DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement Volume 2007

Website: www.aimSciences.org

pp. **000–000**

ENTROPY METHODS FOR REACTION-DIFFUSION SYSTEMS: DEGENERATE DIFFUSION

Laurent Desvillettes

CMLA - ENS de Cachan
61 Av. du Pdt. Wilson
94235 Cachan Cedex, France

Klemens Fellner

Faculty for Mathematics, University of Vienna
Nordbergstr. 15
1090 Wien, Austria

Abstract. In the continuation of [DF05], we study reversible reaction-diffusion systems via entropy methods (based on the free energy functional). We show for a particular model problem with two reacting species in which one of the diffusion constants vanishes that the solutions decay exponentially with explicit rate and constant towards the unique constant equilibrium state.

1. **Introduction.** This work contributes to a quantitative analysis of the large-time behaviour of diffusive and reversible chemical reactions (on bounded domains $\Omega \subset \mathbb{R}^N$, $N \geq 1$) of the type

$$\alpha_1 \mathcal{A}_1 + \cdots + \alpha_q \mathcal{A}_q \rightleftharpoons \beta_1 \mathcal{A}_1 + \cdots + \beta_q \mathcal{A}_q \qquad \alpha_i, \beta_i \in \mathbb{Z}_+ .$$

Our method quantifies the decay of an entropy functional (here the physical free-energy) in terms of a relative entropy (with respect to the entropy minimising equilibrium state). This method, frequently called entropy method, is an alternative to the linearisation around the equilibrium, and has the advantage of being quite robust. This is due to the fact that it mainly relies on functional inequalities which have no direct link with the original PDE. We refer to [DF05, DF06] for comments and links to the vast literature on reaction-diffusion systems e.g. [Zel, CHS, Mas, Rothe, Ama, Web, Mor, FMS, HY88, HY94, FHM, MP] and for references concerning the entropy method e.g. [Grö, GGH, GH, CJMTU, DV01, DV05, FNS].

Here in particular, we study a nonlinear model problem consisting of two diffusive chemicals reacting like $2\,\mathcal{A} \rightleftharpoons \mathcal{B}$ (according to the principle of mass action kinetics) within a smoothly bounded domain Ω, and subject to homogeneous Neumann boundary conditions: Denoting by a, b the concentrations of $\mathcal{A}$ and $\mathcal{B}$, our scaled model system (see [DF05]) reads (after adimensionalizing) as

$$\left.\begin{array}{rcl} \partial_t a - d_a \triangle_x a &=& -2(a^2 - b) \\ \partial_t b - d_b \triangle_x b &=& a^2 - b \end{array}\right\} \qquad x \in \Omega\,,\ |\Omega| = 1\,, \tag{1}$$

$$n(x)\cdot\nabla_x a = 0\,, \quad n(x)\cdot\nabla_x b = 0 \qquad x \in \partial\Omega\,. \tag{2}$$

2000 *Mathematics Subject Classification.* 35B40, 35K57.

Key words and phrases. Reaction-Diffusion, Entropy Method, Exponential Decay, Degenerate Diffusion.

The diffusion rates $d_a \geq 0$, $d_b \geq 0$ are nonnegative constants.

As initial data, we take two nonnegative $L^\infty(\Omega)$ functions $a_0(x) \geq 0$, $b_0(x) \geq 0$. The flow of equations (1) – (2) conserves the total mass (assumed positive)

$$0 < M = \int_\Omega (a(t,x) + 2b(t,x))\,dx = \int_\Omega (a_0(x) + 2b_0(x))\,dx\,. \tag{3}$$

Under the assumption that either $d_a > 0$ or $d_b > 0$, the conservation law (3) determines the unique positive equilibrium state (a_∞, b_∞) as the nonnegative constants satisfying $a_\infty + 2b_\infty = M$ and $a_\infty^2 = b_\infty$, i.e.

$$a_\infty = (-1 + \sqrt{1+8M})/4\,, \qquad b_\infty = (M - a_\infty)/2\,. \tag{4}$$

Global existence of a unique classical solution to (1) – (3) is well known and is a consequence of the following L^∞-bound : for all $t \geq 0$ and $x \in \Omega$,

$$0 \leq a(t,x) \leq L\,, \quad 0 \leq b(t,x) \leq \frac{L}{2}\,, \qquad L = ||a_0||_\infty + 2||b_0||_\infty\,. \tag{5}$$

This is shown either by the maximum principle applied to the single equations (see e.g. [Kir]) or by comparison with the diffusionless system (see e.g. [BH]).

We consider as entropy functional the physical free energy of (1) – (2):

$$E(a,b) = \int_\Omega \left(a\,(\ln a - 1) + b\,(\ln b - 1)\right)\,dx\,. \tag{6}$$

Assuming positive diffusion rates, we have shown in [DF05, theorem 1.1]:

Proposition 1. *Let Ω be a smoothly bounded domain in $\mathbb{R}^N$ ($N \geq 1$), and $d_a > 0$, $d_b > 0$. Let the initial data a_0, $b_0 \in L^\infty(\Omega)$ be nonnegative functions with positive mass $\int_\Omega (a_0 + 2b_0)\,dx = M > 0$ and denote $L = ||a_0||_\infty + 2||b_0||_\infty$. Then, the unique nonnegative solution in $L^\infty(\mathbb{R}_+ \times \Omega)^2$ of equations (1) – (3) decays exponentially in L^1 toward the equilibrium state (4) :*

$$\begin{aligned} &\frac{1}{2}\,||a(t,\cdot) - a_\infty||^2_{L^1(\Omega)} + ||b(t,\cdot) - b_\infty||^2_{L^1(\Omega)} \\ &\leq C_1\,M\,(E(a_0,b_0) - E(a_\infty,b_\infty))\,e^{-\frac{4t}{C_2(L,M)}\min\left\{1,\frac{d_a}{P(\Omega)C_3(M,d_a/d_b)}\right\}}, \end{aligned} \tag{7}$$

where $P(\Omega)$ denotes the Poincaré constant of Ω, $C_2(L,M)$ is stated explicitly in (13), $C_3(M, d_a/d_b)$ in the appendix, and $C_1 \approx 1.51$.

Remark 1. *In fact, exponential decay in all Sobolev norms follows from the smoothing properties of the heat kernel after interpolating the exponential L^1 decay with polynomially growing in time estimates of a and b in H^s with s large enough (see* [DF06]*).*

The aim of this work is to show exponential decay to equilibrium state (4) in the cases of degenerate diffusion rates $d_a > 0, d_b = 0$ or $d_a = 0, d_b > 0$. We obtain this result by deriving functional inequalities (lemma 3 and lemma 4) which quantify how the reaction term transfers diffusion from one specie to the other.

We remark that these functional inequalities are not linked to the considered particular system, and are thus expected to be applicable for more general systems. It is for the sake of clarity that we present them in the context of the particular system (1)–(2).

We prove the two following theorems:

Theorem 1 (Degenerated Diffusion $d_b = 0$)**.** *Let the assumptions of proposition 1 hold except that $d_b = 0$. Then,*

$$\begin{aligned}&\frac{1}{2}\,\|a(t,\cdot) - a_\infty\|^2_{L^1(\Omega)} + \|b(t,\cdot) - b_\infty\|^2_{L^1(\Omega)}\\ &\leq C_1\, M\,(E(a_0,b_0) - E(a_\infty,b_\infty))\,e^{-\frac{C_5(d_a,L,P)}{C_2(L,M)}\,t}\end{aligned} \tag{8}$$

where $C_1 \approx 1.51$, C_2 is stated in (13) and C_5 is estimated explicitly in the appendix.

Theorem 2 (Degenerated Diffusion $d_a = 0$)**.** *Let the assumptions of proposition 1 hold except that $d_a = 0$. Then,*

$$\begin{aligned}&\frac{1}{2}\,\|a(t,\cdot) - a_\infty\|^2_{L^1(\Omega)} + \|b(t,\cdot) - b_\infty\|^2_{L^1(\Omega)}\\ &\leq C_1\, M\,(E(a_0,b_0) - E(a_\infty,b_\infty))\,e^{-\frac{C_6(d_b/P(\Omega),M)}{C_2(L,M)}\,t}\end{aligned} \tag{9}$$

where $C_1 \approx 1.51$, C_2 is stated in (13) and C_6 is estimated explicitly in the appendix.

Notation: We shall indifferently denote A_1 for A and A_2 for B whenever it is convenient and use capital letters for writing the square roots, i.e. $A_i = \sqrt{a_i}$, $i = 1, 2$.

2. **Sketch of the proof of proposition 1.**

We recall here briefly the proof of proposition 1, detailed in [DF05].

Proof of proposition 1. For (1)–(2), we calculate readily that

$$\frac{d}{dt}(E(a(t),b(t)) - E(a_\infty,b_\infty)) = -\,D(a(t),b(t))\,, \tag{10}$$

with $D(a,b)$ denoting the nonnegative entropy dissipation functional

$$D(a,b) = d_a \int_\Omega \frac{|\nabla_x a|^2}{a}\,dx + d_b \int_\Omega \frac{|\nabla_x b|^2}{b}\,dx + \int_\Omega (a^2 - b)\ln\frac{a^2}{b}\,dx\,. \tag{11}$$

Then, the following entropy entropy-dissipation lemma holds

Lemma 1. *For all (measurable) functions $a, b : \Omega \to \mathbb{R}$, which satisfy $0 \leq a \leq L$, $0 \leq b \leq \frac{L}{2}$, and $\int_\Omega (a + 2b) = M$,*

$$D(a,b) \geq \frac{4}{C_2(L,M)}\min\left\{1, \frac{d_a}{P(\Omega)\,C_3(M,d_a/d_b)}\right\}(E(a,b) - E(a_\infty,b_\infty)), \tag{12}$$

where $P(\Omega)$ is the Poincaré constant of Ω, the equilibrium states a_∞, b_∞ are given by (4), and the constant $C_2(L,M)$ is defined as follows: we introduce the function $\Phi(x,y) = (x\,(\ln(x) - \ln(y)) - (x-y))/(\sqrt{x} - \sqrt{y})^2$, which is continuous on $(0,\infty)^2$ with $\Phi(x,x) = 2$. [DF05, lemma 2.1]. *Then*

$$C_2(L,M) = \max\left\{\frac{\Phi(L,a_\infty)}{a_\infty}, \Phi\left(\frac{L}{2}, b_\infty\right)\right\}\,. \tag{13}$$

Moreover, $C_3(M, d_a/d_b) = C_3(\gamma)$ with $C_3(\gamma)$ as define in (17) and for γ chosen below in (18)*. $C_3(M, d_a/d_b)$ is stated explicitly in (35) in the appendix.*

Proof of lemma 1. The proof is outlined in three steps : Firstly, we estimate below the entropy dissipation $D(a,b)$ using the identity $|\nabla_x a_i|^2/a_i = 4\,|\nabla_x A_i|^2$ (with $A_i = \sqrt{a_i}$, $i=1,2$) and Poincaré's inequality as well as the inequality $(a-b)(\ln(a)-\ln(b)) \geq 4(A-B)^2$. We obtain in this way the estimate:

$$D(a,b) \geq 4\left\|A^2 - B\right\|_2^2 + \frac{4d_a}{P(\Omega)}\left\|A-\overline{A}\right\|_2^2 + \frac{4d_b}{P(\Omega)}\left\|B-\overline{B}\right\|_2^2, \tag{14}$$

where $\overline{A}(t) = \int_\Omega A(t,x)\,dx$ and $\overline{B}(t) = \int_\Omega B(t,x)\,dx$ (recall that $|\Omega| = 1$).

Secondly, the continuity of the function $\Phi(x,y)$ (see [DF05, lemma 2.1]) and the L^∞ bound (5) imply an upper bound for the relative entropy

$$E(a,b) - E(a_\infty, b_\infty) \leq C_2(L,M)\left(A_\infty^2\,\|A - A_\infty\|_2^2 + \|B - B_\infty\|_2^2\right), \tag{15}$$

with $C_2(L,M)$ given in (13). Finally, using crucially the conservation of mass (3), equation (15) can be estimated further as (see [DF05, lemma 2.2])

$$\frac{1}{C_2}\left(E(a,b) - E(a_\infty,b_\infty)\right) \leq \|A^2 - B\|_2^2 + C_3\|A - \overline{A}\|_2^2 + C_4\|B - \overline{B}\|_2^2, \tag{16}$$

with the constants (for some $\gamma > 0$ to be chosen)

$$C_3(\gamma) = A_\infty\gamma, \qquad C_4(\gamma) = 4B_\infty + 1 + \frac{A_\infty}{\gamma}. \tag{17}$$

Finally, we choose γ such that $C_3/C_4 = d_a/d_b$ which matches (16) to (14), i.e. we set

$$\gamma = \frac{d_a}{d_b}\left(2A_\infty + \frac{1}{2A_\infty}\right) + \sqrt{\frac{d_a^2}{d_b^2}\left(2A_\infty + \frac{1}{2A_\infty}\right)^2 + \frac{d_a}{d_b}}, \tag{18}$$

and the constant $C_3(M, d_a/d_b)$ (see (35) in the appendix) as used in the lemma is $C_3(\gamma)$ with (18). $\square$

Then, lemma 1 together with the entropy relation (10) implies exponential convergence of the relative entropy, i.e.

$$E(a(t),b(t)) - E(a_\infty,b_\infty) \leq \left(E(a_0,b_0) - E(a_\infty,b_\infty)\right) e^{-\frac{4t}{C_2}\min\{1,\frac{d_a}{P(\Omega)C_3}\}}. \tag{19}$$

Finally, the statement of proposition 1 is a consequence the following analog to the Cziszar-Kullback-Pinsker inequality ([Csi] and [Kul]) in information theory (which we do not prove here):

Lemma 2. *For all (measurable) functions $a, b : \Omega \to \mathbb{R}$ such that $0 \leq a$, $0 \leq b$ and $\int_\Omega (a + 2b) = M$,*

$$C_1\,M\left(E(a,b) - E(a_\infty,b_\infty)\right) \geq \left(\frac{1}{2}\,\|a - a_\infty\|_1^2 + \|b - b_\infty\|_1^2\right), \tag{20}$$

where $C_1 = (6+2\sqrt{2})/(3+2\sqrt{2}) \approx 1.51$, and a_∞, b_∞ are defined in (4).

This completes the sketch of the proof of proposition 1. $\square$

Obviously the above argument fails if either $d_a = 0$ or $d_b = 0$, since the right-hand side of (14) lacks a term to match (16). In the following, we fix that by deriving functional inequalities, which - colloquially speaking - produce spatial diffusion in terms of $\|A_i - \overline{A_i}\|_2^2$, $i = 1,2$ lacking in (14) at the costs of the reaction dissipation term $\|A^2 - B\|_2^2$ and the spatial diffusion $\|A_j - \overline{A_j}\|_2^2$, $j \neq i$ present in (14).

In the sequel, we first consider the case $d_b = 0$ and then the case $d_a = 0$. For the first case, we show a short proof using the L^∞-bounds (5) available for this model. But since L^∞-bounds are not available in general, we show for the second case a proof using only the conservation of mass (3). This proof is somewhat lengthy but uses only elementary tools.

3. **Proof of theorem 1.** We begin by proving the

Lemma 3. *Let A, B denote the square roots of (measurable) nonnegative functions a, b satisfying the global bound (5). Then,*

$$C_7\|A^2 - B\|_2^2 + C_8\|A - \overline{A}\|_2^2 \geq \|B - \overline{B}\|_2^2 \tag{21}$$

for all $\theta \in [0,1)$ and with the constants

$$C_7(\theta) = \frac{2}{1-\theta^2}\,, \qquad C_8(L,\theta) = \frac{16L}{(1+\theta)^2}\,. \tag{22}$$

Proof of lemma 3. For $\theta \in [0,1)$, we expand $\|A^2 - B\|_2^2$ as

$$\begin{aligned}\|A^2 - \overline{B} + \overline{B} - B\|_2^2 &= \|A^2 - \overline{B}\|_2^2 + 2\theta \int_\Omega (A^2 - \overline{B})(\overline{B} - B)\,dx \\ &\quad +2(1-\theta)\int_\Omega (A^2 - \overline{A}^2)(\overline{B} - B)\,dx + \|\overline{B} - B\|_2^2\,,\end{aligned}$$

where we have used that $\int_\Omega \overline{B}(\overline{B} - B)\,dx = \int_\Omega \overline{A}^2(\overline{B} - B)\,dx = 0$. Estimating both integrals with Young's inequality (i.e. $2xy \geq -\gamma^{-1}x^2 - \gamma y^2$ with $\gamma = \theta$ for the first integral, and $\gamma = (1+\theta)/2$ for the second), and comparing the coefficients with (21), the following inequalities have to be satisfied:

$$C_8\|\overline{A} - A\|_2^2 - \frac{2C_7(1-\theta)}{1+\theta}\|A^2 - \overline{A}^2\|_2^2 \geq 0\,, \qquad C_7(-\theta^2 - (1-\theta)\frac{1+\theta}{2} + 1) \geq 1\,.$$

Using the global bound (5) to obtain $\|A^2 - \overline{A}^2\|_2^2 \leq 4L\|\overline{A} - A\|_2^2$, we are led to use the constants (22). □

Continuation of the proof of theorem 1. Let assume firstly that $\frac{d_a}{P} \leq 8L$, which means that (for $\theta = \frac{8LP - d_a}{8LP + d_a}$ and therefore $C_7 = \frac{(8LP + d_a)^2}{(8LP)^2}$) the fraction C_7/C_8 of the constants (22) matches with the coefficients of

$$D \geq 4\|A^2 - B\|_2^2 + 4\frac{d_a}{P}\|A - \overline{A}\|_2^2\,.$$

Then, we may estimate (for all $\mu \in (0,1)$)

$$\begin{aligned} D &\geq 4\mu\|A^2 - B\|_2^2 + 4\mu\frac{d_a}{P}\|A - \overline{A}\|_2^2 + 4(1-\mu)\frac{(8LP)^2}{(8LP + d_a)^2}\|B - \overline{B}\|_2^2 \\ &\geq \frac{C_5}{C_2}(E - E_\infty)\,.\end{aligned}$$

In the second case $\frac{d_a}{P} \geq 8L$, we estimate instead as follows :

$$\begin{aligned} D &\geq 4\mu\|A^2 - B\|_2^2 + 4(\frac{d_a}{P} - (1-\mu)8L)\|A - \overline{A}\|_2^2 \\ &\quad +4(1-\mu)\|A^2 - B\|_2^2 + 4(1-\mu)8L\|A - \overline{A}\|_2^2\,,\end{aligned}$$

and the second line is bounded below by $2(1-\mu)\|B-\overline{B}\|_2^2$ via lemma 3 (with $\theta=0$), which leads to the constant (36) in this case.

In order to conclude, it remains to apply Gronwall's lemma and the Cziszar-Kullback inequality. ☐

4. **Proof of theorem 2.** We now consider the case $d_a = 0$:

Lemma 4. *Let A, B denote the square roots of (measurable) nonnegative functions a, b satisfying the conservation law (3). Then, for all $\eta < 1$,*

$$\frac{1}{1-\eta}C_9\|A^2-B\|_2^2 + \frac{\eta+1}{\eta}C_9\|B-\overline{B}\|_2^2 \geq \|A-\overline{A}\|_2^2\,, \tag{23}$$

with the constant $C_9(M)$ defined by (here, one can take any $\nu > 1$ in order to optimise this constant):

$$C_9(M) = \max\left\{\frac{\nu}{\nu-1}, \frac{2}{M}\frac{\nu}{\nu-1}, 2\nu^2 M + \nu - \frac{1}{2}, \frac{2}{M}\left(4\nu^2+\nu\right) - \frac{1}{2}\right\}. \tag{24}$$

Proof of lemma 4. The proof deals with three cases: 1) $\overline{B}$ is "big" 2a) $\overline{B}$ is "small" and $\overline{A^2}$ is "small", 2b) $\overline{B}$ is "small" and $\overline{A^2}$ is "big".

1) For $\overline{B}$ "big", we apply the ansatz (pointwise for all $x \in \Omega$)

$$A^2(x) = \overline{B}(1+\delta(x))\,, \qquad \delta(x)\in[-1,\infty)\,, \quad \forall x\in\Omega\,. \tag{25}$$

Then, after expanding $\|A^2-B\|_2^2$ as

$$\|\overline{B}-B+\overline{B}\delta\|_2^2 = \|B-\overline{B}\|_2^2 + 2\overline{B}\int_\Omega \delta(\overline{B}-B)\,dx + \overline{B}^2\overline{\delta^2}\,,$$

and using Young's inequality for the above integral:

$$2\overline{B}\int_\Omega \delta(\overline{B}-B)\,dx \geq -\left(1+\frac{1-\eta^2}{\eta}\right)\|B-\overline{B}\|_2^2 - \overline{B}^2\frac{\eta}{1+\eta-\eta^2}\overline{\delta^2}\,,$$

we get for the left hand side of (23) the following estimate

$$\begin{aligned}\tfrac{1}{1-\eta}C_9\left(\|B-\overline{B}\|_2^2 + 2\overline{B}\textstyle\int_\Omega \delta(\overline{B}-B)\,dx + \overline{B}^2\overline{\delta^2}\right) + \tfrac{\eta+1}{\eta}C_9\|B-\overline{B}\|_2^2 \\ \geq C_9\tfrac{1+\eta}{1+\eta-\eta^2}\overline{B}^2\overline{\delta^2} \geq C_9\overline{B}^2\overline{\delta^2}\,,\end{aligned} \tag{26}$$

since $(1+\eta)/(1+\eta-\eta^2) > 1$. Next, for the right hand side of (23), we Taylor-expand $A=\sqrt{\overline{B}}\sqrt{1+\delta}$ for all $x\in\Omega$ as

$$\sqrt{1+\delta} = 1 + \frac{\delta}{2} - \frac{R(\zeta)}{4}\frac{\delta^2}{2}\,, \quad R(\zeta) = \frac{1}{\sqrt{1+\zeta}^3}\,, \quad \zeta(x)\in[0,\delta(x)]\,, \tag{27}$$

and observe that $R(\zeta(\delta))$ is monotone decreasing on $\delta\in[-1,\infty)$ with $R(-1)=4$ and $R(\infty)=0$. We obtain therefore for the right hand side of (23) that

$$\begin{aligned}\overline{A^2} - \overline{A}^2 &= \overline{B}(1+\overline{\delta}) - \overline{B}\left(1+\overline{\delta}+\frac{\overline{\delta}^2}{4} - \frac{1}{4}\overline{\delta^2 R} - \frac{1}{8}\overline{\delta}\,\overline{\delta^2 R} + \frac{1}{64}\overline{\delta^2 R}^2\right) \\ &\leq \frac{\overline{B}}{4}\overline{\delta^2 R} + \frac{\overline{B}}{8}\overline{\delta}\,\overline{\delta^2 R} \;\leq\; \overline{B}\,\overline{\delta^2} + \frac{\overline{B}\,\overline{\delta}}{2}\overline{\delta^2} \leq \overline{\delta^2}\left(\overline{B}+\frac{\overline{B}\,\overline{\delta}}{2}\right) \\ &\leq \overline{\delta^2}\frac{M-\overline{B}^2+\overline{B}}{2}\,,\end{aligned} \tag{28}$$

since $R \leq 4$ and since $\overline{B}\,\delta = \overline{A^2} - \overline{B} = M - \overline{B^2} - \overline{B} \leq M - \overline{B}^2 - \overline{B}$ by Jensen's inequality. Finally, by (26) and (28), equation (23) holds for

$$C_9 \geq \frac{1}{2}\frac{M - \overline{B}^2 + \overline{B}}{\overline{B}^2}\,, \tag{29}$$

and we have to find a different way of estimating in the case that $\overline{B}$ is small.

2) As a preliminary step, we see that

$$\begin{aligned} \|A^2 - B\|_2^2 &= \|A^2 - \overline{B}\|_2^2 + 2\int_\Omega (A^2 - \overline{B})(\overline{B} - B)\,dx + \|B - \overline{B}\|_2^2 \\ &\geq (1-\eta)\|A^2 - \overline{B}\|_2^2 - \left(\frac{1}{\eta} - 1\right)\|B - \overline{B}\|_2^2 \end{aligned}$$

by Young's inequality, and, hence, that it is sufficient to show that

$$C_9\|A^2 - \overline{B}\|_2^2 + C_9\|B - \overline{B}\|_2^2 \geq \|A - \overline{A}\|_2^2\,. \tag{30}$$

Expanding and estimating further by Jensen's inequality ($\overline{A^4} \geq \overline{A^2}^2$), it is sufficient that

$$C_9 \geq \frac{\overline{A^2}}{\overline{A^2}^2 - 2\overline{B}\,\overline{A^2} + \overline{B^2}}\,. \tag{31}$$

2a) Let us firstly assume that $\overline{A^2} \leq M/2$. Then, in (31), we may neglect the quadratic term $\overline{A^2}^2 \geq 0$ and use the conservation law $\overline{B^2} = M - \overline{A^2}$ to estimate (assuming $\overline{B} \leq 1/2\nu$)

$$\frac{\overline{A^2}}{-2\overline{B}\,\overline{A^2} + M - \overline{A^2}} \leq \frac{1}{-1/\nu + M/\overline{A^2} - 1} \leq \frac{1}{-1/\nu + 1} \leq \frac{\nu}{\nu - 1} \qquad \text{for } \overline{B} \leq 1/2\nu\,. \tag{32}$$

2b) In the case when $\overline{A^2} \geq M/2$, we neglect $\overline{B^2} \geq 0$ and estimate (assuming $2\overline{B} \leq M/2\nu \leq \overline{A^2}/\nu$)

$$\frac{1}{\overline{A^2} - 2\overline{B}} \leq \frac{1}{\overline{A^2}\,(1 - 1/\nu)} \leq \frac{2}{M}\frac{\nu}{\nu - 1} \qquad \text{for} \quad \overline{B} \leq M/4\nu\,. \tag{33}$$

Combining the cases, we obtain from (32) and (33) the first two contributions for the constant C_9 (24), which covers the case $\overline{B} \leq \min\{1/2\nu, M/4\nu\}$. In the other case, inserting $\overline{B} \geq \min\{1/2\nu, M/4\nu\}$ into (29) leads to the last two terms in (24).

This concludes the proof of lemma 4. □

Continuation of the proof of theorem 2. By lemma 4, we have

$$\|A^2 - B\|_2^2 + \frac{1-\eta^2}{\eta}\|B - \overline{B}\|_2^2 \geq \frac{1-\eta}{C_9}\|A - \overline{A}\|_2^2\,.$$

Then, if we chose η_1 such that $\frac{d_b}{P} = \frac{1-\eta_1^2}{\eta_1}$, i.e.

$$\eta_1 = -\frac{d_b}{2P} + \sqrt{\frac{d_b^2}{4P^2} + 1}\,, \tag{34}$$

we obtain, for any $\mu \in (0,1)$,

$$
\begin{aligned}
D &\geq 4\mu \|A^2 - B\|_2^2 + 4(1-\mu)\frac{1-\eta_1}{C_9}\|A - \overline{A}\|_2^2 + 4\mu\frac{d_b}{P}\|B - \overline{B}\|_2^2 \\
&\geq C_6\left(\|A^2 - B\|_2^2 + C_3\|A - \overline{A}\|_2^2 + C_4\|B - \overline{B}\|_2^2\right) ,
\end{aligned}
$$

and estimate (15) completes the proof.

Once again, it remains to use Gronwall's lemma and the Cziszar-Kullback inequality. ◻

5. **Appendix: explicit formulas for the decay rates.**

In favor of the readability of the theorems, we collect here the explicit expressions of complex constants.

5.1. **add proposition 1.** The constant $C_3(M, d_a/d_b) = C_3(\gamma)$ as defined in (17) with γ given in (18) and using (4), i.e.

$$
C_3(M, d_a/d_b) = \frac{d_a}{d_b}\frac{\sqrt{1+8M}}{2} + \sqrt{\frac{d_a^2}{d_b^2}\frac{1+8M}{4} + \frac{d_a}{d_b}\frac{\sqrt{1+8M}-1}{4}} . \tag{35}
$$

5.2. **add theorem 1.** The constant $C_5(d_a, L, P(\Omega))$ is defined for all $\mu \in (0,1)$ and $\gamma \in (0,\infty)$ as

$$
C_5(d_a, L, P(\Omega)) = \begin{cases} \min\left\{4\mu, \frac{4\mu d_a}{C_3(\gamma)P(\Omega)}, \frac{(8LP(\Omega))^2}{(8LP(\Omega)+d_a)^2}\frac{4(1-\mu)}{C_4(\gamma)}\right\} & \frac{da}{P(\Omega)} \leq 8L , \\ \min\left\{4\mu, \frac{4(\frac{d_a}{P(\Omega)} - (1-\mu)8L}{C_3(\gamma)}, \frac{2(1-\mu)}{C_4(\gamma)}\right\} & \frac{da}{P(\Omega)} > 8L , \end{cases} \tag{36}
$$

where the constants $C_3(\gamma)$ and $C_4(\gamma)$ are defined in (17) and $C_1 \approx 1.51$. Here, $\gamma \in (0,\infty)$ and $\mu \in (0,1)$ can be chosen in order to maximise C_5.

5.3. **add theorem 2.** The constant $C_6(d_b/P(\Omega), M)$ is defined for all $\mu \in (0,1)$, $\gamma \in (0,\infty)$ and $\nu \in (1,\infty)$ as

$$
C_6(d_b/P(\Omega), M) = \min\left\{4\mu, 4(1-\mu)\frac{1-\eta_1}{C_3(\gamma)}\frac{1}{C_9(\nu)}, 4\mu\frac{d_b}{C_4(\gamma)P(\Omega)}\right\} , \tag{37}
$$

where $\eta_1(d_b/P(\Omega))$ is the constant given in (34) and $C_1 \approx 1.51$. Here, $\mu \in (0,1)$, $\gamma \in (0,\infty)$ in $C_3(\gamma)$ and $C_4(\gamma)$ defined in (17), and $\nu \in (1,\infty)$ in $C_9(\nu)$ defined in (24) can be chosen in order to maximise C_6.

Acknowledgements. This work has been supported by the European IHP network "HYKE-HYperbolic and Kinetic Equations: Asymptotics, Numerics, Analysis", Contract Number: HPRN-CT-2002-00282. K.F. is partially supported by the WWTF project "How do Cells move" and by the Wittgenstein Award 2000 of Peter A. Markowich

REFERENCES

[Ama] H. Amann, *Global existence for semilinear parabolic problems.* J. Reine Angew. Math. **360**, (1985), pp. 47–83.

[BH] D. Bothe, D. Hilhorst, *A reaction-diffusion system with fast reversible reaction.* J. Math. Anal. Appl., **286**, no. 1 (2003), pp. 125–135.

[CJMTU] J. A. Carrillo, A. Jüngel, P. Markowich, G. Toscani, G., A. Unterreiter. *Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities.* Monatsh. Math. *133* (2001), no. 1, pp. 1–82.

[CHS] E. Conway, D. Hoff, J. Smoller, *Large time behaviour of solutions of systems of nonlinear reaction-diffusion equations.* SIAM J. Appl. Math. **35**, no. 1 (1978), pp. 1–16.

[Csi] I. Csiszár, *Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis von Markoffschen Ketten.* Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), pp. 85–108.

[DF05] L. Desvillettes, K. Fellner, *Exponential Decay toward Equilibrium via Entropy Methods for Reaction-Diffusion Equations*, J. Math. Anal. Appl. **319**, (2006), pp. 157–176.

[DF06] L. Desvillettes, K. Fellner, *Entropy methods for Reaction-Diffusion Equations: Slowly Growing A-priori bounds*, preprint 2005.

[DV01] L. Desvillettes, C. Villani, *On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation.* Comm. Pure Appl. Math. **54**, no. 1 (2001), pp. 1–42.

[DV05] L. Desvillettes, C. Villani, *On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation.* Inventiones Mathematicae. **159**, no. 2, (2005) pp. 245–316.

[FNS] K. Fellner, L. Neumann, C. Schmeiser, *Convergence to global equilibrium for spatially inhomogeneous kinetic models of non-micro-reversible processes*, Monatsh. Math. Vol. **141**, no. 4 (2004), pp. 289–299.

[FMS] W. E. Fitzgibbon, J. Morgan, R. Sanders, *Global existence and boundedness for a class of inhomogeneous semilinear parabolic systems.* Nonlinear Analysis, Theory, Methods & application, **19**, no.9 (1992), pp. 885-899.

[FHM] W. Fitzgibbon, S. Hollis, J. Morgan, *Stability and Lyapunov functions for reaction-diffusion systems.* SIAM J. Math. Anal. **28** (1997), no. 3, pp. 595–610.

[Grö] K. Gröger, *Free energy estimates and asymptotic behaviour of reaction-diffusion processes.* Preprint 20, Institut für Angewandte Analysis und Stochastik, Berlin, 1992.

[GGH] A. Glitzky, K. Gröger, R. Hünlich, *Free energy and dissipation rate for reaction-diffusion processes of electrically charged species.* Appl. Anal. **60**, no. 3-4 (1996), pp. 201–217.

[GH] A. Glitzky, R. Hünlich, *Energetic estimates and asymptotics for electro-reaction-diffusion systems.* Z. Angew. Math. Mech. **77** (1997), pp. 823–832.

[HY88] A. Haraux, A. Youkana, *On a result of K. Masuda concerning reaction-diffusion equations.* Tôhoku Math. J. **4** (1988), pp. 159–163.

[HMP] S. Hollis, R. Martin, M. Pierre, *Global existence and boundedness in reaction-diffusion systems.* SIAM J. Math. Anal. **18** (1987), pp. 744–761.

[HY94] H. Hoshino, Y. Yamada, *Asymptotic behavior of global solutions for some reaction-diffusion systems.* Nonlinear Anal. **23** (1994), no. 5, pp. 639–650.

[Mas] K. Masuda, *On the global existence and asymptotic behavior of solution of reaction-diffusion equations.* Hokkaido Math. J. **12**, (1983), pp. 360–370.

[MP] R. H. Martin, M. Pierre, *Nonlinear Reaction-Diffusion Systems.* in Nonlinear Equations in Applied Sciences, W. F. Ames and C. Rogers , eds., Math. Sci. Engrg. 185, Academic Press, New York (1991).

[Mor] J. Morgan, *Global existence for semilinear parabolic systems.* SIAM J. Math. Ana. **20**, (1989), pp. 1128–1144.

[KK] Y. I. Kanel, M. Kirane, *Global solutions of Reaction-Diffusion Systems with a Balance Law and Nonlinearities of Exponential Growth.* J. Diff. Equ. **165**, (2000), pp. 24–41.

[Kir] M. Kirane, *On stabilization of solutions of the system of parabolic differential equations describing the kinetics of an autocatalytic reversible chemical reaction.* Bull. Inst. Mat. Acad. Sin. **18**, no. 4 (1990), pp. 369–377.

[Kul] S. Kullback, Information Theory and Statistics, John Wiley, (1959).

[Rothe] F. Rothe, Global Solutions of Reaction-Diffusion Systems. Lecture Notes in Mathematics, Springer, Berlin, (1984).

[Web] G. F. Webb, Theory of nonlinear agedependent population dynamics. Monographs and Textbooks in Pure and Applied Mathematics, 89. Marcel Dekker, Inc., New York, 1985.

[Zel] T. Zelenyak, *Stabilization of solutions to boundary value pronlems for second order parabolic equations in one space variable.* Differentsial'nye Uravneniya **4** (1968), pp. 34–45.

E-mail address: desville@cmla.ens-cachan.fr; klemens.fellner@univie.ac.at