ON ASYMPTOTICS OF THE BOLTZMANN EQUATION WHEN THE COLLISIONS BECOME GRAZING

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Abstract

We deal in this work with an asymptotics of the Boltzmann equation leading to the Fokker–Planck–Landau equation. We pr ove its mathematical validity in the context of linearized equations and give an extension to the Kac equation.

1 Introduction

The dynamics of a rarefied monoatomic gas is usually described by the Boltzmann equation,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f), \tag{1}$$

where $f(t, x, v)$ is the density of particles which at time t and point x move with velocity v, and Q is a quadratic collision kernel taking in account any collisions preserving momentum and kinetic energy.

When almost all collisions are grazing (i.e., when the difference between velocities before and after all collisions is very small), phenomenological arguments intoduced by Landau in [Li, Pi] or by Chapman and Cowling in [Ch, Co] ensure that the solution of 1 tends to the solution of the Fokker–Planck–Landau equation,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = P(f, f), \tag{2}$$

where

$$P(f,f) = \nabla_v \cdot \int_{v_1 \in I\!R^3} \Gamma(|v - v_1|) \{ I - \frac{(v - v_1) \otimes (v - v_1)}{|v - v_1|^2} \}$$

$$\{ f(v_1) \nabla_v f(v) - f(v) \nabla_{v_1} f(v_1) \} \, dv_1, \qquad (3)$$

and Γ is a nonnegative function depending only on the form of Q.

In section 2, we shall introduce an asymptotics of 1 leading to 2. Moreover, we shall compute the function Γ of 3 in some simple cases.

Section 3 is devoted to the mathematical proof of the above asymptotics, within the context of linearized equations.

Finally, we extend the previous results in section 4 to the case of the Kac equation.

2 Grazing collisions

According to [Ce], [Ch, Co] or [Tr, Mu], the Boltzmann equation writes

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f), \qquad (4)$$

where Q is a quadratic collision kernel acting only on the velocity variable,

$$Q(f,f)(v) = \int_{v_1 \in I\!R^3} \int_{\omega \in S^2} \{ f(v - (\omega \cdot (v - v_1))\,\omega) f(v_1 + (\omega \cdot (v - v_1))\,\omega)$$

$$- f(v)\, f(v_1) \} \, B(\, |v - v_1|, |\omega \cdot \frac{v - v_1}{|v - v_1|}| \,) \, d\omega dv_1, \qquad (5)$$

and B is a nonnegative cross section.

Note that R. Illner and M. Pulvirenti have proved the validity of this equation in the case of a two–dimensional rare gas (Cf. [Il, Pu]).

R.J. DiPerna and P-L. Lions have recently proved in [DP, L 1] that 1 admits a nonnegative global renormalized solution under suitable assumptions on B, including the angular cut–off of Grad (Cf. [Gr]), as soon as the initial datum f_0 satisfies

$$\int_{x \in I\!R^3} \int_{v \in I\!R^3} f_0 \, (\, 1 + |x|^2 + |v|^2 + |\log f_0| \,) \, dv dx < +\infty. \qquad (6)$$

From now on, we shall not write down the dependance of f or Q upon t and x, since these variables play no role in the computation. We shall introduce in (5) the following change of variables,

$$\sigma = 2\,(\omega \cdot \frac{v - v_1}{|v - v_1|})\,\omega - \frac{v - v_1}{|v - v_1|}. \tag{7}$$

Its Jacobian is

$$J(\omega) = \frac{1}{4}\,|\omega \cdot \frac{v - v_1}{|v - v_1|}|. \tag{8}$$

Therefore, the collision kernel Q can be recast in these new variables,

$$Q(f,f)(v) = \int_{v_1 \in I\!R^3} \int_{\sigma \in S^2} \{ f(\frac{v+v_1}{2} + \frac{|v-v_1|}{2}\sigma)\, f(\frac{v+v_1}{2} - \frac{|v-v_1|}{2}\sigma)$$

$$-f(v)\, f(v_1)\}\, C(\,|v - v_1|, |\sigma \cdot \frac{v - v_1}{|v - v_1|}|\,)\, d\sigma dv_1, \tag{9}$$

where

$$C(X,Y) = \frac{4}{\sqrt{\frac{1+Y}{2}}}\, B(\,X, \sqrt{\frac{1+Y}{2}}\,). \tag{10}$$

The angle σ measures the deflection of the velocities after the collision in barycentric coordinates. It can be written under the form,

$$\sigma = \frac{v - v_1}{|v - v_1|} \cos\theta + (\cos\phi\, h_{v,v_1} + \sin\phi\, i_{v,v_1}) \sin\theta, \tag{11}$$

where

$$\left(\frac{v - v_1}{|v - v_1|}, h_{v,v_1}, i_{v,v_1} \right) \tag{12}$$

is an orthonormal basis of $I\!R^3$. Therefore, we shall write

$$Q(f,f)(v) = \int_{v_1 \in I\!R^3} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi}$$

$$\{\ f(v - \frac{1}{2}(v - v_1)(1 - \cos\theta) + \frac{1}{2}|v - v_1|(\cos\phi\, h_{v,v_1} + \sin\phi\, i_{v,v_1}) \sin\theta)$$

$$\times f(v_1 + \frac{1}{2}(v - v_1)(1 - \cos\theta) - \frac{1}{2}|v - v_1|(\cos\phi\, h_{v,v_1} + \sin\phi\, i_{v,v_1}) \sin\theta)$$

$$-f(v)\, f(v_1)\ \}\, D(|v - v_1|, \theta)\, d\phi d\theta dv_1, \tag{13}$$

where

$$D(X,Y) = \sin Y \; C(X, \cos Y). \tag{14}$$

We shall from now on concentrate on grazing collisions. A collision is said to be grazing if the angle θ in (13) is small (i.e., when the difference between velocities before and after the collision is small). Therefore, we shall consider an asymptotics of 1 when the cross section D in (13) concentrates around the value 0 of θ.

For a given nonnegative cross section D defined on $I\!R_+ \times [0,\pi]$, we shall extend D to $I\!R_+ \times I\!R_+$ by setting,

$$\overline{D}(X,Y) = D(X,Y) \quad \text{when } Y \in [0,\pi], \tag{15}$$

$$\overline{D}(X,Y) = 0 \quad \text{elsewhere}. \tag{16}$$

Then, we define the family of cross sections,

$$D^{\epsilon}(X,Y) = \frac{1}{\epsilon^3}\,\overline{D}(X, \frac{Y}{\epsilon}), \tag{17}$$

and the collision kernel

$$Q^{\epsilon}(f,f)(v) = \int_{v_1 \in I\!R^3} \int_{\theta=0}^{\epsilon\pi} \int_{\phi=0}^{2\pi}$$

$$\{ \; f(v - \frac{1}{2}(v-v_1)(1-\cos\theta) + \frac{1}{2}|v-v_1|(\cos\phi\, h_{v,v_1} + \sin\phi\, i_{v,v_1})\sin\theta)$$

$$\times f(v_1 + \frac{1}{2}(v-v_1)(1-\cos\theta) - \frac{1}{2}|v-v_1|(\cos\phi\, h_{v,v_1} + \sin\phi\, i_{v,v_1})\sin\theta)$$

$$- f(v)\, f(v_1)\}\, D^{\epsilon}(|v-v_1|, \theta)\, d\phi d\theta dv_1. \tag{18}$$

The main result of this section is the following:

Theorem 1: *We assume that f is in $C^3(I\!R^3)$ and has a compact support. Moreover, we suppose that the cross section D and its derivative $\nabla_X D$ belong to $L^1_{loc}(I\!R_+ \times [0,\pi])$. Then, the ($L^1_{loc}$) limit of the Boltzmann kernel $Q^{\epsilon}(f,f)(v)$ defined in (18) when ϵ goes to 0 is the Fokker–Planck–Landau collision kernel $P(f,f)$ defined in (3), where*

$$\Gamma(z) = \frac{\pi}{8} z^2 \int_{\theta=0}^{\pi} \theta^2 D(z,\theta)\, d\theta. \tag{19}$$

Remark: Note that $\Gamma(z)$ has to be in $z^2\, D(z,\theta)$ to preserve for P the homogeneity of Q in v (at least if D is homogeneous in its first variable).

Proof of theorem 1: We denote

$$A^\epsilon(v,v_1,\chi,\phi) = -(v-v_1)(1-\cos(\epsilon\chi)) + |v-v_1|(\cos\phi h_{v,v_1} + \sin\phi i_{v,v_1})\sin(\epsilon\chi), \tag{20}$$

and begin the proof of theorem 1 with the following lemma:

Lemma 1: *The operator $Q^\epsilon(f,f)(v)$ defined in (18) satisfies the following asymptotic development,*

$$\begin{aligned} Q^\epsilon(f,f)(v) = \frac{1}{\epsilon^2}\int_{v_1\in I\!R^3} \{ (\nabla_v - \nabla_{v_1})\,(f(v)f(v_1)) \\ \cdot \int_{\chi=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{1}{2} A^\epsilon(v,v_1,\chi,\phi)\, D(|v-v_1|,\chi)\, d\phi d\chi \\ + (\nabla_v - \nabla_{v_1})^2\,(f(v)f(v_1)) \\ : \int_{\chi=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{1}{8} A^\epsilon(v,v_1,\chi,\phi)\otimes A^\epsilon(v,v_1,\chi,\phi)\, D(|v-v_1|,\chi)\, d\phi d\chi \,\}\, dv_1 + O(\epsilon), \end{aligned} \tag{21}$$

where $O(\epsilon)$ may depend on v.

Proof of lemma 1: We make in (18) the following change of variables:

$$\chi = \frac{\theta}{\epsilon}\,. \tag{22}$$

18 becomes:

$$\begin{aligned} Q^\epsilon(f,f)(v) = \frac{1}{\epsilon^2}\int_{v_1\in I\!R^3}\int_{\chi=0}^{\pi}\int_{\phi=0}^{2\pi} \\ \{\, f(v - \frac{1}{2}(v-v_1)(1-\cos(\epsilon\chi)) + \frac{1}{2}|v-v_1|(\cos\phi\, h_{v,v_1} + \sin\phi\, i_{v,v_1})\sin(\epsilon\chi)) \\ \times f(v_1 + \frac{1}{2}(v-v_1)(1-\cos(\epsilon\chi)) - \frac{1}{2}|v-v_1|(\cos\phi\, h_{v,v_1} + \sin\phi\, i_{v,v_1})\sin(\epsilon\chi)) \\ - f(v)\, f(v_1)\} D(|v-v_1|,\chi)\, d\phi d\chi dv_1 \\ = \frac{1}{\epsilon^2}\int_{v_1\in I\!R^3}\int_{\chi=0}^{\pi}\int_{\phi=0}^{2\pi} \{ f(v + \frac{1}{2}A^\epsilon) f(v_1 - \frac{1}{2}A^\epsilon) \\ - f(v)\, f(v_1)\,\}\, D(|v-v_1|,\chi)\, d\phi d\chi dv_1. \end{aligned} \tag{23}$$

But

$$|A^\epsilon(v, v_1, \chi, \phi)| \leq \epsilon R_1(v, v_1), \tag{24}$$

where R_1 is a polynomial in v, v_1 of degree 1.

Then, we expand (23) up to the second order (note that this is possible since f is assumed to be in $C^3(I\!R^3)$).

For ϵ small enough,

$$Q^\epsilon(f,f)(v) = \frac{1}{\epsilon^2} \int_{v_1 \in I\!R^3} \int_{\chi=0}^{\pi} \int_{\phi=0}^{2\pi} \{ \frac{1}{2} A^\epsilon \cdot (\nabla_v - \nabla_{v_1}) \left(f(v) f(v_1) \right)$$

$$+ \frac{1}{8} A^\epsilon \otimes A^\epsilon : (\nabla_v - \nabla_{v_1})^2 \left(f(v) f(v_1) \right) + r_1^\epsilon(v, v_1, \chi, \phi) \} D(|v - v_1|, \chi) \, d\phi d\chi dv_1, \tag{25}$$

with

$$|r_1^\epsilon(v, v_1, \chi, \phi)| \leq \epsilon^3 \, R_2(v, v_1) \, 1_{(\text{ 2 Supp f})}(v_1), \tag{26}$$

where R_2 is a polynomial in v, v_1.

Therefore, formula (21) holds, and lemma 1 is proved. According to lemma 1, we now have to compute:

$$T^\epsilon(v - v_1) = \int_{\chi=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{1}{2} A^\epsilon \, D(|v - v_1|, \chi) \, d\phi d\chi, \tag{27}$$

and

$$U^\epsilon(v - v_1) = \int_{\chi=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{1}{8} A^\epsilon \otimes A^\epsilon \, D(|v - v_1|, \chi) \, d\phi d\chi. \tag{28}$$

Lemma 2: *The function T^ϵ satisfies the following asymptotic development,*

$$T^\epsilon(v - v_1) = -\frac{\pi}{2} \epsilon^2 (v - v_1) \, \zeta(|v - v_1|) + r_3^\epsilon(v, v_1), \tag{29}$$

with

$$\zeta(z) = \int_{\chi=0}^{\pi} \chi^2 D(z, \chi) \, d\chi, \tag{30}$$

and

$$|r_3^\epsilon(v, v_1)| \leq \epsilon^3 R_4(v, v_1), \tag{31}$$

where $R_4(v, v_1) \in L^1_{loc}(I\!R^3 \times I\!R^3)$.

Proof of lemma 2: According to (20) and (27),

$$T^\epsilon(v - v_1) = \int_{\chi=0}^{\pi} \int_{\phi=0}^{2\pi} \{ -\frac{1}{2}(v - v_1)(1 - \cos(\epsilon\chi))$$

$$+\frac{1}{2}|v-v_1|(\cos\phi\, h_{v,v_1}+\sin\phi\, i_{v,v_1})\sin(\epsilon\chi)\,\}D(|v-v_1|,\chi)\,d\phi d\chi$$

$$=\int_{\chi=0}^{\pi}\int_{\phi=0}^{2\pi}\{\,-\frac{1}{2}(v-v_1)(\frac{\epsilon^2\chi^2}{2})+r_2^\epsilon(v,v_1,\chi,\phi)\,\}\,D(|v-v_1|,\chi)\,d\phi d\chi \quad (32)$$

and

$$|r_2^\epsilon(v,v_1,\chi,\phi)|\le\epsilon^3R_3(v,v_1), \quad (33)$$

where R_3 is a polynomial in v,v_1.

Therefore, denoting

$$r_3^\epsilon(v,v_1)=\int_{\chi=0}^{\pi}\int_{\phi=0}^{2\pi}r_2^\epsilon(v,v_1,\chi,\phi)\;D(|v-v_1|,\chi)\,d\phi d\chi, \quad (34)$$

we obtain lemma 2.

Lemma 3: *the function U^ϵ satisfies the following asymptotic development,*

$$U^\epsilon(v-v_1)=\frac{\pi}{8}\epsilon^2\{|v-v_1|^2Id-(v-v_1)\otimes(v-v_1)\}\,\zeta(|v-v_1|)+r_4^\epsilon(v,v_1), \quad (35)$$

with

$$r_4^\epsilon(v,v_1)=\epsilon^3(v-v_1)\otimes(v-v_1)\int_{\chi=0}^{\pi}w_1^\epsilon(\chi)\,D(|v-v_1|,\chi)\,d\chi$$

$$+\,\epsilon^3\,\{\,|v-v_1|^2Id-(v-v_1)\otimes(v-v_1)\,\}\int_{\chi=0}^{\pi}w_2^\epsilon(\chi)\,D(|v-v_1|,\chi)\,d\chi, \quad (36)$$

where w_1^ϵ and w_2^ϵ are bounded in $L^\infty([0,\pi])$.

Proof of lemma 3: According to (20) and (28),

$$U^\epsilon(v-v_1)=\int_{\chi=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{1}{8}\{\,-\frac{1}{2}(v-v_1)(1-\cos(\epsilon\chi))$$

$$+\frac{1}{2}|v-v_1|(\cos\phi\, h_{v,v_1}+\sin\phi\, i_{v,v_1})\sin(\epsilon\chi)\,\}$$

$$\otimes\{\,-\frac{1}{2}(v-v_1)(1-\cos(\epsilon\chi))+\frac{1}{2}|v-v_1|(\cos\phi\, h_{v,v_1}+\sin\phi\, i_{v,v_1})\sin(\epsilon\chi)\,\}$$

$$D(|v-v_1|,\chi)\,d\phi d\chi$$

$$=\frac{\pi}{4}\,(v-v_1)\otimes(v-v_1)\int_{\chi=0}^{\pi}(1-\cos(\epsilon\chi))^2\,D(|v-v_1|,\chi)\,d\chi$$

$$+\frac{1}{8}\,|v-v_1|^2\int_{\chi=0}^{\pi}\int_{\phi=0}^{2\pi}\sin^2(\epsilon\chi)\,\{\,\cos^2\phi\,h_{v,v_1}\otimes h_{v,v_1}+\sin^2\phi\,i_{v,v_1}\otimes i_{v,v_1}$$

$$+\cos\phi\sin\phi\;(h_{v,v_1}\otimes i_{v,v_1}+i_{v,v_1}\otimes h_{v,v_1})\,\}\,D(|v-v_1|,\chi)\,d\phi d\chi. \qquad (37)$$

But

$$\frac{(v-v_1)\otimes(v-v_1)}{|v-v_1|^2}+h_{v,v_1}\otimes h_{v,v_1}+i_{v,v_1}\otimes i_{v,v_1}=Id, \qquad (38)$$

since

$$\left(\frac{v-v_1}{|v-v_1|},h_{v,v_1},i_{v,v_1}\right) \qquad (39)$$

is an orthonormal basis of $\mathbb{R}^3$. Therefore, Defining

$$w_1^\epsilon(\chi)=\epsilon^{-3}\{1-\cos(\epsilon\chi)\}^2, \qquad (40)$$

and

$$w_2^\epsilon(\chi)=\frac{\pi}{8}\,\epsilon^{-3}\{\,\sin^2(\epsilon\chi)-\epsilon^2\chi^2\,\}, \qquad (41)$$

we get lemma 3.

In order to go on, we need the following lemma:

Lemma 4: *If μ belongs to $L^1_{loc}(\mathbb{R}_+)$, then*

$$\nabla\cdot\{(|x|^2 Id-x\otimes x)\,\mu(|x|^2)\}=-2x\mu(|x|^2). \qquad (42)$$

Proof of lemma 4: We first assume that $\mu\in C^1(\mathbb{R}_+)$. We compute

$$A=\nabla\cdot\{(|x|^2 Id-x\otimes x)\,\mu(|x|^2)\}\ |_l$$

$$=\sum_{k=1}^{3}\frac{\partial}{\partial x_k}\{\,[|x|^2\delta_{kl}-x_kx_l]\,\mu(|x|^2)\,\}|_l$$

$$=\sum_{k=1}^{3}\mu(|x|^2)\,\frac{\partial}{\partial x_k}\{|x|^2\delta_{kl}-x_kx_l\,\}|_l+2\sum_{k=1}^{3}x_k\,\mu'(|x|^2)\,\{|x|^2\delta_{kl}-x_kx_l\}|_l$$

$$=\mu(|x|^2)\sum_{k=1}^{3}\,\{2x_k\delta_{kl}-2x_l-2x_l\,\}|_l+\sum_{k=1}^{3}\,\{2x_k|x|^2\mu'(|x|^2)-2x_l|x|^2\mu'(|x|^2)\}|_l$$

$$=-2x_l\mu(|x|^2)|_l. \qquad (43)$$

But this formula does not take in account the derivative of μ, and therefore it remains valid in the sense of distributions when $\mu \in L^1_{\text{loc}}(I\!R_+)$.

Coming back to the proof of theorem 1, we apply lemma 4 to $U^\epsilon(v - v_1)$ (with $\mu(z) = \zeta(\sqrt{z}\,)$). Lemma 3 ensures that

$$(\nabla_v - \nabla_{v_1}) \cdot U^\epsilon(v - v_1) = -\frac{\pi}{2}\epsilon^2\,(v - v_1)\,\zeta(|v - v_1|) + (\nabla_v - \nabla_{v_1}) \cdot r_4^\epsilon(v, v_1). \tag{44}$$

But since D and $\nabla_X D$ are in $L^1_{\text{loc}}(I\!R_+ \times [0, \pi])$,

$$|(\nabla_v - \nabla_{v_1}) \cdot r_4^\epsilon(v, v_1)| \le \epsilon^3 R_5(v, v_1), \tag{45}$$

where $R_5 \in L^1_{\text{loc}}(I\!R^3 \times I\!R^3)$.

According to (44), (45) and lemma 2,

$$(\nabla_v - \nabla_{v_1}) \cdot U^\epsilon(v - v_1) = T^\epsilon(v - v_1) + r_5^\epsilon(v, v_1), \tag{46}$$

with

$$|r_5^\epsilon(v, v_1)| \le \epsilon^3 R_6(v, v_1), \tag{47}$$

where $R_6 \in L^1_{\text{loc}}(I\!R^3 \times I\!R^3)$.

Finally, the previous lemmas and formula (46) ensure that

$$Q^\epsilon(f, f)(v) = \frac{1}{\epsilon^2}\int_{v_1 \in I\!R^3} \{\, (\nabla_v - \nabla_{v_1})\,(f(v) f(v_1))$$

$$\cdot \{\, (\nabla_v - \nabla_{v_1})\, U^\epsilon(v - v_1) - r_5^\epsilon(v, v_1)\,\} + (\nabla_v - \nabla_{v_1})^2\,(f(v) f(v_1))\, U^\epsilon(v - v_1)\,\}\, dv_1$$

$$= \frac{1}{\epsilon^2}\int_{v_1 \in I\!R^3} (\nabla_v - \nabla_{v_1}) \cdot \{\, U^\epsilon(v - v_1)\,(\nabla_v - \nabla_{v_1})\,(f(v) f(v_1))\,\}\, dv_1 + O(\epsilon)$$

$$= \frac{1}{\epsilon^2}\nabla_v \cdot \int_{v_1 \in I\!R^3} U^\epsilon(v - v_1)\,\{\, f(v_1)\nabla_v f(v) - f(v)\nabla_{v_1} f(v_1)\,\}\, dv_1 + O(\epsilon)$$

$$= \nabla_v \cdot \int_{v_1 \in I\!R^3} \frac{\pi}{8}\,\zeta(|v - v_1|)\,\{|v - v_1|^2 Id - (v - v_1) \otimes (v - v_1)\,\}$$

$$\{\, f(v_1)\nabla_v f(v) - f(v)\nabla_{v_1} f(v_1)\,\}\, dv_1 + O(\epsilon), \tag{48}$$

where $O(\epsilon)$ may depend on v.

Denoting

$$\Gamma(z) = \frac{\pi}{8}\, z^2\, \zeta(z), \tag{49}$$

formula (48) becomes

$$Q^\epsilon(f,f)(v) = \nabla_v \cdot \int_{v_1 \in I\!R^3} \Gamma(|v-v_1|) \{ Id - \frac{(v-v_1)\otimes(v-v_1)}{|v-v_1|^2} \}$$

$$\{ f(v_1)\nabla_v f(v) - f(v)\nabla_{v_1} f(v_1) \} \, dv_1 + O(\epsilon), \tag{50}$$

and identity (30) ensures that

$$\Gamma(z) = \frac{\pi}{8} z^2 \int_{\theta=0}^{\pi} \theta^2 \, D(x,\theta) \, d\theta, \tag{51}$$

which concludes the proof of theorem 1.

We now give two computations of Γ when the cross section B in the Boltzmann equation is simple. In the case of hard–sphere gases, the cross section B writes

$$B(X,Y) = X\,Y. \tag{52}$$

Therefore,

$$D(X,Y) = 4\,X\, \sin Y, \tag{53}$$

and

$$\Gamma(x) = \frac{\pi}{8} x^2 \int_{\theta=0}^{\pi} \theta^2 \, 4x \, \sin\theta \, d\theta$$

$$= \frac{\pi}{2} \, (\pi^2 - 4) \, x^3. \tag{54}$$

We now look to the case of repulsion between two particles depending only on the distance r between them, the interaction coming out from a potential of the type

$$U(r) = \frac{k}{r^{s-1}}, \tag{55}$$

where k is a strictly positive number, and s is a real number. According to [Ce], the cross section B writes,

$$B(X,Y) = X^{\frac{s-5}{s-1}} \, \zeta(Y), \tag{56}$$

where ζ is a function defined implicitly.

Therefore,

$$D(X,Y) = \frac{4\sin Y}{\sqrt{\frac{1+\cos Y}{2}}} \, \zeta\Big(\sqrt{\frac{1+\cos Y}{2}}\Big) \, X^{\frac{s-5}{s-1}}, \tag{57}$$

and

$$\Gamma(x) = x^{\frac{3s-7}{s-1}} \int_{\theta=0}^{\pi} \pi\theta^2 \sin\frac{\theta}{2}\,\zeta(\cos\frac{\theta}{2})\,d\theta. \tag{58}$$

Note that the function ζ given by the physics is locally bounded on $[0, \frac{\pi}{2}[$ and as a singularity in $\theta = \frac{\pi}{2}$ of the form,

$$\zeta(x) \underset{x\to\frac{\pi}{2}}{\sim} (\frac{\pi}{2} - x)^{-\frac{s+1}{s-1}}. \tag{59}$$

Therefore, the previous analysis makes sense as soon as $s > 2$. However, when $s = 2$ (that is in the case of coulombian repulsion between the particles), this analysis yields

$$\Gamma(x) = x^{-1} \int_{\theta=0}^{\pi} \pi\theta^2 \sin\frac{\theta}{2}\,\zeta(\cos\frac{\theta}{2})\,d\theta, \tag{60}$$

and the integral over θ in formula (60) does not converge (Cf. [Li, Pi]). A physical analysis is then required in order to give a sense to 60. Note that in this case, a more precise asymptotics of the Boltzmann kernel can be performed. This is done by P. Degond and B. Lucquin–Desreux in [Dg, Lu].

3 The case of linearized equations

We proved in the previous section that for f regular enough, the family of Boltzmann kernels $Q^\epsilon(f,f)(v)$ converges towards the Fokker–Planck–Landau kernel $P(f,f)(v)$.

However, it is not easy to prove rigorously the convergence of a solution f^ϵ of the Boltzmann problem,

$$\frac{\partial f^\epsilon}{\partial t} + v\cdot\nabla_x f^\epsilon = Q^\epsilon(f^\epsilon, f^\epsilon), \tag{61}$$

$$f^\epsilon(0,x,v) = f_0(x,v), \tag{62}$$

towards a solution f of the Fokker–Planck–Landau problem

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = P(f,f), \tag{63}$$

$$f(0,x,v) = f_0(x,v), \tag{64}$$

since global existence for those equations needs a renormalization (Cf. [DP, L] and [L]). Note also that the results on the Boltzmann equation obtained by

R. Illner and M. Shinbrot (Cf. [Il, Shi]) cannot be easily extended to the case of the Fokker–Planck–Landau equation since they are based on the facts that the collision term of the Boltzmann equation includes no derivatives and has some properties of monotony. Therefore, we shall concentrate in this section on the linearized problems associated to (61) – (64). Following a classical technique, we linearize 61 and 63 around a given Maxwellian

$$M(v) = \frac{\rho}{(2\pi T)^{3/2}} \exp\{ - \frac{|v-u|^2}{2T} \}, \tag{65}$$

where the averaged velocity u belongs to $I\!R^3$, and the density and temperature ρ and T belong to $I\!R_+^*$. Note that ρ, u and T do not depend on t and x.

Then, we write f^ϵ and f under the form:

$$f^\epsilon = M(1+g^\epsilon), \tag{66}$$

$$f = M(1+g), \tag{67}$$

where g^ϵ and g are assumed to be small. Casting the second order terms in g^ϵ and g, equations (61) – (64) become

$$\frac{\partial g^\epsilon}{\partial t} + v \cdot \nabla_x g^\epsilon = M^{-1}\, Q^\epsilon(M, Mg^\epsilon), \tag{68}$$

$$g^\epsilon(0,x,v) = g_0(x,v), \tag{69}$$

$$\frac{\partial g}{\partial t} + v \cdot \nabla_x g = M^{-1}\, P(M, Mg), \tag{70}$$

$$g(0,x,v) = g_0(x,v), \tag{71}$$

where Q^ϵ and P are considered as symmetric bilinear operators.

We shall from now on denote

$$L_\epsilon = M^{-1}\, Q^\epsilon(M, M\,\cdot\,), \tag{72}$$

and

$$K = M^{-1}\, P(M, M\,\cdot\,). \tag{73}$$

We now give for the sake of completeness some classical results (at least for the Boltzmann equation).

Lemma 5: *For all h in $L^2(M^{\frac{1}{2}}(v)dv)$,*

$$L_\epsilon h(v) = \frac{1}{2}\int_{v_1\in I\!\!R^3}\int_{\omega\in S^2} M(v_1)\,\{\,h(v-(\omega\cdot(v-v_1))\,\omega) + h(v_1+(\omega\cdot(v-v_1))\,\omega)$$

$$-\,h(v)\;-\;h(v_1)\,\}\,B^\epsilon(|v-v_1|,\,|\omega\cdot\frac{v-v_1}{|v-v_1|}|\,)\,d\omega dv_1, \qquad (74)$$

and

$$Kh(v) = \frac{1}{2}\int_{v_1\in I\!\!R^3} M(v_1)\,\{\nabla_v-\nabla_{v_1}\}\,\{\,\Gamma(|v-v_1|)\,(\,I-\frac{(v-v_1)\otimes(v-v_1)}{|v-v_1|^2}\,)\,\}$$

$$\{\nabla_v-\nabla_{v_1}\}\,\{h(v)+h(v_1)\}\,dv_1. \qquad (75)$$

Proof of lemma 5: According to 72,

$$L_\epsilon h(v) = M^{-1}(v)\,Q^\epsilon(M,Mh)(v)$$

$$= \frac{1}{2}M^{-1}(v)\int_{v_1\in I\!\!R^3}\int_{\omega\in S^2}\{\,M(v-(\omega\cdot(v-v_1))\,\omega)M(v_1+(\omega\cdot(v-v_1))\,\omega)$$

$$\times\,h(v_1+(\omega\cdot(v-v_1))\,\omega)\;+\;M(v_1+(\omega\cdot(v-v_1))\,\omega)M(v-(\omega\cdot(v-v_1))\,\omega)$$

$$\times\,h(v-(\omega\cdot(v-v_1))\,\omega)\;-\;M(v)M(v_1)h(v_1)\;-\;M(v_1)M(v)h(v)\,\}$$

$$\times\,B^\epsilon(|v-v_1|,\,|\omega\cdot\frac{v-v_1}{|v-v_1|}|\,)d\omega dv_1. \qquad (76)$$

But

$$M(v_1+(\omega\cdot(v-v_1))\,\omega)\,M(v-(\omega\cdot(v-v_1))\,\omega)\;=\;M(v)\,M(v_1) \qquad (77)$$

since M is a Maxwellian, and therefore

$$L_\epsilon h(v) = \frac{1}{2}\int_{v_1\in I\!\!R^3}\int_{\omega\in S^2} M(v_1)\,\{h(v-(\omega\cdot(v-v_1))\,\omega) + h(v_1+(\omega\cdot(v-v_1))\,\omega)$$

$$-h(v)-h(v_1)\}B^\epsilon(|v-v_1|,|\omega\cdot\frac{v-v_1}{|v-v_1|}|\,)\,d\omega dv_1. \qquad (78)$$

Moreover,

$$Kh(v) = M^{-1}(v)\,P(M,Mh)(v). \qquad (79)$$

Therefore,

$$Kh(v) = \frac{1}{2} M^{-1}(v)\,\nabla_v \cdot \int_{v_1 \in I\!R^3} \Gamma(|v-v_1|)\,\{I - \frac{(v-v_1)\otimes(v-v_1)}{|v-v_1|^2}\}$$

$$\{\,M(v_1)\,\nabla_v(M(v)h(v))\;+\;M(v_1)\,h(v_1)\,\nabla_v M(v)$$

$$-M(v)\,\nabla_{v_1}(M(v_1)h(v_1))\;-\;M(v)\,h(v)\,\nabla_{v_1}M(v_1)\,\}\,dv_1$$

$$= \frac{1}{2} M^{-1}(v)\,\nabla_v \cdot \int_{v_1 \in I\!R^3} \Gamma(|v-v_1|)\,\{\,I - \frac{(v-v_1)\otimes(v-v_1)}{|v-v_1|^2}\,\}$$

$$\{\,(h(v)+h(v_1))\,(M(v_1)\nabla_v M(v) - M(v)\nabla_{v_1}M(v_1))$$

$$+\,M(v)\,M(v_1)\,\{\nabla_v h(v) - \nabla_{v_1}h(v_1)\}\,\}\,dv_1. \qquad (80)$$

But

$$\{\,I - \frac{(v-v_1)\otimes(v-v_1)}{|v-v_1|^2}\,\}\,\{\,M(v_1)\nabla_v M(v) - M(v)\nabla_{v_1}M(v_1)\,\} = 0, \qquad (81)$$

since M is a Maxwellian and therefore

$$Kh(v) = \frac{1}{2} M^{-1}(v)\,\nabla_v \cdot \int_{v_1 \in I\!R^3} M(v)\,M(v_1)\,\Gamma(|v-v_1|)$$

$$\{\,I - \frac{(v-v_1)\otimes(v-v_1)}{|v-v_1|^2}\,\}\,\{\,\nabla_v - \nabla_{v_1}\,\}\,\{\,h(v)+h(v_1)\,\}\,dv_1$$

$$= \frac{1}{2}\int_{v_1 \in I\!R^3} M(v_1)\,\nabla_v \cdot \{\,\Gamma(|v-v_1|)\,\{\,I - \frac{(v-v_1)\otimes(v-v_1)}{|v-v_1|^2}\,\}$$

$$\{\nabla_v - \nabla_{v_1}\}\,(h(v)+h(v_1))\,\}\,dv_1 \;+\; \frac{1}{2} M^{-1}(v)\int_{v_1 \in I\!R^3} -\frac{v-u}{T}\,M(v)\,M(v_1)$$

$$\Gamma(|v-v_1|)\,\{\,I - \frac{(v-v_1)\otimes(v-v_1)}{|v-v_1|^2}\,\}\,\{\,\nabla_v - \nabla_{v_1}\,\}\,\{\,h(v)+h(v_1)\,\}\,dv_1$$

$$= \frac{1}{2}\int_{v_1 \in I\!R^3} M(v_1)\,\nabla_v \cdot \{\,\Gamma(|v-v_1|)\,\{\,I - \frac{(v-v_1)\otimes(v-v_1)}{|v-v_1|^2}\,\}$$

$$\{\,\nabla_v - \nabla_{v_1}\,\}\,(h(v)+h(v_1))\,\}\,dv_1 \;+\; \frac{1}{2}\int_{v_1 \in I\!R^3} -\frac{v_1-u}{T}\,M(v_1)\,\Gamma(|v-v_1|)$$

$$\{\,I - \frac{(v-v_1)\otimes(v-v_1)}{|v-v_1|^2}\,\}\,\{\,\nabla_v - \nabla_{v_1}\,\}\,\{h(v)+h(v_1)\,\}\,dv_1$$

$$+\,\frac{1}{2}\int_{v_1 \in I\!R^3} \frac{v_1-v}{T}\,M(v_1)\,\Gamma(|v-v_1|)\{\,I - \frac{(v-v_1)\otimes(v-v_1)}{|v-v_1|^2}\,\}$$

$$\{ \nabla_v - \nabla_{v_1} \} \{ h(v) + h(v_1) \} \, dv_1$$

$$= \frac{1}{2} \int_{v_1 \in I\!R^3} M(v_1) \, \nabla_v \cdot \{ \Gamma(|v - v_1|) \{ I - \frac{(v - v_1) \otimes (v - v_1)}{|v - v_1|^2} \}$$

$$\{ \nabla_v - \nabla_{v_1} \} (h(v) + h(v_1)) \} \, dv_1 \; - \; \frac{1}{2} \int_{v_1 \in I\!R^3} M(v_1) \, \nabla_{v_1} \cdot \{ \Gamma(|v - v_1|)$$

$$\{ I - \frac{(v - v_1) \otimes (v - v_1)}{|v - v_1|^2} \} \{ \nabla_v - \nabla_{v_1} \} (h(v) + h(v_1)) \} \, dv_1 \qquad (82)$$

which ends the proof of lemma 5.

Lemma 6: *For every function h in $L^2(M^{\frac{1}{2}}(v)dv)$ and for every test function ψ in $D(I\!R^3)$,*

$$\int_{v \in I\!R^3} L_\epsilon h(v) \, \psi(v) \, M(v) \, dv = -\frac{1}{8} \int_{v \in I\!R^3} \int_{v_1 \in I\!R^3} \int_{\omega \in S^2} M(v) \, M(v_1)$$

$$\{ h(v - (\omega \cdot (v - v_1)) \, \omega) + h(v_1 + (\omega \cdot (v - v_1)) \, \omega) - h(v) - h(v_1) \}$$

$$\{ \psi(v - (\omega \cdot (v - v_1)) \, \omega) + \psi(v_1 + (\omega \cdot (v - v_1)) \, \omega) - \psi(v) - \psi(v_1) \}$$

$$B^\epsilon(|v - v_1|, \, |\omega \cdot \frac{v - v_1}{|v - v_1|}| \,) \, d\omega dv_1 dv, \qquad (83)$$

and

$$\int_{v \in I\!R^3} K h(v) \, \psi(v) \, M(v) \, dv$$

$$= -\frac{1}{4} \int_{v \in I\!R^3} \int_{v_1 \in I\!R^3} M(v) \, M(v_1) \, \Gamma(|v - v_1|) \, \{ \nabla_v - \nabla_{v_1} \} \, (\psi(v) + \psi(v_1))$$

$$\{ I - \frac{(v - v_1) \otimes (v - v_1)}{|v - v_1|^2} \} \{ \nabla_v - \nabla_{v_1} \} \, (h(v) + h(v_1)) \, dv_1. \qquad (84)$$

Proof of lemma 6: According to lemma 5, formula (83) is obtained after the changes of variables $(v, v_1) \to (v_1, v)$ and $(v, v_1) \to (v - (\omega \cdot (v - v_1)) \, \omega, v_1 - (\omega \cdot (v - v_1)) \, \omega)$.

Formula (84) is simply obtained after the change of variables $(v, v_1) \to (v_1, v)$.

Lemmas 5 and 6 immediately yield the following results,

Corollary 1: *For every function h in $L^2(M^{\frac{1}{2}}(v)dv)$ and for every test function ψ in $D(I\!R^3)$,*

$$\int_{v\in I\!R^3} L_\epsilon h(v)\,\psi(v)\,M(v)\,dv = \int_{v\in I\!R^3} h(v)\,L_\epsilon\,\psi(v)\,M(v)\,dv, \tag{85}$$

and

$$\int_{v\in I\!R^3} Kh(v)\,\psi(v)\,M(v)\,dv = \int_{v\in I\!R^3} h(v)\,K\psi(v)\,M(v)\,dv. \tag{86}$$

Corollary 2: *For every function h in $L^2(M^{\frac{1}{2}}(v)dv)$,*

$$\int_{v\in I\!R^3} L_\epsilon h(v)\,h(v)\,M(v)\,dv \le 0. \tag{87}$$

It is now classical that under reasonable assumptions on the collision cross section B of Q (for example, if B belongs to $L^1_{\text{loc}}(I\!R_+ \times [0,\pi])$ and is at most quadratic in the first variable), 68 admits a unique solution g^ϵ belonging to $L^\infty([0,+\infty[; L^2(dx \otimes M^{\frac{1}{2}}(v)dv))$ as soon as g_0 belongs to $L^2(dx \otimes M^{\frac{1}{2}}(v)dv)$. The main result of this section is the following:

Theorem 2: *We assume that g_0 belongs to $L^2(dx \otimes M^{\frac{1}{2}}(v)dv)$, and that B and $\nabla_X B$ belong to $L^1_{loc}(I\!R_+ \times [0,\pi])$. Moreover, we suppose that*

$$M(v)\,M(v_1)\int_{\theta=0}^{\pi} B(|v-v_1|,\theta)\,d\theta \tag{88}$$

and

$$M(v)\,M(v_1)\int_{\theta=0}^{\pi} \nabla_X B(|v-v_1|,\theta)\,d\theta \tag{89}$$

have superalgebraic decay (i.e., decrease faster than the inverse of any polynomial in v, v_1) when v, v_1 go to infinity.

*Then, if g^ϵ is the solution of 68 belonging to $L^\infty([0,+\infty[; L^2(dx\otimes M^{\frac{1}{2}}(v)dv))$, it is possible to extract from g^ϵ a subsequence still denoted by g^ϵ converging weakly * in $L^\infty([0,+\infty[; L^2(dx \otimes M^{\frac{1}{2}}(v)dv))$ to a function g satisfying 70.*

Remark: The assumptions of theorem 2 are satisfied by all hard potentials with the angular cut–off of Grad or by hard–spheres.

Proof of theorem 2: Multiplying each term of 68 by g_ϵ and integrating the result over $I\!R^3 \times I\!R^3$ against $M(v)dvdx$, corollary 2 of lemma 6 ensures that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{x\in I\!R^3}\int_{v\in I\!R^3}(g^\epsilon)^2(t,x,v)\,M(v)\,dvdx \le 0. \tag{90}$$

Therefore, initial data (69) ensure that

$$||g^\epsilon(t)||_{L^2(dx\otimes M^{\frac{1}{2}}(v)dv)} \le ||g_0||_{L^2(dx\otimes M^{\frac{1}{2}}(v)dv)}. \tag{91}$$

According to estimate (91), the family g^ϵ is uniformly bounded in $L^\infty(dt; L^2(dx\otimes M^{\frac{1}{2}}(v)dv))$, and we can extract from g^ϵ a subsequence still denoted by g^ϵ which converges weakly * in $L^\infty(dt; L^2(dx\otimes M^{\frac{1}{2}}(v)dv))$ towards a function denoted by g. But corollary 1 of lemma 6 ensures that for every test function ψ in $D([0,+\infty[\times I\!R^3\times I\!R^3)$,

$$\int_{t=0}^{+\infty}\int_{x\in I\!R^3}\int_{v\in I\!R^3}\{\,\frac{\partial g^\epsilon}{\partial t}+v\cdot\nabla_x g^\epsilon - L_\epsilon g^\epsilon\,\}\,\psi(t,x,v)\,M(v)\,dvdxdt$$

$$=\int_{t=0}^{+\infty}\int_{x\in I\!R^3}\int_{v\in I\!R^3}\{\,-\frac{\partial \psi}{\partial t}-v\cdot\nabla_x \psi - L_\epsilon \psi\,\}\,g^\epsilon(t,x,v)\,M(v)\,dvdxdt. \tag{92}$$

Therefore, since g^ϵ converges to g weakly in $L^2_{loc}(dt; L^2(dx\otimes M^{\frac{1}{2}}(v)dv))$ and since corollary 1 holds, we only have to prove that $L_\epsilon\psi$ converges towards $K\psi$ strongly in $L^2_{loc}(dt; L^2(dx\otimes M^{\frac{1}{2}}(v)dv))$.

According to lemma 5,

$$(L_\epsilon\psi - K\psi)(t,x,v)$$

$$=\frac{1}{2}\int_{v_1\in I\!R^3}M(v_1)\{\,\frac{1}{\epsilon^2}\int_{\chi=0}^{\pi}\int_{\phi=0}^{2\pi}(\psi(t,x,v+\frac{1}{2}A^\epsilon)+\psi(t,x,v_1-\frac{1}{2}A^\epsilon)$$
$$-\psi(t,x,v)-\psi(t,x,v_1)\,)\,D(|v-v_1|,\chi)d\phi d\chi\;-\;\{\,\nabla_v-\nabla_{v_1}\,\}$$
$$(\,\Gamma(|v-v_1|)\,\{\,I-\frac{(v-v_1)\otimes(v-v_1)}{|v-v_1|^2}\,\}\,)$$
$$\{\,\nabla_v-\nabla_{v_1}\,\}\,(\psi(t,x,v)+\psi(t,x,v_1))\,\}\,dv_1$$
$$=\frac{1}{2}\int_{v_1\in I\!R^3}M(v_1)\{\,\frac{1}{\epsilon^2}\,(\,\frac{1}{2}A^\epsilon\cdot\{\,\nabla_v-\nabla_{v_1}\,\}\,(\psi(t,x,v)+\psi(t,x,v_1))$$
$$+\frac{1}{8}A^\epsilon\otimes A^\epsilon\,:\,(\,\nabla_v-\nabla_{v_1}\,)^2\,(\psi(t,x,v)+\psi(t,x,v_1))+\lambda_1(t,x,v,v_1,\epsilon)\,)$$

$$- \{ \nabla_v - \nabla_{v_1} \} (\Gamma(|v - v_1|) \{ I - \frac{(v - v_1) \otimes (v - v_1)}{|v - v_1|^2} \})$$

$$\{ \nabla_v - \nabla_{v_1} \} (\psi(t,x,v) + \psi(t,x,v_1)) \} \, dv_1, \tag{93}$$

with

$$|\lambda_1(t,x,v,v_1,\epsilon)| \leq \epsilon^3 R_7(v,v_1) \int_{\chi=0}^{\pi} D(|v - v_1|, \chi) \, d\chi, \tag{94}$$

R_7 being a polynomial in v, v_1. Therefore, 44, 46 and estimates (93), (94) ensure that

$$(L_\epsilon \psi - K\psi)(v) \leq \frac{1}{2} \epsilon \int_{v_1 \in I\!R^3} M(v_1) \, \lambda_2(t,x,v,v_1,\epsilon) \, dv_1, \tag{95}$$

where

$$\lambda_2(t,x,v,v_1,\epsilon) \leq R_8(v,v_1) \int_{\chi=0}^{\pi} (D + |\nabla_X D|)(|v - v_1|, \chi) \, d\chi, \tag{96}$$

R_8 being a polynomial in v, v_1. According to estimates (95) and (96),

$$\int_{t=0}^{+\infty} \int_{x \in I\!R^3} \int_{v \in I\!R^3} M(v) \, |(L_\epsilon \psi - K\psi)(v)|^2 \, dv dx dt$$

$$\leq \frac{\epsilon^2}{2} \int \int_{(t,x) \in \text{ Supp } \psi} \int_{v \in I\!R^3} M(v)$$

$$| \int_{v_1 \in I\!R^3} R_8(v,v_1) \{ \int_{\chi=0}^{\pi} (D + |\nabla_X D|)(|v - v_1|, \chi) \, d\chi \} \, M(v_1) \, dv_1|^2 \, dv dx dt$$

$$\leq C \epsilon^2, \tag{97}$$

where C is a strictly positive constant, since the decay at infinity of

$$M(v) \, M(v_1) \int_{\chi=0}^{\pi} (D + |\nabla_X D|)(|v - v_1|, \chi) \, d\chi \tag{98}$$

is superalgebraic. Therefore, $L_\epsilon \psi$ tends to $K\psi$ strongly in $L^2_{loc}(dt; L^2(dx \otimes M^{\frac{1}{2}}(v) dv))$, which ends the proof of theorem 2.

4 The case of Kac equation

A simplified model of the Boltzmann equation was introduced by Kac in [K], and deals with a gas evolving in a one–dimensional space. In this model, kinetic energy is conserved, but not momentum. Moreover, all collisions conserving kinetic energy have the same probability to appear. Therefore, the form of the equation is

$$\frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial x} = Q'(f,f), \tag{99}$$

where

$$Q'(f,f)(t,x,v) = \int_{v_1\in I\!R}\int_{\theta=-\pi}^{\pi} \{\, f(t,x,v\cos\theta - v_1\sin\theta) f(t,x,v\sin\theta + v_1\cos\theta)$$

$$- f(t,x,v)\, f(t,x,v_1)\,\}\,\frac{d\theta}{2\pi} dv_1. \tag{100}$$

The physics underlying this model is described in [MK].

We shall concentrate here on a slightly different model. We assume no longer that all collisions conserving kinetic energy have the same probability to appear. Therefore, we introduce a collision cross section $B(\theta)$ and denote

$$Q'_B(f,f)(t,x,v) = \int_{v_1\in I\!R}\int_{\theta=-\pi}^{\pi} \{\, f(t,x,v\cos\theta - v_1\sin\theta)$$

$$\times f(t,x,v\sin\theta + v_1\cos\theta) - \; f(t,x,v)\, f(t,x,v_1)\,\}\, B(\theta)\,\frac{d\theta}{2\pi} dv_1. \tag{101}$$

From now on, we shall not write down the dependence of f or Q'_B upon t and x, since these variables play no role in the computation. In this model, the grazing collisions are those for which θ is near 0. Therefore, we define:

$$\overline{B}(\theta) \;=\; B(\theta) \quad \text{when } \theta \in [-\pi,\pi], \tag{102}$$

$$\overline{B}(\theta) \;=\; 0 \quad \text{elsewhere.} \tag{103}$$

Then, we denote:

$$B_1^\epsilon(\theta) = \frac{1}{\epsilon^2}\, B(\frac{\theta}{\epsilon}), \tag{104}$$

$$B_2^\epsilon(\theta) = \frac{1}{\epsilon^3}\, B(\frac{\theta}{\epsilon}), \tag{105}$$

and

$$Q'_{B_1^\epsilon}(f,f)(v) = \int_{v_1 \in I\!R} \int_{\theta=-\pi}^{\pi} \{ f(v\cos\theta - v_1\sin\theta)\, f(v\sin\theta + v_1\cos\theta)$$

$$- f(v)\, f(v_1) \}\, B_1^\epsilon(\theta)\, \frac{d\theta}{2\pi} dv_1, \tag{106}$$

$$Q'_{B_2^\epsilon}(f,f)(v) = \int_{v_1 \in I\!R} \int_{\theta=-\pi}^{\pi} \{ f(v\cos\theta - v_1\sin\theta)\, f(v\sin\theta + v_1\cos\theta)$$

$$- f(v)\, f(v_1) \}\, B_2^\epsilon(\theta)\, \frac{d\theta}{2\pi} dv_1. \tag{107}$$

The main result of this section is the following:

Theorem 3: *Let f be in $C^3(I\!R)$ with a compact support and B be in $L^1([-\pi,\pi])$. Then, when ϵ goes to 0,*
if

$$A = \int_{\theta=-\pi}^{\pi} \theta\, B(\theta)\, \frac{d\theta}{2\pi} \neq 0, \tag{108}$$

the collision kernel $Q'_{B_1^\epsilon}(f,f)(v)$ tends (in L^1_{loc}) to

$$Q'_1(f,f)(v) = -A \int_{v_1 \in I\!R} v_1\, f(v_1)\, dv_1\, \frac{\partial f}{\partial v}(v); \tag{109}$$

if $A = 0$, the collision kernel $Q'_{B_2^\epsilon}(f,f)(v)$ tends (in L^1_{loc}) to

$$Q'_2(f,f)(v) = A' \left(\frac{1}{2} \frac{\partial (v f(v))}{\partial v} \int_{v_1 \in I\!R} f(v_1)\, dv_1 + \frac{1}{2} \frac{\partial^2 f(v)}{\partial v^2} \int_{v_1 \in I\!R} v_1^2\, f(v_1)\, dv_1 \right), \tag{110}$$

where

$$A' = \int_{\theta=-\pi}^{\pi} \theta^2\, B(\theta)\, \frac{d\theta}{2\pi}. \tag{111}$$

Proof of theorem 3: We compute:

$$\int_{v_1 \in I\!R} \int_{\theta=-\pi}^{\pi} \{ f(v\cos\theta - v_1\sin\theta)\, f(v\sin\theta + v_1\cos\theta)$$

$$- f(v)\, f(v_1) \}\, B(\frac{\theta}{\epsilon})\, \frac{d\theta}{2\pi} dv_1$$

$$= \int_{v_1 \in I\!R} \int_{\theta=-\pi}^{\pi} \{\, f(v\cos(\epsilon\theta) - v_1 \sin(\epsilon\theta))\, f(v\sin(\epsilon\theta) + v_1\cos(\epsilon\theta))$$

$$- f(v)\, f(v_1)\,\}\, B(\theta)\, \frac{d\theta}{2\pi} dv_1$$

$$= \int_{v_1 \in I\!R} \int_{\theta=-\pi}^{\pi} \{\, f(v - \epsilon\theta v_1 - \frac{1}{2}\epsilon^2\theta^2 v + O(\epsilon^3))\, f(v_1 + \epsilon\theta v - \frac{1}{2}\epsilon^2\theta^2 v_1 + O(\epsilon^3))$$

$$- f(v)\, f(v_1)\,\}\, B(\theta)\, \frac{d\theta}{2\pi} dv_1$$

$$= \int_{v_1 \in I\!R} \int_{\theta=-\pi}^{\pi} \{\, (f(v) + (-\epsilon\theta v_1 - \frac{1}{2}\epsilon^2\theta^2 v)\frac{\partial f}{\partial v} + \frac{1}{2}\epsilon^2\theta^2 v_1^2 \frac{\partial^2 f}{\partial v^2} + O(\epsilon^3))$$

$$(f(v_1) + (\epsilon\theta v - \frac{1}{2}\epsilon^2\theta^2 v_1)\frac{\partial f}{\partial v_1} + \frac{1}{2}\epsilon^2\theta^2 v^2 \frac{\partial^2 f}{\partial v_1^2} + O(\epsilon^3))$$

$$- f(v)\, f(v_1)\,\}\, B(\theta)\, \frac{d\theta}{2\pi} dv_1$$

$$= \epsilon\, A \int_{v_1 \in I\!R} \{\, v\, f(v)\, \frac{\partial f}{\partial v_1} \; - \; v_1\, f(v_1)\, \frac{\partial f}{\partial v}\,\}\, dv_1$$

$$+ \frac{1}{2}\epsilon^2\, A' \int_{v_1 \in I\!R} \{ v_1\, f(v)\, \frac{\partial f}{\partial v_1} \; + \; v^2\, f(v)\, \frac{\partial^2 f}{\partial v_1^2}$$

$$- v\, f(v_1)\, \frac{\partial f}{\partial v} \; - \; 2\, v\, v_1\, f(v)\, \frac{\partial f}{\partial v_1} \; + \; v_1^2\, f(v_1)\, \frac{\partial^2 f}{\partial v^2}\,\}\, dv_1 \; + \; O(\epsilon^3)$$

$$= -\epsilon\, A \int_{v_1 \in I\!R} v_1\, f(v_1)\, dv_1\, \frac{\partial f}{\partial v}$$

$$+ \epsilon^2 A' \{\, \frac{1}{2} \frac{\partial v f(v)}{\partial v} \int_{v_1 \in I\!R} f(v_1)\, dv_1 \; + \; \frac{1}{2}\, \frac{\partial^2 f(v)}{\partial v^2} \int_{v_1 \in I\!R} v_1^2\, f(v_1)\, dv_1\,\} \; + \; O(\epsilon^3), \tag{112}$$

and therefore, theorem 3 holds.

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