ENTROPY DISSIPATION RATE AND CONVERGENCE IN KINETIC EQUATIONS

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Abstract

We give a lower bound of the entropy dissipation rate of Kac, Boltzmann and Fokker–Planck–Landau equations. We apply this estimate to the problem of the speed of convergence to equilibrium in large time for the Boltzmann equation.

1 Introduction

Rarefied gas dynamics is usually described by the Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \tag{1}$$

where $f(t,x,v)$ is the density of particles which at time t and point x, move with velocity v, and Q is a quadratic collision term described in [Ce], [Ch, Co] and [Tr, Mu].

A simpler one–dimensional model has been introduced by Kac in [K],

$$\partial_t f + v\,\partial_x f = \tilde{Q}(f,f), \tag{2}$$

where $\tilde{Q}$ is defined in [K] or[MK].

The asymptotics of the Boltzmann equation when the grazing collisions become predominent formally leads to the Fokker–Planck–Landau equation

$$\partial_t f + v \cdot \nabla_x f = Q'(f,f), \tag{3}$$

where Q' is still a quadratic collision term. The formal derivation of this equation and the form of Q' can be found in [Ch, Co] or [Li, Pi]. The corresponding asymptotics is described in [De 1] and [Dg, Lu].

According to Boltzmann's H–theorem, the entropy dissipation rate is nonpositive for all f such that it is defined,

$$E_Q(f) = \int_{v \in I\!R^3} Q(f,f)(v)\, \log f(v)\, dv \leq 0. \tag{4}$$

Moreover, when $f \in L^1(I\!R^3_v)$, it is equal to 0 if and only if f is a Maxwellian function of v (Cf. [Tr, Mu]),

$$E_Q(f) = 0 \qquad \Longleftrightarrow \qquad \forall v \in I\!R^3, \quad Q(f,f)(v) = 0$$

$$\Longleftrightarrow \qquad \exists \rho \geq 0, T > 0,\ u \in I\!R^3, \qquad f(v) = \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|v-u|^2}{2T}}. \tag{5}$$

The same property holds for Q', and for $\tilde{Q}$ with the additional prescription that the Maxwellian is of bulk velocity $u = 0$.

Therefore, in order to get a better understanding of the phenomena appearing when the entropy dissipation term tends to 0, it is useful to obtain a lower bound of it in terms of some distance from f to the space of Maxwellians.

Such an estimate measures the speed of convergence to equilibrium when the entropy dissipation $E_Q(f)$ tends to 0. This situation occurs for example when the time t tends to infinity in equations (1), (2) and (3), or when the mean free path ϵ tends to zero in the Hilbert expansion,

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = \frac{1}{\epsilon} Q(f_\epsilon, f_\epsilon). \tag{6}$$

Therefore, in section 2, we give a lower bound of the entropy dissipation term for the Kac equation in terms of a distance to the Maxwellian states of the type:

$$L_{\tilde{Q}}(f) = \inf_{m \in \tilde{\Gamma}} \int |\log f(v) - m(v)| \, dv, \tag{7}$$

where $\tilde{\Gamma}$ is the space of logarithms of Maxwellians with zero bulk velocity.

We extend this result in section 3 to the case of the Boltzmann collision kernel Q, and in section 4 to the case of the Fokker–Planck–Landau collision kernel Q'.

Finally, we explain in section 6 how to apply the previous estimates to investigate the long time behavior of the Boltzmann equation and the Chapman–Enskog expansion associated to (6).

2 On Kac's collision kernel

The Kac equation (2) models a one–dimensional gas in which all collisions conserving the energy are equiprobable (Cf. [K] or [MK]). Therefore, its collision kernel $\tilde{Q}$ writes

$$\tilde{Q}(f,f)(v) = \int_{v_1 \in I\!R} \int_{\theta=0}^{2\pi} \{ f(v')f(v_1') - f(v)f(v_1) \} \frac{d\theta}{2\pi} dv_1, \tag{8}$$

where

$$v' \quad = v\cos\theta + v_1 \sin\theta, \tag{9}$$

$$v_1' \quad = -v\sin\theta + v_1\cos\theta. \tag{10}$$

The theorem below is a partial answer to a question of McKean (Cf. [MK], p. 365; 13 b). Note that new results on this topic are also to be found in [Ga].

We shall denote by U the set of all convex, continuous and even functions from $I\!R$ to $I\!R$ such that for all x in $I\!R$,

$$0 \le \phi(x) \le x\,(e^x - 1). \tag{11}$$

Moreover, if B is a function from $\mathbb{R}_+^*$ to $\mathbb{R}_+^*$, we introduce the set

$$L_B^p(\mathbb{R}^N) = \{f \in L^p(\mathbb{R}^N)/ \quad \text{for all } v \text{ such that } |v| \le R, \quad f(v) \ge B(R)\}.$$

Finally, $\tilde{\Gamma}$ is the space of logarithms of Maxwellians with zero bulk velocity:

$$\tilde{\Gamma} = \{a + bv^2 / \quad a, b \in \mathbb{R}\}.$$

Theorem 1: *Let $\tilde{Q}$ be Kac's collision kernel, and R be strictly positive. Then there exists $K_R > 0$ such that for all f in $L_B^1(\mathbb{R})$ and all ϕ in U,*

$$-E_{\tilde{Q}}(f) = -\int_{v \in \mathbb{R}} \tilde{Q}(f,f)(v) \, \log f(v) \, dv$$

$$\ge \frac{1}{4} B(R)^2 \pi R^2 \, \phi\bigg(\frac{K_R}{\pi R^2} \inf_{m \in \tilde{\Gamma}} \int_{|v| \le R} |\log f(v) - m(v)| \, dv \bigg). \tag{12}$$

Corollary 1: *Let $\tilde{Q}$ be Kac's collision kernel, and R be strictly positive. Then there exists $K_R' > 0$ such that for all f in $L_B^1(\mathbb{R})$ satisfying $-E_{\tilde{Q}}(f) \le 1$, the following estimate holds,*

$$-E_{\tilde{Q}}(f) \ge K_R' \bigg(\inf_{m \in \tilde{\Gamma}} \int_{|v| \le R} |\log f(v) - m(v)| \, dv \bigg)^2. \tag{13}$$

Remark: Corollary 1 is a straightforward consequence of theorem 1 when one takes (for example)

$$\phi(x) = \frac{x^2}{1 + |x|}. \tag{14}$$

Proof of theorem 1: Boltzmann's H–theorem ensures that

$$-E_{\tilde{Q}}(f) = -\int_{v \in \mathbb{R}} \tilde{Q}(f,f)(v) \, \log f(v) \, dv$$

$$= \frac{1}{4} \int_{v \in \mathbb{R}} \int_{v_1 \in \mathbb{R}} \int_{\theta=0}^{2\pi} \{f(v')f(v_1') - f(v)f(v_1)\}$$

$$\times \{ \log (f(v')f(v_1')) - \log (f(v)f(v_1)) \} \frac{d\theta}{2\pi} dv_1 dv$$

$$\geq \frac{1}{4} B(R)^2 \int_{v^2+v_1^2 \leq R^2} \int_{\theta=0}^{2\pi} \lambda\Big(\log f(v') \,+\, \log f(v_1')$$

$$- \log f(v) \,-\, \log f(v_1) \Big) \frac{d\theta}{2\pi} dv_1 dv \tag{15}$$

where

$$\lambda(x) = x\,(e^x - 1). \tag{16}$$

Therefore, according to Jensen's inequality,

$$-E_{\tilde{Q}}(f) = -\int_{v \in I\!R} \tilde{Q}(f,f)(v)\, \log f(v)\, dv$$

$$\geq \frac{1}{4} B(R)^2 \pi R^2\, \phi\Big(\frac{1}{\pi R^2} \int_{v^2+v_1^2 \leq R^2} \int_{\theta=0}^{2\pi} |\log f(v') + \log f(v_1')$$

$$- \log f(v) - \log f(v_1)| \frac{d\theta}{2\pi} dv_1 dv \Big). \tag{17}$$

In order to complete the proof of theorem 1, we need the following lemmas,

Lemma 1: *Let $\tilde{M}$ be the space of all functions $T(v, v_1) \in L^1(v^2 + v_1^2 \leq R^2)$ which depend only upon $v^2 + v_1^2$. Then, for all functions $f \in L^1_B$,*

$$\int_{v^2+v_1^2 \leq R^2} \int_{\theta=0}^{2\pi} |\log f(v') + \log f(v_1') - \log f(v) - \log f(v_1)| \frac{d\theta}{2\pi} dv_1 dv$$

$$\geq \inf_{T \in \tilde{M}} \int_{v^2+v_1^2 \leq R^2} |\log f(v) + \log f(v_1) - T(v^2 + v_1^2)|\, dv_1 dv. \tag{18}$$

Proof of lemma 1: Let us denote by R_ψ the rotation of angle ψ, and

$$g(v, v_1) = \log f(v) + \log f(v_1). \tag{19}$$

We compute

$$\int_{\theta=0}^{2\pi} |\log f(v') + \log f(v_1') - \log f(v) - \log f(v_1)| \frac{d\theta}{2\pi}$$

$$= \int_{\theta=0}^{2\pi} |g(R_\theta(v, v_1)) - g(v, v_1)| \frac{d\theta}{2\pi}$$

$$\geq \Big| \int_{\theta=0}^{2\pi} g(R_\theta(v,v_1))\,\frac{d\theta}{2\pi} - g(v,v_1)\,\Big|. \tag{20}$$

But

$$T(v,v_1) = \int_{\theta=0}^{2\pi} g(R_\theta(v,v_1))\,\frac{d\theta}{2\pi} \tag{21}$$

depends only on $v^2+v_1^2$, and lemma 1 is proved.

Lemma 2: *For all $R>0$, there exists $K_R>0$ such that*

$$\inf_{T\in\tilde{M}} \int_{v^2+v_1^2\leq R^2} |\log f(v) + \log f(v_1) - T(v^2+v_1^2)|\,dv_1 dv$$

$$\geq K_R \inf_{m\in\tilde{\Gamma}} \int_{|v|\leq R} |\log f(v) - m(v)|\,dv. \tag{22}$$

Proof of lemma 2: Let $\tilde{L}$ be the following operator,

$$\tilde{L} : t \in L^1(|v|\leq R)/\tilde{\Gamma} \to \tilde{L}t(v,v_1) = t(v)+t(v_1) \in L^1(v^2+v_1^2\leq R^2)/\tilde{M}. \tag{23}$$

The operator $\tilde{L}$ is clearly linear and one–one (Cf. [Ce] or [Tr, Mu] in the more complicated case of Boltzmann's collision kernel).

Observe that for all t in $L^1(|v|\leq R)$,

$$\inf_{T\in\tilde{M}} \int_{v^2+v_1^2\leq R^2} |t(v)+t(v_1) - T(v^2+v_1^2)|\,dv_1 dv$$

$$\leq \inf_{a,b\in\mathbb{R}} \int_{v^2+v_1^2\leq R^2} |t(v)+t(v_1) - a(v^2+v_1^2) - 2b|\,dv_1 dv$$

$$\leq 4R \inf_{m\in\tilde{\Gamma}} \int_{|v|\leq R} |t(v) - m(v)|\,dv. \tag{24}$$

Therefore, the operator $\tilde{L}$ is continuous.

In order to apply the open mapping theorem, we still have to prove that the image of $\tilde{L}$ is closed.

Assume that there exists a sequence t_n in $L^1(|v|\leq R)/\tilde{\Gamma}$ and t in $L^1(v^2+v_1^2\leq R^2)/\tilde{M}$ such that $t_n(v)+t_n(v_1)$ tends to $t(v,v_1)$ in $L^1(v^2+v_1^2\leq R^2)/\tilde{M}$. Then, there exist a sequence k_n in $L^1(|v|\leq R)$, a sequence T_n in $\tilde{M}$ and g in $L^1(v^2+v_1^2\leq R^2)$ such that t_n is the natural projection of k_n on $L^1(|v|\leq R)/\tilde{\Gamma}$, t is the natural projection of g on $L^1(v^2+v_1^2\leq R^2)/\tilde{M}$ and

$$k_n(v) + k_n(v_1) + T_n(v^2+v_1^2) \to g(v,v_1) \tag{25}$$

in $L^1(v^2+v_1^2 \leq R^2)$.

Then, we introduce the differential operator

$$\tilde{\nabla} = v_1 \frac{\partial}{\partial v} - v \frac{\partial}{\partial v_1}, \tag{26}$$

which has the following property:

$$\tilde{\nabla}\, T(v^2+v_1^2) = 0. \tag{27}$$

According to eq. (27), the sequence

$$\tilde{\nabla}\,(k_n(v) + k_n(v_1) + T_n(v^2+v_1^2)) = v_1 k_n'(v) - v k_n'(v_1) \tag{28}$$

converges in $W^{-1,1}$.

Taking the double partial derivative of this expression with respect to v, v_1, we prove that

$$k_n''(v) - k_n''(v_1) \tag{29}$$

converges in $W^{-3,1}$.

Therefore, there exists a sequence of real numbers b_n such that

$$k_n''(v) + b_n \tag{30}$$

converges in $W^{-3,1}$.

Then, there exists a sequence of real numbers c_n such that

$$k_n'(v) + b_n v + c_n \tag{31}$$

converges in $W^{-2,1}$ and, according to eq. (28),

$$k_n'(v) + b_n v \tag{32}$$

converges in $W^{-2,1}$.

But eq. (28) also ensures that:

$$\frac{\partial}{\partial v_1}\{v_1 k_n'(v) - v k_n'(v_1)\} = k_n'(v) - v k_n''(v_1) \tag{33}$$

converges in $W^{-2,1}$.

Then, properties (32) and (33) imply that

$$k_n''(v) + b_n \tag{34}$$

converges in $W^{-2,1}$.

According to property (32), the convergence in $W^{-1,1}$ of

$$k_n'(v) + b_n v \tag{35}$$

holds and therefore there exists a sequence of real numbers a_n such that

$$k_n(v) + \frac{1}{2} b_n v^2 + a_n \tag{36}$$

converges in L^1.

Therefore, t_n converges in $L^1(|v| \le R)/\tilde{\Gamma}$, and the image of $\tilde{L}$ is closed. Applying the theorem of the open mapping to $\tilde{L}$, we obtain a strictly positive K_R such that:

$$\begin{aligned} &\inf_{T \in \tilde{M}} \int_{v^2+v_1^2 \le R^2} |t(v) + t(v_1) - T(v^2 + v_1^2)| \, dv_1 dv \\ &\qquad \ge K_R \inf_{m \in \tilde{\Gamma}} \int_{|v| \le R} |t(v) - m(v)| \, dv. \end{aligned} \tag{37}$$

Injecting $t = \log f$ in this estimate, we obtain lemma 2.

The proof of theorem 1 easily follows from lemmas 1, 2, and estimate (17).

3 On Boltzmann's collision kernel

For the derivation of Boltzmann's collision kernel, we refer to [Ce], [Ch, Co] or [Tr, Mu]. We recall that

$$Q(f,f)(v) = \int_{v_1 \in I\!\!R^3} \int_{\omega \in S^2} \{f(v')f(v_1') - f(v)f(v_1)\} \, B(v, v_1, \omega) \, d\omega dv_1, \tag{38}$$

where

$$v' = v + ((v_1 - v) \cdot \omega) \, \omega, \tag{39}$$

$$v_1' = v_1 - ((v_1 - v) \cdot \omega) \, \omega, \tag{40}$$

and B is a nonnegative collision cross section depending only upon $|v - v_1|$ and $|(v - v_1) \cdot \omega|$.

According to Boltzmann's H–theorem, property (5) holds as soon as B is strictly positive a.e. In order to obtain an estimate of the form (7), we need a stronger assumption on B. From now on, we shall assume that for all $R > 0$, there exists $C_R > 0$ such that

$$B(v, v_1, \omega) \geq C_R \, |\omega \cdot \frac{v_1 - v}{|v_1 - v|}| \tag{41}$$

as soon as $v^2 + v_1^2 \leq R^2$. Note that this assumption is satisfied in the classical cases of soft potentials with or without the angular cut–off assumption (Cf. [Ce], [Ch, Co], [Gr] and [Tr, Mu]). Note also that the case of hard potentials, which is not covered by this work, is now treated in [We].

We keep in this section the notations of section 2. Moreover, we introduce the space Γ of logarithms of Maxwellians,

$$\Gamma = \{av^2 + b \cdot v + c / \quad a, c \in I\!R, \, b \in I\!R^3\}.$$

We denote by $|A|$ the Lebesgue measure of the set A, and by S^N the sphere of dimension N.

The main result of this section is the following:

Theorem 2: *Let Q be Boltzmann's collision kernel with a cross section B satisfying assumption(41) and let R be a strictly positive number. Then, there exists $K_R > 0$ such that for all f in $L^1_B(I\!R^3)$ and all ϕ in U,*

$$-E_Q(f) = - \int_{v \in I\!R^3} Q(f,f)(v) \, \log f(v) \, dv$$

$$\geq \frac{1}{4} B(R)^2 |S^5||S^2| R^6 C_R \, \phi\Big(\frac{K_R}{|S^5||S^2|R^6} \inf_{m \in \Gamma} \int_{|v| \leq R} |\log f(v) - m(v)| \, dv \Big). \tag{42}$$

Corollary 2: *Let Q be Boltzmann's collision kernel with a cross section B satisfying assumption (41) and let R be a strictly positive number. Then, there exists $K'_R > 0$ such that for all f in $L^1_B(I\!R^3)$ satisfying $-E_Q(f) \leq 1$, the following estimate holds,*

$$-E_Q(f) \geq K'_R \Big(\inf_{m \in \Gamma} \int_{|v| \leq R} |\log f(v) - m(v)| \, dv \Big)^2. \tag{43}$$

Remark: Corollary 2 is a straightforward consequence of theorem 2 when one takes (for example)

$$\phi(x) = \frac{x^2}{1 + |x|}. \tag{44}$$

Proof of theorem 2: Because of Boltzmann's H–theorem,

$$-E_Q(f) = -\int_{v\in I\!R^3} Q(f,f)(v)\log f(v)\,dv$$

$$= \frac{1}{4}\int_{v\in I\!R^3}\int_{v_1\in I\!R^3}\int_{\omega\in S^2}\{f(v')f(v_1') - f(v)f(v_1)\}$$

$$\{\log(f(v')f(v_1')) - \log(f(v)f(v_1))\}\,B(v,v_1,\omega)\,d\omega dv_1 dv$$

$$\geq \frac{1}{4}B(R)^2 C_R\int_{v^2+v_1^2\leq R^2}\int_{\omega\in S^2}\lambda\Big(\log f(v')+\log f(v_1')-\log f(v)-\log f(v_1)\Big)$$

$$|\omega\cdot\frac{v_1-v}{|v_1-v|}|\,d\omega dv_1 dv, \tag{45}$$

with

$$\lambda(x) = x\,(e^x-1). \tag{46}$$

Therefore, because of Jensen's inequality,

$$-E_Q(f) \geq \frac{1}{4}B_R^2 C_R|S^2||S^5|R^6\,\phi\Big(\frac{1}{|S^2||S^5|R^6}\int_{v^2+v_1^2\leq R^2}\int_{\omega\in S^2}$$

$$|\log f(v') + \log f(v_1') - \log f(v) - \log f(v_1)|\,|\omega\cdot\frac{v_1-v}{|v_1-v|}|\,d\omega dv_1 dv\Big). \tag{47}$$

In the sequel, we need the following lemmas:

Lemma 3: *Let M be the space of all functions $T(v,v_1)\in L^1(v^2+v_1^2\leq R^2)$ which depend only upon $v+v_1$ and $v^2+v_1^2$. Then, for all $f\in L^1_B$,*

$$\int_{v^2+v_1^2\leq R^2}\int_{\omega\in S^2}|\log f(v')+\log f(v_1')-\log f(v)-\log f(v_1)||\omega\cdot\frac{v_1-v}{|v_1-v|}|d\omega dv_1 dv$$

$$\geq \inf_{T\in M}\int_{v^2+v_1^2\leq R^2}|\log f(v)+\log f(v_1) - T(v+v_1, v^2+v_1^2)|\,dv_1 dv. \tag{48}$$

Proof of lemma 3: We introduce the notation

$$g(v,v_1) = \log f(v) + \log f(v_1), \tag{49}$$

and compute

$$\int_{v^2+v_1^2\leq R^2}\int_{\omega\in S^2}|\log f(v')+\log f(v_1')-\log f(v)-\log f(v_1)|\,|\omega\cdot\frac{v_1-v}{|v_1-v|}|\,d\omega dv_1 dv$$

$$=\int_{v^2+v_1^2\leq R^2}\int_{\omega\in S^2}|g(v',v_1')-g(v,v_1)|\,|\omega\cdot\frac{v_1-v}{|v_1-v|}|\,d\omega dv_1 dv. \tag{50}$$

Then, we consider the change of variables

$$\sigma=S(\omega), \tag{51}$$

with

$$S(\omega)=2\,(\omega\cdot\frac{v_1-v}{|v_1-v|})\,\omega-\frac{v_1-v}{|v_1-v|}\ . \tag{52}$$

The Jacobian of S is

$$J(\omega)=|\omega\cdot\frac{v_1-v}{|v_1-v|}|^{-1}. \tag{53}$$

Denoting

$$(v',v_1')=U_\sigma(v,v_1) \tag{54}$$

with

$$U_\sigma(v,v_1)=\frac{1}{2}(v+v_1+|v-v_1|\sigma,v+v_1-|v-v_1|\sigma), \tag{55}$$

we get the following estimate:

$$\int_{v^2+v_1^2\leq R^2}\int_{\omega\in S^2}|\log f(v')+\log f(v_1')-\log f(v)-\log f(v_1)|\,|\omega\cdot\frac{v_1-v}{|v_1-v|}|\,d\omega dv_1 dv$$

$$\geq\int_{v^2+v_1^2\leq R^2}\int_{\sigma\in S^2}|g(U_\sigma(v,v_1))-g(v,v_1)|\,d\sigma dv_1 dv$$

$$\geq\int_{v^2+v_1^2\leq R^2}\left|\int_{\sigma\in S^2}g(U_\sigma(v,v_1))d\sigma-g(v,v_1)\right|dv_1 dv. \tag{56}$$

But $U_\sigma(v,v_1)$ depends only on $v+v_1$ and $v^2+v_1^2$. Therefore,

$$\int_{v^2+v_1^2\leq R^2}\int_{\omega\in S^2}|\log f(v')+\log f(v_1')-\log f(v)-\log f(v_1)|\,|\omega\cdot\frac{v_1-v}{|v_1-v|}|\,d\omega dv_1 dv$$

$$\geq\inf_{T\in M}\int_{v^2+v_1^2\leq R^2}|g(v,v_1)-T(v+v_1,v^2+v_1^2)|\,dv_1 dv$$

$$\geq\inf_{T\in M}\int_{v^2+v_1^2\leq R^2}|\log f(v)+\log f(v_1)-T(v+v_1,v^2+v_1^2)|\,dv_1 dv, \tag{57}$$

which concludes the proof of the lemma.

lemma 4: *For all $R > 0$, there exists $K_R > 0$ such that when $f \in L^1_B$,*

$$\inf_{T\in M} \int_{v^2+v_1^2\le R^2} |\log f(v) + \log f(v_1) - T(v+v_1, v^2+v_1^2)|\, dv_1 dv$$

$$\ge K_R \inf_{m\in\Gamma} \int_{|v|\le R} |\log f(v) - m(v)| dv. \tag{58}$$

Proof of lemma 4: Let L be the following operator:

$$L : t \in L^1(|v| \le R)/\Gamma \to Lt(v, v_1) = t(v) + t(v_1) \in L^1(v^2 + v_1^2 \le R^2)/M. \tag{59}$$

The operator L is clearly linear and one–one (Cf. [Ce] or [Tr, Mu]) .

Observe that for all t in $L^1(|v| \le R)$,

$$\inf_{T\in M} \int_{v^2+v_1^2\le R^2} |t(v) + t(v_1) - T(v^2 + v_1^2)|\, dv_1 dv$$

$$\le \inf_{a,c\in\mathbb{R}, b\in\mathbb{R}^3} \int_{v^2+v_1^2\le R^2} |t(v) + t(v_1) - a(v^2+v_1^2) - b\cdot(v+v_1) - 2c|\, dv_1 dv$$

$$\le 16\, R^3 \inf_{m\in\Gamma} \int_{|v|\le R} |\log t(v) - m(v)|\, dv. \tag{60}$$

Therefore, the operator L is continuous.

In order to apply the open mapping theorem, we still have to prove that the image of L is closed.

Suppose that there exist a sequence t_n in $L^1(|v| \le R)/\Gamma$ and t in $L^1(v^2 + v_1^2 \le R^2)/M$ such that $t_n(v)+t_n(v_1)$ tends to $t(v, v_1)$ in $L^1(v^2+v_1^2 \le R^2)/M$. Then, there exist a sequence k_n in $L^1(|v| \le R)$, a sequence T_n in M and g in $L^1(v^2 + v_1^2 \le R^2)$ such that t_n is the natural projection of k_n on $L^1(|v| \le R)/\Gamma$, t is the natural projection of g on $L^1(v^2 + v_1^2 \le R^2)/M$, and

$$k_n(v) + k_n(v_1) + T_n(v + v_1, v^2 + v_1^2) \to g(v, v_1) \tag{61}$$

in $L^1(v^2 + v_1^2 \le R^2)$.

From now on, we shall write $v = (x_1, x_2, x_3)$, and $v_1 = (y_1, y_2, y_3)$. We introduce the following differential operator:

$$\overline{\nabla} = \begin{pmatrix} (y_2 - x_2)(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}) - (y_1 - x_1)(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_2}) \\ \\ (y_3 - x_3)(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_2}) - (y_2 - x_2)(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_3}) \\ \\ (y_1 - x_1)(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_3}) - (y_3 - x_3)(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}) \end{pmatrix}. \tag{62}$$

Note that

$$\overline{\nabla} T(v + v_1, v^2 + v_1^2) = 0. \tag{63}$$

Therefore,

$$\overline{\nabla} k_n(v) + \overline{\nabla} k_n(v_1) \tag{64}$$

converges in $W^{-1,1}$, which means that

$$\begin{aligned} &(y_2 - x_2)\frac{\partial k_n}{\partial 1}(x_1, x_2, x_3) - (y_1 - x_1)\frac{\partial k_n}{\partial 2}(x_1, x_2, x_3) \\ &-(y_2 - x_2)\frac{\partial k_n}{\partial 1}(y_1, y_2, y_3) + (y_1 - x_1)\frac{\partial k_n}{\partial 2}(y_1, y_2, y_3) \end{aligned} \tag{65}$$

converges in $W^{-1,1}$. Moreover, the same formula holds if we change the indices 1, 2, and 3 by circular permutation.

Taking the double partial derivative of this expression with respect to x_1, y_1, we obtain the convergence in $W^{-3,1}$ of

$$-\frac{\partial^2 k_n}{\partial 1 \partial 2}(x_1, x_2, x_3) - \frac{\partial^2 k_n}{\partial 1 \partial 2}(y_1, y_2, y_3). \tag{66}$$

Taking also its double partial derivative with respect to x_1, y_2, we obtain the convergence in $W^{-3,1}$ of

$$\frac{\partial^2 k_n}{\partial 1^2}(x_1, x_2, x_3) - \frac{\partial^2 k_n}{\partial 2^2}(y_1, y_2, y_3). \tag{67}$$

Therefore, there exists a sequence of real numbers a_n such that $\frac{\partial^2 k_n}{\partial 1 \partial 2}$ and $\frac{\partial^2 k_n}{\partial 1^2} + a_n$ converge in $W^{-3,1}$.

Moreover, the same convergences hold with the same sequence a_n when we change the indices 1, 2 and 3 by circular permutation.

Therefore, there exist three sequences of real numbers b_n^1, b_n^2, b_n^3 such that

$$\frac{\partial k_n}{\partial i} + a_n x_i + b_n^i \tag{68}$$

converges in $W^{-2,1}$.

Differentiating eq. (65) with respect to x_1, we get the convergence in $W^{-2,1}$ of the sequence

$$(y_2 - x_2)\frac{\partial^2 k_n}{\partial 1^2}(x_1, x_2, x_3) - (y_1 - x_1)\frac{\partial^2 k_n}{\partial 1 \partial 2}(x_1, x_2, x_3)$$
$$+\frac{\partial k_n}{\partial 2}(x_1, x_2, x_3) - \frac{\partial k_n}{\partial 2}(y_1, y_2, y_3). \tag{69}$$

Injecting $y_2 = x_2$ in formula (69), eq. (68) ensures the convergence in $W^{-2,1}$ of $\frac{\partial^2 k_n}{\partial 1 \partial 2}$. Injecting also $y_1 = x_1$ in formula (69), eq. (68) ensures that $\frac{\partial^2 k_n}{\partial 1^2} + a_n$ converges in $W^{-2,1}$.

Therefore, there exists a sequence c_n of real numbers such that

$$k_n(v) + \frac{1}{2}a_n v^2 + b_n \cdot v + c_n \tag{70}$$

converges in L^1 (b_n being the vector of components b_n^i). Finally, the sequence k_n converges in $L^1(|v| \leq R)/\Gamma$, and the image of L is closed. Thus we can apply the open mapping theorem to L in order to obtain a strictly positive K_R such that

$$\inf_{T \in M} \int_{v^2+v_1^2 \leq R^2} |t(v) + t(v_1) - T(v + v_1, v^2 + v_1^2)| \, dv dv_1$$
$$\geq K_R \inf_{m \in \Gamma} \int_{|v| \leq R} |t(v) - m(v)| \, dv. \tag{71}$$

Injecting $t = \log f$ in this estimate, we obtain lemma 4.

The proof of theorem 2 easily follows from lemmas 3 and 4 together with estimate (47).

4 On Fokker–Planck–Landau's collision kernel

The derivation of Fokker–Planck–Landau's collision kernel can be found in [Ch, Co] or [Li, Pi]. It writes

$$Q'(f,f) = \mathrm{div}_v \int_{w \in \mathbb{R}^3} (\frac{1}{|v-w|}) \{ I - \frac{(v-w) \otimes (v-w)}{|v-w|^2} \}$$
$$\{ f(w) \nabla_v f(v) - f(v) \nabla_w f(w) \} \, dw, \tag{72}$$

where I is the identity tensor.

We keep in this section the notations of sections 2 and 3. Moreover, we introduce the space Γ' of derivatives of logarithms of Maxwellians:

$$\Gamma' = \{a + bv / \quad a \in I\!\!R^3,\, b \in I\!\!R\},$$

and the set

$$H^1_{\log}(I\!\!R^3) = \{f \in L^2(I\!\!R^3) / \quad \log f \in H^1(I\!\!R^3)\}.$$

The main result of this section is the following:

Theorem 3: *Let Q' be Fokker–Planck–Landau's collision kernel and R be a strictly positive number. Then, there exists $K_R > 0$ such that for all f in $L^2_B(I\!\!R^3) \cap H^1_{log}(I\!\!R^3)$,*

$$-E_{Q'}(f) = -\int_{v \in I\!\!R^3} Q'(f,f)(v)\, \log f(v)\, dv$$

$$\geq \frac{B(R)^2}{2R} K_R \inf_{m \in \Gamma'} \int_{|v| \leq R} |\nabla_v \log f(v) - m(v)|^2\, dv. \tag{73}$$

Proof of theorem 3: According to Boltzmann's H–theorem,

$$-E_{Q'}(f) = -\int_{v \in I\!\!R^3} Q'(f,f)(v) \log f(v)\, dv$$

$$= \frac{1}{2} \int_{v \in I\!\!R^3} \int_{w \in I\!\!R^3} \frac{f(v) f(w)}{|v-w|} \{\nabla_v \log f(v) - \nabla_w \log f(w)\}$$

$$\{I - \frac{(v-w) \otimes (v-w)}{|v-w|^2}\} \{\nabla_v \log f(v) - \nabla_w \log f(w)\}\, dwdv$$

$$\geq \frac{B(R)^2}{4R} \int_{|v| \leq R} \int_{|w| \leq R} \{\nabla_v \log f(v) - \nabla_w \log f(w)\}$$

$$\{I - \frac{(v-w) \otimes (v-w)}{|v-w|^2}\} \{\nabla_v \log f(v) - \nabla_w \log f(w)\}\, dwdv. \tag{74}$$

But the eigenvalues of the symmetric tensor

$$\mathcal{T}(v-w) = I - \frac{(v-w) \otimes (v-w)}{|v-w|^2} \tag{75}$$

are 1 with order $n-1$ and 0 with order 1. Moreover, the eigenvector corresponding to the eigenvalue 0 is $v-w$. Therefore, for all x in $I\!R^3$,

$$(I - \frac{(v-w)\otimes(v-w)}{|v-w|^2})\, x\cdot x \geq \inf_{\lambda\in I\!R} |x+\lambda\,(v-w)|^2. \tag{76}$$

Therefore, if we denote by M' the space of all functions $T(v,w)\in L^1(|v|\leq R, |w|\leq R; I\!R^3)$ such that $T(v,w)$ is always parallel to $v-w$, we get

$$\begin{aligned}
-E_{Q'}(f) &= -\int_{v\in I\!R^3} Q'(f,f)(v)\,\log f(v)\,dv \\
&\geq \frac{B(R)^2}{4R} \inf_{T\in M'} \int_{|v|\leq R}\int_{|w|\leq R} \\
&|\nabla_v \log f(v) - \nabla_w \log f(w) + T(v,w)|^2\, dwdv.
\end{aligned} \tag{77}$$

Before going further in the proof, we need the following lemma:

lemma 5: *For all $R>0$, there exists $K_R>0$ such that for all $f\in L^2_B(I\!R^3)\cap H^1_{log}(I\!R^3)$,*

$$\begin{aligned}
&\inf_{T\in M'} \int_{|v|\leq R}\int_{|w|\leq R} |\nabla_v \log f(v) - \nabla_w \log f(w) + T(v,w)|^2\, dwdv \\
&\geq K_R \inf_{m\in\Gamma'} \int_{|v|\leq R} |\nabla_v \log f(v) - m(v)|^2\, dv.
\end{aligned} \tag{78}$$

Proof of lemma 5: Let L' be the following operator:

$$\begin{aligned}
L' : t \in L^2(|v|\leq R;\ I\!R^3)/\Gamma' \to \\
L't(v,w) = t(v)-t(w) \in L^2(|v|\leq R, |w|\leq R;\ I\!R^3)/M'.
\end{aligned} \tag{79}$$

The operator L' is clearly linear and one–one (Cf. [Li, Pi]).

Observe that for all t in $L^2(|v|\leq R;\ I\!R^3)$,

$$\inf_{T\in M'} \int_{|v|\leq R}\int_{|w|\leq R} |t(v)-t(w)+T(v,w)|^2\, dwdv$$

$$\leq \inf_{a\in I\!R,\ b\in I\!R^3} \int_{|v|\leq R}\int_{|w|\leq R} |t(v)-t(w)+a(v-w)+b-b|^2\, dwdv$$

$$\leq 32\ R^3 \inf_{m\in\Gamma'} \int_{|v|\leq R} |t(v) - m(v)|\, dv. \tag{80}$$

Therefore, the operator L' is continuous.

In order to apply the open mapping theorem, we still have to prove that the image of L' is closed.

Suppose that there exist a sequence t_n in $L^2(|v| \leq R;\ I\!R^3)/\Gamma'$ and t in $L^2(|v| \leq R, |w| \leq R;\ I\!R^3)/M'$ such that $t_n(v) - t_n(w)$ tends to $t(v,w)$ in $L^2(|v| \leq R, |w| \leq R;\ I\!R^3)/M'$. Then, there exists a sequence k_n in $L^2(|v| \leq R;\ I\!R^3)$, a sequence T_n in M' and g in $L^2(|v| \leq R, |w| \leq R;\ I\!R^3)$ such that t_n is the natural projection of k_n on $L^2(|v| \leq R;\ I\!R^3)/\Gamma'$, t is the natural projection of g on $L^2(|v| \leq R, |w| \leq R;\ I\!R^3)/M'$ and

$$k_n(v) - k_n(w) + T_n(v,w) \to g(v,w) \tag{81}$$

in $L^2(|v| \leq R, |w| \leq R;\ I\!R^3)$.

Therefore, if we set $k_n = (k_n^1, k_n^2, k_n^3)$, $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$, the sequence

$$(k_n^i(v) - k_n^i(w))\,(v_j - w_j)\ -\ (k_n^j(v) - k_n^j(w))\,(v_i - w_i) \tag{82}$$

converges in $L^2(|v| \leq R, |w| \leq R;\ I\!R^3)$ for all i, j in $\{1,2,3\}$.

Taking the double partial derivative of this expression with respect to v_i, w_i, we obtain the convergence in H^{-2} of $\frac{\partial k_n^j}{\partial i}(v) + \frac{\partial k_n^j}{\partial i}(w)$.

Taking also the double partial derivative of formula (82) with respect to v_i, w_j, when $i \neq j$, we obtain the convergence in H^{-2} of $-\frac{\partial k_n^i}{\partial i}(v) + \frac{\partial k_n^j}{\partial j}(w)$. Therefore, $\frac{\partial k_n^j}{\partial i}$ converges in H^{-2} for all $i \neq j$ and there exists a sequence of real numbers a_n such that $\frac{\partial k_n^i}{\partial i} + a_n$ converges in H^{-2}.

Thus, there exist three sequences of real numbers b_n^1, b_n^2, b_n^3 such that

$$k_n^i + a_n v_i + b_n^i \tag{83}$$

converges in H^{-1}.

Differentiating formula (82) with respect to v_i, we obtain the convergence in H^{-1} of

$$\frac{\partial k_n^i}{\partial i}(v)\,(v_j - w_j) - (k_n^j(v) - k_n^j(w)) - (v_i - w_i)\,\frac{\partial k_n^j}{\partial i}(v). \tag{84}$$

Injecting $v_j = w_j$ in formula (84), we get the convergence of $\frac{\partial k_n^j}{\partial i}$ in H^{-1} .

In the same way, injecting $v_i = w_i$ in formula (84), we get the convergence of $\frac{\partial k_n^i}{\partial i} + a_n$ in H^{-1} .

Finally, the sequence

$$k_n^i + a_n v_i + b_n^i \tag{85}$$

converges in L^2. Therefore, k_n converges in $L^2(|v| \leq R)/\Gamma'$, which ensures that the image of L' is closed. Thus we can apply the open mapping theorem to L' in order to obtain a strictly positive K_R such that

$$\inf_{T \in M'} \int_{|v| \leq R} \int_{|w| \leq R} |t(v) - t(w) + T(v,w)|^2 \, dwdv$$

$$\geq K_R \inf_{m \in \Gamma'} \int_{|v| \leq R} |t(v) - m(v)|^2 \, dv. \tag{86}$$

Injecting $t = \nabla_v \log f$ in this estimate, we obtain lemma 5.

The proof of theorem 3 easily follows from lemma 5 together with estimate (77).

5 Applications of the previous estimates

The reader can find a survey on the subject of convergence towards equilibrium for the Boltzmann equation in [De 3]. We recall however the result of Arkeryd (Cf. [A]) of strong and exponential convergence in an L^1 setting for the solution of the homogeneous Boltzmann equation with hard potentials towards its Maxwellian limit.

Note also, in a similar context, the bounds on $E_Q(f)$ recently given by E.A. Carlen (Cf. [Cl]) in terms of the relative entropy of f.

The estimates given in this paper are of course much rougher, but they can still be applied in complex situations (for example when the equation is non homogeneous, or when force terms are involved). We prove here on an example how, in some sense, the convergence towards the Maxwellian state holds in $O(\frac{1}{\sqrt{t}})$ in a large context. More precisely, we state the

Theorem 4: *Let f be a renormalized solution of the Boltzmann equation in a bounded domain Ω (Cf. [DP, L] and [Ha]) with a cross section B satisfying (41) (and the necessary hypothesis which guarantee the existence of a renormalized solution (Cf. [DP, L])) and such that f is in $L^1_B(\mathbb{R}^3)$. Then, for all $R > 0$, there exists $K_R > 0$ such that*

$$\int_t^{2t} \int_{x \in \mathbb{R}^3} \inf_{m \in \Gamma} \int_{|v| \leq R} |\log f - m| \, dv \, dx \frac{dt}{t} \geq \frac{K_R}{\sqrt{t}}. \tag{87}$$

Proof of theorem 4: According to [DP, L] and [Ha],

$$-\int_{t=0}^{+\infty} \int_{x\in\Omega} \int_{v\in I\!R^3} Q(f,f)(t,x,v)\, \log f(t,x,v)\, dvdxdt < +\infty, \tag{88}$$

and therefore,

$$-\int_{t}^{2t} \int_{x\in\Omega} \int_{v\in I\!R^3} Q(f,f)(t,x,v)\, \log f(t,x,v)\, dvdx \frac{dt}{t} \le \frac{C}{t} \tag{89}$$

for some constant $C > 0$. Then using corollary 2, we get theorem 4.

Remark: The same kind of theorem holds in the case of the Fokker–Planck–Landau equation.

Remark: An application of corollary 2 in the context of the Chapman–Enskog expansion can also be found in [De 4].

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