ is not included in an affine hyperplane, therefore $C_0 = 0, c_2 = 0, \Lambda_0 = 0$. Finally,

$$m(t,x,v) = r_0 \exp -\nu\, v^2, \tag{66}$$

where $\nu = -c_1 > 0$ since $m \in L^1$ and $r_0 = \log D_0$.

4.2 States of equilibrium in a bounded domain with specular reflexion at the wall

This subsection is devoted to the description of the states of equilibrium in a bounded domain Ω when the function satisfies the boundary condition (9), which models the specular reflexion of all particles at the wall.

Its main result is the following:

Theorem 5: *Let Ω be a bounded, regular, and simply connected open set of $I\!R^2$ or $I\!R^3$, and let f be a renormalized solution of the Boltzmann equation (1) with boundary condition (9) under assumptions 1 and 2 (or of the B.G.K equation (2) with the same boundary condition and under assumptions 2 and (20)). Then, for every sequences t_n going to infinity, there exist a subsequence t_{n_k} and a local, time–independant Maxwellian $m(x,v)$ such that $f_{n_k}(t,x,v) = f(t_{n_k}+t,x,v)$ converges weakly in $L^1([0,T]\times\Omega\times I\!R^N)$ to $m(x,v)$ for any $T>0$.*

1. *If $N=2$ and Ω is not a disc, or if $N=3$ and $\partial\Omega$ is not a surface of revolution, there exist $r_0 \geq 0$ and $\nu > 0$ such that*

$$m(x,v) = r_0 \exp -\nu\, v^2. \tag{67}$$

2. *If $N=2$ and Ω is a disc, or if $N=3$ and Ω is a ball, which may be assumed to be centered at 0 without loss of generality, there exist $r_0 \geq 0$, $\nu > 0$, and a constant skew–symmetric tensor Λ_0 such that*

$$m(x,v) = r_0 \exp\{-\nu\, v^2 + \Lambda_0(x)\cdot v\}. \tag{68}$$

3. *Finally, if $N=3$, and $\partial\Omega$ is a surface of revolution which is not a sphere, and which axis of revolution may be assumed to contain 0 without loss of generality, there exist $r_0 \geq 0$, $\nu > 0$, and a constant number λ_0 such that*

$$m(x,v) = r_0 \exp\{-\nu\, v^2 + (\lambda_0\, z \times x)\cdot v\}, \tag{69}$$

where z is a unit vector parallel to the axis of revolution of Ω.

Moreover, each Maxwellian given in the theorem may be obtained as a limit when the time t goes to infinity of some renormalized solution of the Boltzmann equation.

Remark: The non–global Maxwellians appearing in eq. (68) and eq. (69) reflect the fact that when the domain Ω is very symmetric, some extra global conservations can appear for the Boltzmann equation. Namely, in the case of the ball, the global kinetic momentum is conserved, and when $\partial\Omega$ is a surface of revolution, one component of this global kinetic momentum is conserved.

Proof of theorem 5: According to theorem 1 and 3 (or 2 and 3), m is a Maxwellian of the type (35) satisfying the boundary condition (9) as soon as $f_0 \neq 0$ (once again, the trivial case when $f = 0$ must be treated independently). But for a Maxwellian of the type (35), eq. (9) ensures that for all t in $[0, T]$, x in $\partial\Omega$,

$$-(C_1 \cdot n(x))\, t - 2\, c_3\, t\, (x \cdot n(x)) + (C_0 \cdot n(x)) \\ + c_2\, (x \cdot n(x)) + (\Lambda_0(x) \cdot n(x)) = 0. \tag{70}$$

Then, eq. (70) ensures that for all x in $\partial\Omega$,

$$(C_1 + 2\, c_3\, x) \cdot n(x) = 0, \tag{71}$$

$$(C_0 + c_2\, x + \Lambda_0(x)) \cdot n(x) = 0. \tag{72}$$

Now consider the curve defined by

$$\frac{dx}{ds}(s) = 2\, c_3\, x(s) + C_1, \tag{73}$$

$$x(0) = x_0 \in \partial\Omega. \tag{74}$$

According to eq. (71), for all $s \in I\!R$, $x(s) \in \partial\Omega$. But the solutions of eq. (73) are given by

$$x(s) = (x_0 + \frac{C_1}{2c_3})\, \exp\{2\, c_3\, s\} - \frac{C_1}{2c_3}, \tag{75}$$

if $c_3 \neq 0$. Therefore $\partial\Omega$ is not bounded, which is impossible. Now if $c_3 = 0$, it also implies that $C_1 = 0$, or $\partial\Omega$ would be included in an affine hyperplane.

Now consider the curve defined by

$$\frac{dx}{ds}(s) = \Lambda_0(x(s)) + c_2\, x(s) + C_0, \tag{76}$$

$$x(0) = x_0 \in \partial\Omega. \tag{77}$$

According to eq. (72), for all $s \in I\!R$, $x(s) \in \partial\Omega$. But $\partial\Omega$ is bounded, and therefore $c_2 = 0$.

In a first step, we concentrate on the case $N = 2$.

If $\Lambda_0 \neq 0$, there exists y in $I\!R^2$ such that $\Lambda_0(y) = C_0$. According to eq. (76),

$$x(s) = \exp\{s\Lambda_0\}(x_0 + y) - y \tag{78}$$

is in $\partial\Omega$ for all s in $I\!R$, and thus, $\partial\Omega$ is a union of circles of center $-y$.

If Ω is not a disc, its boundary is not a union of circles because Ω is simply connected, therefore $\Lambda_0 = 0$, and $C_0 = 0$ because of eq. (72). In this case, formula (67) holds.

If Ω is a disc, we can assume that it is centered at 0, without loss of generality. Then eq. (72) becomes

$$C_0 \cdot x = 0 \tag{79}$$

for all x in $\partial\Omega$. It means that $C_0 = 0$, and formula (68) holds.

In a second step, we assume that $N = 3$.

If $\Lambda_0 \neq 0$, then there exists z in $I\!R^3$ such that $\Lambda_0(x) = z \times x$. Now C_0 can be written under the form

$$C_0 = \delta\, z + w \tag{80}$$

where δ is a real number and w is orthogonal to z. Moreover, there exists y in $I\!R^3$ such that $\Lambda_0(y) = w$. According to eq. (72) and eq. (76),

$$x(s) = \exp\{s\,\Lambda_0\}(x_0 + y) - y + \delta\, s\, z \tag{81}$$

is in $\partial\Omega$ for all s in $I\!R$. Therefore $\delta = 0$ because $\partial\Omega$ is bounded. Moreover, this formula implies (if $\Lambda_0 \neq 0$) that $\partial\Omega$ is a surface of revolution of axis Δ parallel to z and containing $-y$.

If $\partial\Omega$ is not a surface of revolution, we obtain that $\Lambda_0 = 0$ and $U = 0$ because of eq. (72), and formula (67) holds.

If $\partial\Omega$ is a surface of revolution which is not a sphere, it has a unique axis of revolution (parallel to z) which may be assumed to contain 0 without loss of generality. Then eq. (72) implies that Λ_0 is of the form:

$$\Lambda_0(x) = \lambda_0\, z \times x, \tag{82}$$

where λ_0 is a real constant and $C_0 = 0$. Therefore, formula (69) holds.

Finally, if Ω is a ball which may be assumed to be centered at 0, eq. (72) implies that $C_0 = 0$, and formula (68) holds.

5 On strong convergence

In this section, we prove that the convergence stated in Theorem 2 for a solution of the B.G.K model is in fact strong in L^1.

Our goal is to prove the following proposition:

Theorem 6: *Let $f(t,x,v)$ be a solution of the B.G.K equation (2) with boundary condition (8) or (9) under assumptions 2 and (20). Then, for every sequences t_n going to infinity, there exist a subsequence t_{n_k} and a local time–dependant Maxwellian $m(t,x,v)$ satisfying the conclusions of theorem 4 or 5 such that $f_{n_k}(t,x,v) = f(t_{n_k}+t,x,v)$ converges strongly in $L^1([0,T]\times \Omega \times I\!R^N)$ to $m(t,x,v)$ for any $T>0$.*

Proof of theorem 6: According to theorems 2, 4 and 5, the only thing to prove is that the convergence holds strongly in L^1. We already know that $M_{f_{n_k}}$ converges strongly in L^1 to m (Cf. the proof of theorem 2). Moreover, using estimate (23), we can prove that

$$\int_0^T \int_{x\in\Omega} \int_{I\!R^N} |\sqrt{M_{f_{n_k}}} - \sqrt{f_{n_k}}|^2 dt dx dv \quad \underset{k\to+\infty}{\longrightarrow} \quad 0. \tag{83}$$

Therefore, $||\sqrt{M_{f_{n_k}}} - \sqrt{f_{n_k}}||_{L^2}$ tends to 0 when $k\to+\infty$. Now the fact that $||M_{f_{n_k}} - m||_{L^1}$ tends to 0 together with the inequality

$$|\sqrt{x} - \sqrt{y}|^2 \leq |x-y|, \tag{84}$$

ensures that $||\sqrt{M_{f_{n_k}}} - \sqrt{m}||_{L^2}$ tends to 0. Finally we obtain the strong convergence in L^2 of $\sqrt{f_{n_k}}$ to $\sqrt{m}$. Extracting another subsequence n_{k_p}, we prove the convergence a.e. of $f_{n_{k_p}}$ to m.

We conclude the proof by using the result of weak L^1 convergence stated in theorem 2.

Remark: It is now proved that the strong convergence to equilibrium also holds in the case of renormalized solutions of the Boltzmann equation for a large class of cross sections B. This is a consequence of the properties of compactness of the positive term Q_+ of Boltzmann's collision kernel (Cf. [L]). It can also be proved using non–standard techniques (Cf. [Ar 2])

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