Study of a degenerate reaction-diffusion system arising in particle dynamics with aggregation effects

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Abstract

In this paper we study a degenerate parabolic system of reaction-diffusion equations arising in cellular biology models. Its specificity lies in the fact that one of the concentrations does not diffuse. Under realistic conditions on the reaction term, we prove existence and uniqueness of a nonnegative solution to the considered system, and we study its regularity. Moreover, we discuss the existence and linear stability of the steady solutions (equilibria), and give a sufficient condition on the reaction term for Turing-like instabilities to be triggered. These results are finally illustrated by some numerical simulations.

Keys words: degenerate parabolic system, existence of solutions, stability, pattern formation, aggregation

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1 Introduction

The study of the adhesion processes between cells is an active and complex research subject in biology, leading to interesting questions in mathematics. The adhesion process plays a significant role in tissues organization, cancer growth, neuronal connections in embryos, etc., see for example [3], [5], [6], [8], [11] and [16]. The growth of adhesion junctions between cells is mediated by a class of molecules called cadherins, which diffuse on the membrane when they are free. Cadherins however can bind, thus stop to diffuse, and form intra-cellular contacts. If the contact force is not large enough, the bound may break up and cadherins turn back to the free state, and diffuse again. Otherwise, they form an aggregate which is itself linked to the actin cytoskeleton, but we do neglect this part of the modeling in this study. From now on, we consider cadherins as particles and we study their spatiotemporal dynamics on a substrate by means of a mathematical model defined at the macroscopic scale. The model describes the evolution of two particles densities : the first one concerns the diffusing particles, and the second one is related to the fixed ones. For a more detailed description we refer to [9].

We perform the mathematical analysis in an n-dimensional space domain, although the biological problem as well as numerics are to be considered only in a 2-dimensional space. We define an open, bounded and regular domain $\Omega \subset \mathbb{R}^n$ and two density functions $u := u(t,x) \geq 0$ and $v := v(t,x) \geq 0$, with $x \in \Omega$ and $t \geq 0$, respectively representing the space distribution of the diffusing population of particles and of the fixed one. Particles in both populations may change their status from free (diffusing) to fixed and vice-versa. The space-time evolution of the densities u and v can then be described by the following system of reaction-diffusion equations :

$$\begin{cases} \partial_t u - \sigma \Delta_x u = -Q(u,v), \\ \partial_t v = Q(u,v), \end{cases} \tag{1}$$

in $]0,+\infty[\times\Omega$, where $\sigma > 0$ is the diffusion coefficient, and $Q(u,v)$ is the reaction term, describing how particles switch from one state to the other one. The reaction term $Q(u,v)$ counts the gain and loss for each particle population, and is therefore defined as the difference between a gain term $Q^+(u,v)$ and a loss term $Q^-(u,v)$.

Diffusing particles can bind only if targets are available on the domain Ω. Moreover, the probability of adhesion of a diffusing particle to a target is increased by the local presence of other fixed particles. This effect is then taken into account by a monotonous (increasing) function $F := F(v) \geq 0$. Thus, the gain term $Q^+(u,v)$ writes $Q^+(u,v) = u\,(\rho - v)\,F(v)$, where $\rho > 0$ is the target density.

On the other hand, the loss term $Q^-(u,v)$ is only proportional to the unbinding rate $G := G(v) \geq 0$, which depends only on the presence of other fixed particles, so that $Q^-(u,v) = v\,G(v)$. Note that the concentration u has no influence on the binding or unbinding rates.

Finally, the reaction term $Q := Q(u,v)$ reads :

$$Q(u,v) := Q^+(u,v) - Q^-(u,v) = u(\rho - v)F(v) - vG(v). \tag{2}$$

Note that the fixed particles density v cannot be larger than the targets density ρ, so that we expect that $0 \leq v(x,t) \leq \rho$ for all $x \in \Omega$ and for all $t \geq 0$.

As we shall see, the definition of the rate functions F and G may radically change the behavior of the solution to (1), leading or not to Turing like instabilities, see [13]. For the study of the time dependent problem, we shall only assume that F and G are such that :

$$\begin{aligned} &a)\ \forall v \in [0,\rho], \quad 0 < F(v), G(v) \leq 1, \\ &b)\ v \mapsto F(v) \text{ and } v \mapsto G(v) \text{ are Lipschitz-continuous on } [0,\rho]. \end{aligned}$$

Assumption a) reflects the fact that $F(v)$ and $G(v)$ are bounded rates (the bound 1 can be obtained thanks to a simple rescaling in time from any other strictly positive bound), while b) is the minimal regularity assumption needed in the sequel.

Moreover, when needed, we shall continuously extend the definition of F and G to the whole real space $\mathbb{R}$, thanks to the formulas:

$$\begin{aligned} &c)\ \forall\ v \leq 0, \quad F(v) = F(0)\ , \quad G(v) = G(0), \\ &d)\ \forall\ v \geq \rho, \quad F(v) = F(\rho)\ , \quad G(v) = G(\rho). \end{aligned}$$

In the stability study, we will moreover assume F and G to be of class C^1 by parts, and we will consider the following real-valued function $h := h(v)$ for $v \in [0,\rho[$:

$$h(v) = \frac{v\,G(v)}{(\rho - v)\,F(v)}, \tag{3}$$

and its derivative :

$$h'(v) = \frac{\rho F(v)G(v) + v(\rho - v)\left(F(v)G'(v) - F'(v)G(v)\right)}{(\rho - v)^2 F(v)^2}, \tag{4}$$

where the derivatives of F and G are replaced by both right and left derivatives at points of discontinuity of F' and G'.

Although a large number of parabolic, degenerate or not, systems were already studied (see for example [1, 2, 10, 15, 14], as well as the references therein), the existing results do not directly apply to (1) because of the particular form of our reaction term $Q(u,v)$, and of the degeneracy of the second equation. Note, in particular, that $Q(u,v)$ given by (2) does not satisfy the bound of [2]. Therefore, under hypothesis a)-b), we are first interested in the theoretical study of the solution to (1)-(2) (endowed with suitable boundary conditions and initial data). Our main results in this direction concern the existence, uniqueness and smoothness of the solutions to the degenerate reaction-diffusion system (1)-(2).

In order to shed light on aggregates formation, we also discuss the existence and stability of solutions to the stationary problem. We consider for this study that F and G are C^1 by parts functions. We compute the stationary solutions to the problem, and give a criterion for the linear stability of a given spatially homogeneous equilibrium. We show examples in which this criterion is satisfied or not. When it is not satisfied, we study the appearance of Turing-like instabilities. From a biological point of view, this means that aggregates formation is possible in such a situation.

The paper is organized as follows. In section 2, we detail the mathematical model describing the particles dynamics and we state the main results. Section 3 is devoted to the proofs of the Theorems concerning existence, uniqueness and smoothness of the solution to the evolution problem. In section 4, we study the existence of solutions to the stationary problem and their stability. Finally, in section 5, we illustrate by some numerical simulations the study of the stationary problem.

2 The model and main results

The unknown of our problem is the couple of density functions $(u, v) := (u(t, x), v(t, x))$ of resp. diffusing and fixed particles, where $x \in \Omega$ is an n-dimensional vector point, and $t \geq 0$ represents time. We assume that the targets density on the substrate is constant in time and space, and we denote it by ρ so that, up to a rescaling of the concentration v, $0 < \rho \leq 1$. Considering the model description detailed in Section 1, the density functions u and v are assumed to satisfy the degenerate system of reaction-diffusion equations (1)-(2).

Since Ω is bounded, the first equation in (1) needs to be endowed with some boundary conditions. We consider here the homogeneous Neumann boundary condition:

$$\forall\ (t, x) \in]0, +\infty[\times \partial\Omega, \qquad \frac{\partial u}{\partial \nu}(t, x) = 0, \tag{5}$$

where ν denotes the exterior normal to the boundary $\partial\Omega$. It describes at the biological level a situation in which one keeps a zero flux on the boundary of the space domain.

Finally, we complete system (1) by the initial condition,

$$\forall\ x \in \Omega, \qquad (u(0, x), v(0, x)) = (u^{in}(x), v^{in}(x)), \tag{6}$$

representing the initial distribution of both diffusing and fixed particles, and we assume that

$$\forall\ x \in \Omega, \qquad 0 \leq u^{in}(x) \quad \text{and} \quad 0 \leq v^{in}(x) \leq \rho. \tag{7}$$

In the following we shall prove that (7) holds for any time $t \geq 0$ also for the solution (u, v) to (1)-(2) (see Prop. 1 below).

Since particles are neither lost nor created, but just change their status from diffusing to fixed and vice-versa, we expect the following conservation property:

$$\int_{\Omega} (u(t,x) + v(t,x))\, dx = \int_{\Omega} (u^{in}(x) + v^{in}(x))\, dx := M\,|\Omega|, \qquad \forall\, t \geq 0. \tag{8}$$

This identity is obtained at the formal level from (1)-(2), (5) by summing up the two equations and by integrating over Ω.

We now state the results that will be proven in the next sections. We start with an *a priori* estimate that will be used in the proofs of existence:

Proposition 1 *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $\rho, \sigma > 0$, F and G satisfy a)-b), and assume that the initial data (u^{in}, v^{in}) satisfy (7) and are continuous on $\overline{\Omega}$. Then any classical solution (u,v) (that is, such that u, v, $\partial_t u$, $\partial_t v$, $\partial_{x_i} u$ and $\partial_{x_i x_j} u$ are continuous on $\overline{\Omega}$ for all $i,j = 1,..,n$) to (1), (2), (5) and (6) satisfies, $\forall\, t > 0$ and $x \in \Omega$:*

$$0 \leq u(t,x) \leq \mu + \rho t\ , \qquad 0 \leq v(t,x) \leq \rho, \tag{9}$$

where

$$\mu := \sup_{x \in \overline{\Omega}} u(0,x). \tag{10}$$

Our first main result is concerned with the existence of weak solutions to system (1)-(6) :

Theorem 1 *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $\rho, \sigma > 0$, F and G satisfy a)-b), and assume that the initial data (u^{in}, v^{in}) satisfy (7) and $u^{in} \in L^\infty(\Omega)$. Then there exist two nonnegative functions $u \in L^2_{loc}([0,+\infty[; H^1(\Omega)) \cap L^\infty_{loc}(\mathbb{R}_+; L^\infty(\Omega))$ and $v \in L^\infty([0,+\infty) \times \Omega)$ (more precisely $0 \leq v \leq \rho$) which are weak solutions to system (1), (2), (5) and (6), in the following sense: $Q(u,v) \in L^\infty_{loc}(\mathbb{R}_+; L^\infty(\Omega))$, and for all $\phi, \psi \in C^2_c([0,+\infty[\times\overline{\Omega})$ such that $\frac{\partial\phi}{\partial\nu} = 0$ on $[0,+\infty[\times\partial\Omega$, the following identities hold:*

$$-\int_0^\infty \int_\Omega u\, \partial_t \phi\, dx\, dt - \int_\Omega u^{in}\, \phi(0,\cdot)\, dx - \sigma \int_0^\infty \int_\Omega u\, \Delta_x \phi\, dx\, dt = -\int_0^\infty \int_\Omega Q(u,v)\, \phi\, dx\, dt,$$

$$-\int_0^\infty \int_\Omega v\, \partial_t \psi\, dx\, dt - \int_\Omega v^{in}\, \psi(0,\cdot)\, dx = \int_0^\infty \int_\Omega Q(u,v)\, \psi\, dx\, dt.$$

When the initial data are smooth enough, the solutions defined above are in fact strong (classical) and unique, as stated in the theorem below:

Theorem 2 *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $\rho, \sigma > 0$, F and G satisfy a)-b), and assume that the initial data (u^{in}, v^{in}) satisfy (7) and belong respectively to $C^2(\overline{\Omega})$ and $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in]0,1[$. Assume also that u^{in} satisfies Neumann boundary condition. Then there exists a unique classical solution (u,v) of (1), (2), (5) and (6), that is a solution $u, v \geq 0$ such that $u, \partial_t u, \partial_{x_k} u, \partial_{x_k x_l} u \in C([0,+\infty[\times\overline{\Omega})$ for $k,l = 1,..,n$ and $v, \partial_t v \in C([0,+\infty[\times\overline{\Omega})$.*

We consider in section 4 the existence and stability of steady solutions (equilibria). We thus look for solutions to the associated stationary problem :

$$\begin{cases} \Delta_x u = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ Q(u,v) = 0 \quad \text{in} \quad \Omega, \\ u \geq 0, \qquad 0 \leq v \leq \rho, \\ \displaystyle\int_\Omega (u(x) + v(x))\, dx = M\, |\Omega|. \end{cases} \tag{11}$$

We prove that there exists at least one homogeneous in space solution to (11), denoted by (U, V). We also present a uniqueness result for such a stationary solution, under an extra condition. More precisely, we state the :

Proposition 2 *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, and $\rho, \sigma, M > 0$. Let also F and G satisfy a)-b), and assume moreover that F and G are C^1 by parts. Then all strong (that is, such that $u \in C^2(\overline{\Omega})$ and $v \in L^\infty(\Omega)$) solutions to (11), are such that u is spatially homogeneous (does not depend on x). Moreover, at least one spatially homogeneous solution (that is, both u and v do not depend on x) exists. Finally, all spatially homogeneous solutions (U, V) satisfy the strict inequality $0 < U$, $0 < V < \rho$, and the system*

$$U = \frac{V G(V)}{(\rho - V) F(V)}, \qquad U + V = M. \tag{12}$$

When the extra condition

$$\forall v \in [0, \rho], \quad h'(v) + 1 \geq 0, \tag{13}$$

(with h' given by (4)) is satisfied, there is a unique spatially homogeneous solution (U, V) to system (11). Note that condition (13) always holds if the following simpler condition is satisfied:

$$\forall v \in [0, \rho], \qquad F(v) G'(v) - F'(v) G(v) > 0. \tag{14}$$

Finally, if for some real number $U > 0$, there are at least two distinct real numbers $V_1 \in]0, \rho[$ and $V_2 \in]0, V_1[$ such that

$$U = \frac{V_1 G(V_1)}{(\rho - V_1) F(V_1)} = \frac{V_2 G(V_2)}{(\rho - V_2) F(V_2)} \tag{15}$$

and $M - U \in]V_2, V_1[$, then system (11) has an infinite number of spatially inhomogeneous solutions defined as $(U, V_1 1_A + V_2 1_{A^c})$, where A is a measurable subset of Ω (with $|A| \neq 0$ and $|A^c| \neq 0$) such that $M - U = V_1 \frac{|A|}{|\Omega|} + V_2 \frac{|A^c|}{|\Omega|}$ (that is, $|A| = |\Omega| \frac{M-U-V_2}{V_1 - V_2}$).

We then discuss the linear stability of a given spatially homogeneous solution of (11), and give conditions for the appearance of Turing-like instabilities.

Proposition 3 *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, and $\rho, \sigma, M > 0$. Let also F and G satisfy a)-b), and assume moreover that F and G are C^1 by parts. We consider a spatially homogeneous solution of (11), that is a couple of real numbers (U, V) such that $0 < U$, $0 < V < \rho$ and (12) holds (according to Prop. 2, at least one such solution exists). Then if F and G are C^1 at point V and*

$$h'(V) + 1 > 0, \tag{16}$$

the equilibrium (U, V) is linearly (and thus locally nonlinearly) stable for the ODE system

$$\begin{cases} \dot{u} = -Q(u, v), \\ \dot{v} = Q(u, v). \end{cases} \tag{17}$$

If F and G are C^1 at point V and

$$h'(V) + 1 < 0, \tag{18}$$

the equilibrium (U, V) is linearly unstable for the ODE system (17).

Note at this level that the stability condition (16) is always satisfied when the condition of uniqueness (13) holds.

Moreover, if

$$h'(V) > 0, \tag{19}$$

the equilibrium (U, V) is linearly stable for the PDE system (1), and if

$$h'(V) < 0, \tag{20}$$

the equilibrium (U, V) is linearly unstable for the PDE system (1).

Section 5 is devoted to the numerical simulations (when $n = 2$) illustrating the results on the large time behavior of our system.

As in [9], we consider that the binding process is described by rate functions F and G defined as follows :

$$F(v) = \frac{a + \tanh(v)}{a + \tanh(\rho)}\ , \quad G(v) = 1 - \tanh(\alpha v), \quad \forall\ v \in]0, \rho[, \tag{21}$$

with $0 < a \leq 1$ and $\alpha > 0$ real parameters. Parameter a represents the rate at which a particle naturally binds on a target (i.e. if no other particle is fixed nearby), and α measures the influence of the unbinding. As we shall see, it is possible to find values a and α such that a stable homogeneous steady state exists for the system of PDEs, and other values a and α such that no such stable homogeneous steady state exists. In that last case, Turing-like patterns appear (that is, aggregation occurs).

3 Existence, uniqueness and smoothness

We first consider the positivity and boundedness of classical solutions (u, v) to (1), (2), (5) and (6), assuming that they exist. In a second time, we prove the existence of weak solutions by constructing two Cauchy sequences. And finally we deal with the existence, uniqueness, and smoothness of strong solutions.

3.1 Proof of Proposition 1

We present below the proof of Prop. 1 and divide it in three steps. First we prove the upper bound for v. Second we prove the bound for u. Third we prove the nonnegativity of both u and v.

Proof:
Step I. We first prove by contradiction the upper bound : $v \leq \rho$. Let us define the set

$$A := \{t > 0 \ : \ \exists\, x \in \Omega \text{ such that } v(t,x) > \rho\}.$$

We now define (if A is not empty) $t_0 := \inf A$ in $\mathbb{R}_+$.
Note that, by means of continuity arguments, $t_0 > 0$. Indeed if $v(0,x) = \rho$, then $\partial_t v(t_0,x) = -G(\rho)\rho < 0$. Then there exists a point $x_0 \in \overline{\Omega}$ such that $v(t_0,x_0) = \rho$.
Now, considering the equation for v in (1) and computing its value at the point (t_0,x_0), we get, because of assumption a),

$$\partial_t v(t_0,x_0) = -G(\rho)\rho < 0.$$

Therefore, the function $t \mapsto v(t,x_0)$ is strictly decreasing in a neighborhood of t_0. Hence, there exists $\eta > 0$ such that for all $t \in [t_0-\eta, t_0]$, $v(t,x_0) > v(t_0,x_0) = \rho$, and (using a point of Ω in a neighbourhood of x_0 if $x_0 \in \partial\Omega$), $t_0 - \eta \in A$. But $t_0 - \eta < t_0$, contradicting the fact that $t_0 = \inf A$. Hence, A is empty.

Step II. We now consider the upper bound $u(t,x) \leq \mu + \rho t$ (remembering that, $\forall\, t \geq 0$ and $\forall\, x \in \Omega$, $v(t,x) \leq \rho$).
We wish to prove that the function $(u(t,x)-\mu-\rho t)^+ = \max(0, u(t,x)-\mu-\rho t)$ is equal to zero for all time $t \geq 0$ and all $x \in \Omega$, where μ is defined by (10).
Note that if $(u(t,x)-\mu-\rho t)^+ \neq 0$, then $u(t,x) \geq \mu + \rho t$, so that $u(t,x) > 0$. Using the equation for u in (1), and since, from a), $G(v) \leq 1$ and, from Step I, $v \leq \rho$, we have :

$$\begin{aligned}
&\partial_t(u-\mu-\rho t)(u-\mu-\rho t)^+ - \sigma\Delta_x(u-\mu-\rho t)(u-\mu-\rho t)^+ \\
&\qquad = (\partial_t u - \sigma\Delta_x u)(u-\mu-\rho t)^+ - \rho(u-\mu-\rho t)^+ \\
&\qquad = -u(\rho - v)F(v)(u-\mu-\rho t)^+ + (vG(v)-\rho)(u-\mu-\rho t)^+ \leq 0.
\end{aligned}$$

Hence, integrating over Ω, (and using $[(x^+)^2]' = 2\,x^+\,x'$):

$$\frac{1}{2}\frac{d}{dt}\int_\Omega ((u-\mu-\rho t)^+)^2 \, dx + \sigma \int_\Omega |\nabla_x(u-\mu-\rho t)^+|^2 \, dx \leq 0,$$

so that, for all $t \geq 0$,

$$\|(u-\mu-\rho t)^+\|_{L^2}^2 \leq \|(u(0,x)-\mu)^+\|_{L^2}^2 = 0,$$

which concludes the second step of the proof.

Step III. Let (u,v) be a solution of (1), under hypothesis (7) we prove that $v \geq 0$ and $u \geq 0$ for all times $t \geq 0$. Multiplying equations of (1) respectively by $u^- = \min(u,0)$ and $v^- = \min(v,0)$, integrating over Ω (and using $[(x^-)^2]' = 2\,x^-\,x'$), we obtain :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega [(u^-)^2 + (v^-)^2]dx + \sigma\int_\Omega |\nabla_x(u^-)|^2 dx \\ = -\int_\Omega (u^-)^2(\rho - v)F(v)dx + \int_\Omega v^+ G(v)u^- dx + \int_\Omega v^- G(v)u^- dx \\ + \int_\Omega u^+ v^-(\rho - v)F(v)dx + \int_\Omega u^- v^-(\rho - v)F(v)dx - \int_\Omega (v^-)^2 G(v)dx,\end{aligned}$$

where we have used that $u = u^+ + u^-$ and $v = v^+ + v^-$. Because of a), c) and d), of Step I, and since $u^+v^- \leq 0$ and $u^-v^+ \leq 0$, we get :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega [(u^-)^2 + (v^-)^2]dx \\ \leq \int_\Omega (\rho - v)F(v)[-(u^-)^2 + u^-v^-]dx + \int_\Omega G(v)[-(v^-)^2 + u^-v^-]dx.\end{aligned}$$

But $u^-v^- \leq [(u^-)^2 + (v^-)^2]$ implies :

$$\frac{d}{dt}\int_\Omega [(u^-)^2 + (v^-)^2]dx \leq \int_\Omega (\rho - v)F(v)(v^-)^2 dx + \int_\Omega G(v)(u^-)^2 dx. \tag{22}$$

Recalling that $v \in C([0,\infty[\times\overline{\Omega})$, we denote, for any $T > 0$, by $m := m(T)$ the minimum of v on $[0,T] \times \overline{\Omega}$, and using the upper bound in a), we deduce from (22) that the following estimate holds on $[0,T]$:

$$\frac{d}{dt}\int_\Omega [(u^-)^2 + (v^-)^2]dx \leq \max(\rho - m, 1)\int_\Omega [(u^-)^2 + (v^-)^2]dx.$$

Since $u(0,\cdot) \geq 0$ and $v(0,\cdot) \geq 0$, Gronwall's Lemma allows us to conclude that :

$$\int_\Omega [(u^-)^2 + (v^-)^2]dx \leq 0.$$

Hence, $u, v \geq 0$ on $[0,+\infty[\times\overline{\Omega}$. ■

3.2 Proof of Theorem 1 and Theorem 2

We prove here first the existence of weak solutions to (1), (2), (5) and (6) stated in Theorem 1, by constructing two Cauchy sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$, converging in $L^\infty_{loc}(\mathbb{R}_+; L^2(\Omega))$ towards u and v, weak solutions to our problem.

Let $\sigma, \rho > 0$, let u^{in} be a nonnegative initial datum in $L^\infty(\Omega)$, and v^{in} be a nonnegative initial datum such that $v^{in} \leq \rho$. We first define u_n^{in} and v_n^{in}, a sequence of approximated nonnegative and $C^{0,\alpha}$ (on $\overline{\Omega}$, for some $\alpha \in]0,1[$) initial data satisfying $0 \leq u_n^{in} \leq \mu := ||u^{in}||_{L^\infty(\Omega)}$ and $0 \leq v_n^{in} \leq \rho$, and such that u_n^{in} and v_n^{in} converge towards u^{in} and v^{in} a.e. Moreover, we assume that $||u_n^{in} - u^{in}||^2_{L^2(\Omega)} + ||v_n^{in} - v^{in}||^2_{L^2(\Omega)} \leq 1/(n!)$. For all $(t,x) \in [0,+\infty[\times\overline{\Omega}$, we then define two sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ by:

$$\begin{cases} \partial_t u_{n+1} - \sigma \Delta_x u_{n+1} = -Q(u_{n+1}, v_n) \quad \text{in }]0,+\infty[\times\Omega \\ \partial_t v_{n+1} = Q(u_n, v_{n+1}) \quad \text{in }]0,+\infty[\times\Omega \\ \dfrac{\partial u_{n+1}}{\partial \nu} = 0 \quad \text{on }]0,+\infty[\times\partial\Omega \\ u_{n+1}(0,.) = u_{n+1}^{in} \quad \text{in } \Omega \\ v_{n+1}(0,.) = v_{n+1}^{in} \quad \text{in } \Omega, \end{cases} \tag{23}$$

with

$$u_0(t,x) = u_0^{in}(x) \quad \text{and} \qquad v_0(t,x) = v_0^{in}(x) \quad \forall (t,x) \in (0,T)\times\Omega. \tag{24}$$

Note that those functions are well defined and $C^{0,\alpha}$ (on $\overline{\Omega}$, and for $\alpha \in]0,1[$ introduced in the assumption of the theorem). Indeed, thanks to an induction, the first equation can be seen as a linear heat equation in u_{n+1} with a $C^{0,\alpha}$ coefficient $(\rho - v_n)F(v_n)$ and a $C^{0,\alpha}$ source term $-v_n\, G(v_n)$, with u_{n+1}, $\partial_t u_{n+1}$, $\partial_{x_i} u_{n+1}$, $\partial_{x_i x_j} u_{n+1}$ in $C^{0,\alpha}$, see [7], and the second one can be seen as a Riccati ODE in v_{n+1}, where x is a ($C^{0,\alpha}$ regular) parameter.

As a consequence, using the properties of the heat equation, $\partial_t u_{n+1}$, $\partial_{x_i} u_{n+1}$, $\partial_{x_i x_j} u_{n+1}$ are continuous on $[0,+\infty[\times\overline{\Omega}$, and even $C^{0,\alpha}$ on the same space. The same obviously holds for $\partial_t v_{n+1}$.

We prove that $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$, defined by (23) and (24), are two Cauchy sequences in $L^\infty_{loc}(\mathbb{R}_+; L^2(\Omega))$, converging to u and v (in the same space), which are weak solutions to (1), (2), (5) and (6). We first need some technical results.

Proposition 4 *Let u_n and v_n be defined by (23) and (24). Then, under the assumptions of Theorem 1, both functions u_n and v_n satisfy the bound (9), that is:*

$$\forall\ (t,x) \in]0,T[\times\Omega, \qquad 0 \leq u_n(t,x) \leq \mu + \rho t \quad \textit{and} \qquad 0 \leq v_n(t,x) \leq \rho. \tag{25}$$

Proof:
This proposition can be proven by induction using the same arguments as in the proof of Prop. 1. ∎

We introduce now a final time $T > 0$ (the estimates that we shall present will all blow up when $T \to +\infty$). For the sake of simplicity, for $n \in \mathbb{N}^*$ and $t \in]0,T[$, we define by $U_n(t)$ and $V_n(t)$

the following squares of L^2-norms: $U_n(t) := \|u_{n+1}(t,\cdot) - u_n(t,\cdot)\|^2_{L^2(\Omega)}$ and $V_n(t) := \|v_{n+1}(t,\cdot) - v_n(t,\cdot)\|^2_{L^2(\Omega)}$. Hence:

$$U_n(t) = \int_\Omega |u_{n+1}(t,x) - u_n(t,x)|^2 dx \ , \qquad V_n(t) = \int_\Omega |v_{n+1}(t,x) - v_n(t,x)|^2 dx. \tag{26}$$

We first show the following technical estimate:

Proposition 5 *Under the assumptions of Theorem 1, there exists a constant $k > 1$ (depending only on F, G, ρ, T and μ), such that for all $n \in \mathbb{N}^*$ and $t \in]0, T[$:*

$$U_n'(t) + 2\sigma \int_\Omega |\nabla(u_{n+1} - u_n)|^2 dx \leq 3k\, U_n(t) + k\, V_{n-1}(t), \tag{27}$$

and

$$V_n'(t) \leq 3k\, V_n(t) + k\, U_{n-1}(t). \tag{28}$$

Proof :
The assumption of Lipschitz-continuity of F, G on $[0, \rho]$ implies that the function $(r, s) \mapsto Q(r, s)$ (defined by (2)) is a Lipschitz-continuous function on $[0, \gamma] \times [0, \rho]$, where

$$\gamma = \mu + \rho T.$$

Therefore there exists a constant $k > 1$ such that for all $((r, s), (r', s')) \in ([0, \gamma] \times [0, \rho])^2$:

$$|Q(r, s) - Q(r', s')| \leq k(|r - r'| + |s - s'|). \tag{29}$$

Since both functions u_n and u_{n+1} satisfy (23), we have :

$$\partial_t(u_{n+1} - u_n) - \sigma\Delta(u_{n+1} - u_n) = -Q(u_{n+1}, v_n) + Q(u_n, v_{n-1}) \tag{30}$$

in $]0, T[\times\Omega$. Multiplying (30) by $(u_{n+1} - u_n)$ and integrating on Ω, we obtain for all $t \in]0, T[$:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_{n+1} - u_n)^2 dx + \sigma \int_\Omega |\nabla(u_{n+1} - u_n)|^2 dx \tag{31}$$

$$= \int_\Omega \left(Q(u_n, v_{n-1}) - Q(u_{n+1}, v_n)\right)(u_{n+1} - u_n)\, dx.$$

Thanks to (29) and (26), we deduce from (31) that:

$$\frac{1}{2}U_n'(t) + \sigma\int_\Omega |\nabla(u_{n+1} - u_n)|^2 dx \leq k \int_\Omega \left(|u_{n+1} - u_n| + |v_n - v_{n-1}|\right)|u_{n+1} - u_n| dx \tag{32}$$

$$\leq k \int_\Omega |u_{n+1} - u_n|^2 dx + \frac{k}{2}\int_\Omega \left(|v_n - v_{n-1}|^2 + |u_{n+1} - u_n|^2\right) dx \tag{33}$$

$$\leq \frac{3k}{2}U_n(t) + \frac{k}{2}V_{n-1}(t), \tag{34}$$

which implies (27). The proof of estimate (28) is almost identical since the second term of the first member of (31) doesn't play any role in the proof above. ■

The differential inequalities (27) and (28) yield upper bounds for U_n and V_n thanks to a variant of Gronwall's lemma explained below:

Proposition 6 *Under the assumptions of Theorem 1, we have for all $n \in \mathbb{N}^*$ and $t \in]0, T[$:*

$$(U_n + V_n)(t) \leq \sum_{q=0}^{n-1} \frac{4\, e^{3k\,T}}{q!\,(n-q)!}\,(k\, e^{3k\,T}\, t)^q + (2\,\rho^2 + 2\,(\mu + \rho\, T)^2)\,|\Omega|\,\frac{(k\, e^{3k\,T}\, t)^n}{n!},$$

where $k \geq 1$ is the Lipschitz constant of Q defined in (29), and $|\Omega|$ is the Lebesgue's measure of Ω.

Proof:
We first treat the case when $u^{in}, v^{in} \in C^{0,\alpha}(\overline{\Omega})$ (for some $\alpha \in]0,1[$), so that one can take $u_n^{in} = u^{in}$, $v_n^{in} = v^{in}$, and $U_n(0) = V_n(0) = 0$.

We observe that for $t \in [0, T]$, $n \geq 1$,

$$(U_n + V_n)'(t) \leq 3k\,(U_n + V_n)(t) + k\,(U_{n-1} + V_{n-1})(t),$$

so that a first application of Gronwall's lemma yields (for $t \in [0, T]$, $n \geq 1$, and remembering that $U_n(0) = V_n(0) = 0$),

$$(U_n + V_n)(t) \leq k\, e^{3k\,T} \int_0^t (U_{n-1} + V_{n-1})(s)\, ds.$$

A direct induction shows then that

$$(U_n + V_n)(t) \leq \frac{(k\, e^{3k\,T}\, t)^n}{n!} \sup_{s \in [0,T]} (U_0 + V_0)(s).$$

Note finally that (for $t \in [0, T]$),

$$(U_0 + V_0)(t) = \int_\Omega \left(|u_1 - u_0|^2 + |v_1 - v_0|^2\right)\, dx$$

$$\leq (2\,\rho^2 + 2\,(\mu + \rho\, T)^2)\,|\Omega|.$$

We now briefly explain how to proceed when we don't assume anymore the identity $u_n^{in} = u^{in}$, $v_n^{in} = v^{in}$, but only the estimate $||u_n^{in} - u^{in}||^2_{L^2(\Omega)} + ||v_n^{in} - v^{in}||^2_{L^2(\Omega)} \leq 1/(n!)$. Then $U_n(0) + V_n(0) \leq 4/(n!)$, so that

$$(U_n + V_n)(t) \leq 4\, e^{3k\,T}/(n!) + k\, e^{3k\,T} \int_0^t (U_{n-1} + V_{n-1})(s)\, ds.$$

A direct induction shows then that (for $t \in [0,T]$)

$$(U_n + V_n)(t) \le \sum_{q=0}^{n-1} \frac{4\, e^{3kT}}{q!\,(n-q)!}\,(k\, e^{3kT}\, t)^q + (2\,\rho^2 + 2\,(\mu+\rho\, T)^2)\,|\Omega|\,\frac{(k\, e^{3kT}\, t)^n}{n!}.$$

■

We now can prove that $(u_n)_{n\ge 0}$ and $(v_n)_{n\ge 0}$ are Cauchy sequences in $L^\infty([0,T];L^2(\Omega))$, as well as Theorem 1.

Proof of Theorem 1 :
Let $t \in]0,T[$. We have for all $(n,m) \in (\mathbb{N}^*)^2$, $n > m$:

$$\|u_n(t,.) - u_m(t,.)\|_{L^2(\Omega)} \le \sum_{j=m}^{n-1} \|u_{j+1}(t,.) - u_j(t,.)\|_{L^2(\Omega)} \le \sum_{j=m}^{n-1} (U_j(t))^{1/2}.$$

We deduce that (for some constant $K > 0$ depending on T and the data of the problem)

$$\|u_n(t,.) - u_m(t,.)\|_{L^2(\Omega)} \le K \sum_{j=m+1}^{n} \left(\frac{(\max(1, ke^{3kT}T))^j}{([j/2]-1)!} \right)^{1/2}.$$

Since $\sum_{j=2}^\infty \left(\frac{(\max(1,ke^{3kT}T))^j}{([j/2]-1)!} \right)^{1/2}$ converges, we see that $(u_n)_{n\ge 0}$ is a Cauchy sequence in $L^\infty([0,T];L^2(\Omega))$. Similarly we get that $(v_n)_{n\ge 0}$ is a Cauchy sequence in $L^\infty([0,T];L^2(\Omega))$. Therefore, there exist two functions u and v in $L^\infty_{loc}(\mathbb{R}_+;L^2(\Omega))$ such that for all $T > 0$:

$$\|u_n - u\|_{L^\infty([0,T];L^2(\Omega))} \to 0\ , \quad \text{and} \quad \|v_n - v\|_{L^\infty([0,T];L^2(\Omega))} \to 0.$$

Using now the boundedness and Lipschitz-continuity of F, G and $v \mapsto \rho - v$ on $[0,\rho]$, we see that $Q(u_{n+1},v_n)$ as well as $Q(u_n,v_{n-1})$ converges to $Q(u,v)$ in $L^\infty_{loc}(\mathbb{R}_+;L^2(\Omega))$. It is then clear that one can pass to the limit in all the terms defining the weak solutions defined in Theorem 1.

Integrating in time estimate (32), we see that

$$2\sigma \int_0^t \int_\Omega |\nabla u_{n+1}(s,x) - \nabla u_n(s,x)|^2 dx ds \le \frac{4}{n!} + 3k \int_0^t \Big[U_n(s) + V_n(s) + U_{n-1}(s) + V_{n-1}(s) \Big]\, ds.$$

Summing over n and using the summability property of (U_n), (V_n), we see that (∇u_n) is a Cauchy sequence in $L^2([0,T]\times\Omega)$, and therefore $\nabla u \in L^2([0,T]\times\Omega)$.

■

We now present the

Proof of Theorem 2 :
We first prove uniqueness. Let (u_1, v_1) and (u_2, v_2) be two classical solutions (that is, such that $u_i, \partial_t u_i, \partial_{x_k} u_i, \partial_{x_k x_l} \in C([0,+\infty[\times\overline{\Omega})$ for $k,l = 1,..,n$ and $v_i, \partial_t v_i \in C([0,+\infty[\times\overline{\Omega}))$ of:

$$\begin{cases} \partial_t u_i - \sigma\Delta_x u_i = -Q(u_i, v_i) \quad \text{ in }]0,T[\times\Omega, \\ \partial_t v_i = Q(u_i, v_i) \quad \text{ in }]0,T[\times\Omega, \\ \dfrac{\partial u_i}{\partial \nu} = 0 \quad \text{ on }]0,T[\times\partial\Omega, \\ u_i(0,.) = u^{in} \quad \text{ in } \Omega, \\ v_i(0,.) = v^{in} \quad \text{ in } \Omega, \end{cases} \tag{35}$$

for $i = 1, 2$. Then

$$\partial_t(u_1 - u_2)(u_1 - u_2) - \sigma\Delta_x(u_1 - u_2)(u_1 - u_2) = [Q(u_2, v_2) - Q(u_1, v_1)](u_1 - u_2),$$

and

$$\partial_t(v_1 - v_2)(v_1 - v_2) = [-Q(u_2, v_2) + Q(u_1, v_1)](v_1 - v_2)$$

on $(0,T)\times\Omega$. Integrating both equalities and using an integration par parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_1 - u_2)^2 dx + \sigma\int_\Omega |\nabla_x(u_1 - u_2)|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega (v_1 - v_2)^2 dx$$
$$= \int_\Omega [Q(u_1, v_1) - Q(u_2, v_2)][(v_1 - v_2) - (u_1 - u_2)]dx.$$

We recall (cf. proof of the previous theorem) that Q is Lipschitz-continuous on $[0, u_{max}] \times [0, \rho]$ for all $u_{max} > 0$, so that when $t \in [0,T]$, using $u_{max} := \sup_{t\in[0,T]} ||u(t,\cdot)||_\infty$, we can write

$$|Q(u_1, v_1) - Q(u_2, v_2)| \le K_1(|u_1 - u_2| + |v_1 - v_2|). \tag{36}$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [(u_1 - u_2)^2 + (v_1 - v_2)^2]dx \le K_1\int_\Omega [(u_1 - u_2)^2 + (v_1 - v_2)^2]dx.$$

But $u_1(0,x) = u_2(0,x)$ and $v_1(0,x) = v_2(0,x)$ for all $x \in \Omega$, so that thanks to Gronwall's Lemma, we get the identities $u_1 = u_2$ and $v_1 = v_2$ on $[0,T]$. Since T can be taken arbitrarily large, this concludes the proof of uniqueness.

We now show that under the extra assumptions on the regularity of initial data, the weak solution (u, v) obtained in the previous theorem is in fact classical.

Indeed, we already know that $Q(u, v) \in L^\infty([0,T] \times \Omega)$ for all $T > 0$. As a consequence, thanks to maximal regularity estimates, we get that $\partial_t u, \partial_{x_i} u, \partial_{x_i x_j} u \in L^p([0,T] \times \Omega)$ for all $p \in [1, +\infty[$, $T \ge 0$. Then $u \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in]0,1[$. But v now solves for all x a Riccati equation, and the dependence of the parameters of the equation w.r.t. x is of class $C^{0,\alpha}(\overline{\Omega})$. Thanks to the assumption on the initial datum v^{in}, we get that $v \in C^1(\mathbb{R}_+; C^{0,\alpha}(\overline{\Omega}))$. As a consequence, we see that

$Q(u,v) \in C^{0,\alpha}(\overline{\Omega})$, and finally thanks to Schauder's estimates, we get that $\partial_t u, \partial_{x_i} u, \partial_{x_i x_j} u \in C^{0,\alpha}(\overline{\Omega})$, so that (u,v) is a classical solution of the system. ∎

4 Stability of solutions

We consider in this section the behavior of the stationary problem (11). In particular, we prove Propositions 2 and 3, concerning the existence of stationary solutions to (11) and their stability.

Proof of Proposition 2 : Multiplying the first equation in (11) by u, integrating over Ω and applying Green formula, we obtain that its solution u must be spatially homogeneous : $u(x) = U \in \mathbb{R}$, $\forall\, x \in \Omega$.

Remembering the definition of h given in (3) and considering the behavior of the function $v \mapsto h(v) + v - M$ at the boundaries of the interval $[0,\rho[$, we have : $h(0) + 0 - M = -M < 0$ and $lim_{v \to \rho^-} [h(v) + v - M] = +\infty$. Thanks to the continuity of h on $[0,\rho[$, we conclude that there exists at least one $V \in]0,\rho[$, such that : $h(V) + V = M$. Using (3) and defining $U := h(V) = M - V$, we end up with the sytem (12).

Moreover, if (16) holds, then $v \mapsto h(v) + v - M$ is strictly increasing in $]0,\rho[$, concluding the proof of uniqueness.

Let us now assume that for some real number $U > 0$, there are at least two distinct real numbers $V_1 \in]0,\rho[$ and $V_2 \in]0, V_1[$ such that (15) holds and $M - U \in]V_2, V_1[$. Then, for any A measurable subset of Ω, the couple $(u(x), v(x)) = (U, V_1\, 1_A(x) + V_2\, 1_{A^c}(x))$ clearly satisfies $\Delta_x u = 0$ and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. It also satisfies $Q(u,v) = 0$ and, thanks to the assumption that $M - U = V_1 \frac{|A|}{|\Omega|} + V_2 \frac{|A^c|}{|\Omega|}$, one can check that

$$\int_\Omega (u(x) + v(x))\, dx = M\, |\Omega|.$$

∎

In Proposition 3, we study the stability of a given spatially homogeneous stationary solution (U,V). Its proof is detailed below.

Proof of Proposition 3 : We first study the stability of a given couple of real numbers (U,V) such that $U > 0$, $V \in]0,\rho[$ which is an equilibrium for the system of two ODEs (17), i.e. such that (12) holds.

First, because of the conservation law $\dfrac{d}{dt}(u+v) = 0$, we observe that the system (17) rewrites

$$\dot{v} = (M - v)\,(\rho - v)\,F(v) - v\,G(v) =: q(v),$$

where $M = u(0) + v(0)$.

Computing

$$q'(v) = -(\rho - v)\,F(v) - (M - v)\,F(v) + (M - v)\,(\rho - v)\,F'(v) - v\,G'(v) - G(v),$$

and observing that $(M-V)\,(\rho-V)\,F(V) = V\,G(V)$, we get

$$q'(V) = V\,\frac{F'(V)\,G(V) - F(V)\,G'(V)}{F(V)} - \frac{\rho}{\rho - V}\,G(V) - (\rho - V)\,F(V),$$

so that $q'(V) < 0$ is equivalent to (16) (and $q'(V) > 0$ is equivalent to (18)).
We thus get the result of stability for the ODE system (17) stated in the proposition.
We then turn to the linear stability of the system (1), at a given spatially homogeneous equilibrium (U, V) such that $U > 0$, $V \in]0, \rho[$ (that is, a couple satisfying (12)).

We first compute

$$A := \partial_u Q(U, V) \quad \text{and} \quad B := \partial_v Q(U, V),$$

remembering that U is related to V by (12). We get :

$$A = (\rho - V)\,F(V) > 0,$$
$$B = \frac{-V(\rho - V)\,(F(V)G'(V) - F'(V)G(V)) - \rho F(V)G(V)}{(\rho - V)F(V)}.$$

Denoting by $\lambda_0 = 0$ and $\lambda_1 > 0, \lambda_2 > 0, ..., \lambda_k > 0, ...$ the eigenvalues of $-\Delta$ on Ω with homogeneous Neumann boundary conditions, we see that the matrix associated to the k-th mode in the linearization of (1) around (U, V) is

$$M_k := \begin{pmatrix} -A - \sigma\,\lambda_k & -B \\ A & B \end{pmatrix}.$$

Note that its trace is $Tr(M_k) = -A + B - \sigma\,\lambda_k$ and its determinant is $Det(M_k) = -\sigma\,\lambda_k\,B$.
Assume now that (20) holds. Then if (18) also holds, we already know that (U, V) is linearly unstable for the system of ODEs (17), so that it is *a fortiori* unstable for the system of PDEs (1). If (18) does not hold, we see that $-A + B \leq 0$ (this is in fact equivalent). Thus, $Tr(M_k) < 0$ and, since (20) is equivalent to $B > 0$, $Det(M_k) < 0$ (for any $k \geq 1$), so that (U, V) is linearly unstable for the system of PDEs (1).
Assume finally that (19) holds. This is equivalent to say that $B < 0$. Remembering that $A > 0$, we get that for any $k \geq 1$, $Tr(M_k) < 0$ and $Det(M_k) > 0$, so that (U, V) is linearly stable for the system of PDEs (1). ∎

5 Numerical illustrations

We present here a few simulations for equations (1), (2), (5), when the functions F and G are given by formulas (21). The numerical values of the parameters that we consider are $\sigma = 3.3 \cdot 10^{-}2$, $\rho = 1$, and

- Case 1: $a = 0.5, \qquad \alpha = 1,$
- Case 2: $a = 0.005, \qquad \alpha = 1.8.$

We also use the domain $\Omega = [0, 10] \times [0, 10]$ and the initial data

$$v^{in}(x) = V + 10^{-3} \sum_{i=1}^{4} \exp\left(\frac{(x - x_i)^2}{10^{-4}}\right), \qquad u^{in}(x) = 1 - \frac{1}{|\Omega|} \int_\Omega v^{in}(x)\, dx, \tag{37}$$

with x_i, for $i = 1, \dots 4$, approximatively the following points : (2.5, 2.5), (2.5, 7.5), (7.5, 2.5) and (7.5, 7.5), and so that the parameter M defined by (8) is $M = 1$.

We use an explicit centered discretization for the Laplace operator, choose a space step $\Delta x = 0.05$ on both directions x_1 and x_2, and define the time step $\Delta t = \dfrac{0.1 \Delta x^2}{4\sigma}$, so that the CFL condition is satisfied.

We start with Case 1. We look for the solutions of (12). We find $U \cong 0.5964596$ and $V \cong 0.4035404$. Then we check the condition (19), and see that it is fulfilled.

We present the results obtained numerically in this case. More precisely, we present the curves w.r.t. time of $t \mapsto \min_{x\in\Omega} u(t, x)$ and $t \mapsto \max_{x\in\Omega} u(t, x)$ in figure 1, and the curves w.r.t. time of $t \mapsto \min_{x\in\Omega} v(t, x)$ and $t \mapsto \max_{x\in\Omega} v(t, x)$ in figure 2. As can be seen, one can conjecture that exponential convergence towards (U, V) holds in this case.

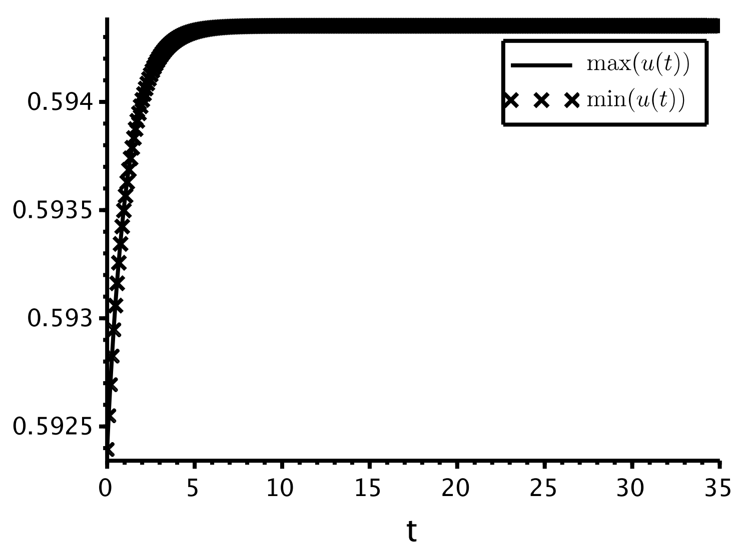

Figure 1: Time evolutions of $\min_{x\in\Omega} u(t, x)$ and $\max_{x\in\Omega} u(t, x)$. Both curves converge to the value 0.5943519.

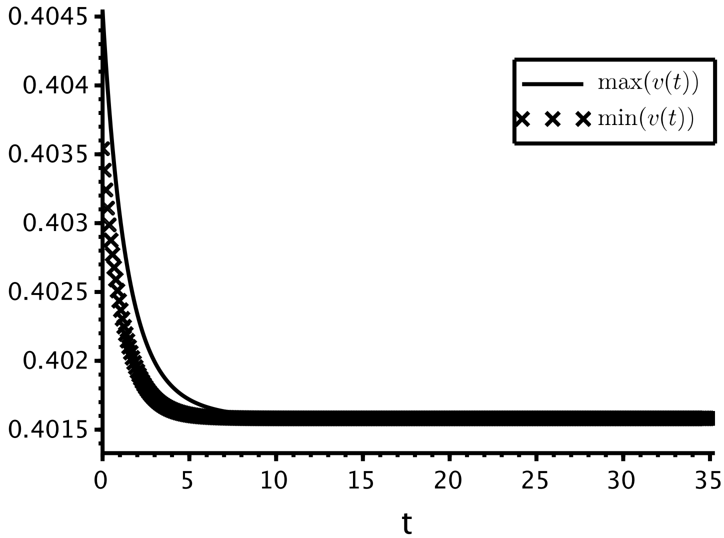

Figure 2: Time evolutions of $\min_{x\in\Omega} v(t,x)$ and $\max_{x\in\Omega} v(t,x)$. Both curves converge to the value 0.4015822.

Moreover, since in figure 1 the curves showing $\max(u(t))$ and $\min(u(t))$ are almost indistinguishable in figure 3, we show the difference $\max(u(t)) - \min(u(t))$ (this difference is in fact smaller than $1.3 \cdot 10^{-5}$).

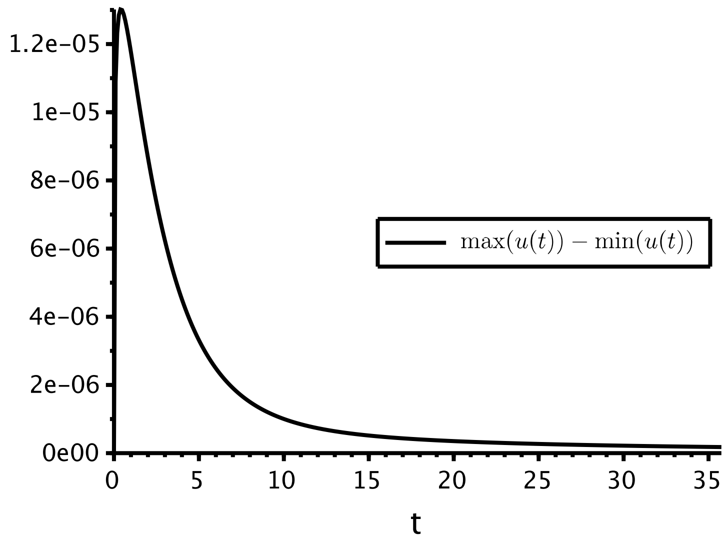

Figure 3: Time evolution of $\max_{x\in\Omega} u(t,x) - \min_{x\in\Omega} u(t,x)$.

Case 1 corresponds to a situation in which a (linearly) stable homogeneous steady state exists for the system of PDEs. The numerical simulation illustrates this fact, and suggests that this stability is also nonlinear, global, and exponential.

We now turn to Case 2. We look for the solutions of (12). We find $U \cong 0.4496058$ and $V \cong 0.5503942$. Then we check the condition (19), and see that it is not fulfilled. In fact, two others equilibria appears for V, $V_1^* \cong 0.0073564$ and $V_2^* \cong 0.7526096$, with a corresponding $U^* \cong 0.4536921$ for both case. One can check that (15) holds for these values.

We present the results obtained numerically in this case. More precisely, we present the curves w.r.t. time of $t \mapsto \min_{x\in\Omega} u(t,x)$ and $t \mapsto \max_{x\in\Omega} u(t,x)$ in figure 4 and the curves w.r.t. time of $t \mapsto \min_{x\in\Omega} v(t,x)$ and $t \mapsto \max_{x\in\Omega} v(t,x)$ in figure 5.

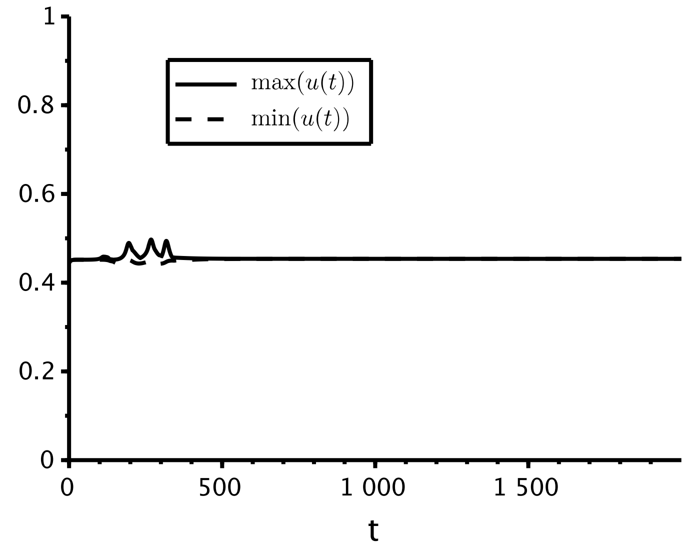

Figure 4: Time evolutions of $\min_{x\in\Omega} u(t,x)$ and $\max_{x\in\Omega} u(t,x)$. Both curves converge to the value $U^* \cong 0.4536921$.

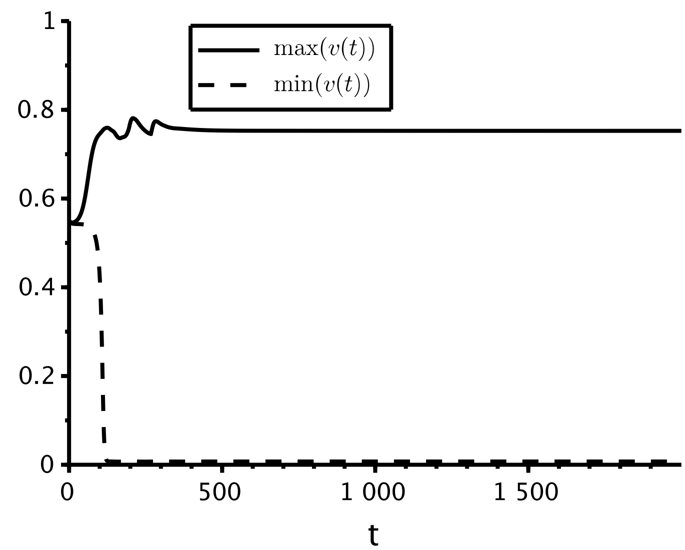

Figure 5: Time evolutions of $\min_{x\in\Omega} v(t,x)$ and $\max_{x\in\Omega} v(t,x)$. The $\min_{x\in\Omega} v(t,x)$ curve converges to the value $V_1^* \cong 0.0073564$, while the $\max_{x\in\Omega} v(t,x)$ one converges to $V_2^* \cong 0.7526096$.

We clearly see in figure 6 that a pattern appears for v when $t \to \infty$, corresponding to a state described at the end of Prop. 2, with U^*, V_1^*, V_2^* described above. Thus, the numerical simulation illustrates the theoretical results showing the linear instability of homogeneous steady states in this case. We recall, see [9] for more details, that the patterns that are observed are coherent with the experiments in which aggregation of cadherines occurs.

Figure 6: The bi-dimensional distribution $v(T, x)$ for T very large. Patterns induced by the initial data defined by (37) clearly appear.

In conclusion, the numerical tests that we present are in agreement with the linear stability study described by Propositions 2 and 3. Moreover, various numerical tests (not presented in this work) show that the kind of patterns that we get strongly depends on the initial data $(u^{in}(x), v^{in}(x))$.

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References

[1] H. Amann, Highly degenerate quasilinear parabolic systems, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 18(1) (1991), 135–166.

[2] H. Amann, Global existence for a class of highly degenerate parabolic systems, Japan J. Indust. Appl. Math., 8 (1) (1991),143–151.

[3] K. Anguige and C. Schmeiser, A one-dimensional model for cell diffusion and aggregation, incorporating volume filling and cell-to-cell adhesion, J. Math. Biol. 58 (2009), 395–427.

[4] N. Bedjaoui and P. Souplet, Critical blowup exponents for a system of reaction-diffusion equations with absorption, Zeitschrift für angewandte Mathematik und Physik ZAMP 53 (2002), 197–210.

[5] "Cell mechanics: From single scale-based models to multi-scale modeling". Editors: A. Chauviere, L. Preziosi and C. Verdier, Publisher: Taylor & Francis Group, Chapman & Hall/CRC Mathematical and Computational Biology Series, 2010.

[6] T. Colin, M.-C. Durrieu, J. Joie, Y. Lei, Y. Mammeri, C. Poignard and O. Saut, Modeling of the migration of endothelial cells on bioactive micro-patterned polymers, Mathematical Biosciences and engineering, 10(4) (2013), 997 - 1015.

[7] O. Ladyzhenskaja, V. A. Solonnikov and N. N. Ural'ceva "Linear and quasi-linear equations of parabolic type", Transl. Math. AMS, Providence R.I., 1968.

[8] M. Lambert, O. Thoumine, J. Brevier, D. Choquet, D. Riveline and R.-M. Mège, Nucleation and growth of cadherin adhesions, Exp. Cell Res., 313 (19) (2007), 4025–4040.

[9] S. Mancini, R.M. Mège, B. Sarels, P.O. Strale, A phenomenological model of cell-cell adhesion mediated by cadherins, J. Math. Biol. (2016) doi:10.1007/s00285-016-1072-7

[10] M. Marion, Finite-dimensional attractors associated with partly dissipative reaction-diffusion systems, SIAM J. Math. Anal., 20(4) (1989), 816–844.

[11] R.M. Mège, J. Gavard and M. Lambert, Regulation of cell-cell junctions by the cytoskeleton, Current Opinion in Cell Biology, 18 (5) (2006), 541–548.

[12] J.D. Murray, "Mathematical Biology I. An Introduction", vol. 17, Springer-Verlag New York, 2002.

[13] J.D. Murray, "Mathematical Biology II. Spatial Models and Biomedical Applications", vol. 18, Springer-Verlag New York, 2003.

[14] P. Quittner and P. Souplet, "Blow-up, Global Existence and Steady States", Birkhäuser Advanced Texts Basler Lehrbcher, XI, (2007).

[15] A. Rodriguez-Bernal, B. Wang, Attractors for partly dissipative reaction diffusion systems in $\mathbb{R}^n$, J. Math. Anal. Appl., 252(2) (2000), 790–803.

[16] Y. Wu, J. Vendome, L. Shapiro, A. Ben-Shaul and B. Honig, Transforming binding affinities from three dimensions to two with application to cadherin clustering, Nature, 475 (2011), 510–513.

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