A FORMAL PASSAGE FROM A SYSTEM OF BOLTZMANN EQUATIONS FOR MIXTURES TOWARDS A VLASOV-EULER SYSTEM OF COMPRESSIBLE FLUIDS

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Abstract. A formal asymptotics leading from a system of Boltzmann equations for mixtures towards either Vlasov-Navier Stokes or Vlaov-Stokes equations of incompressible fluids was established by the same authors and Etienne Bernard in [1] and [2]. With the same starting point but with a different scaling, we establish here a formal asymptotics leading to the Vlasov-Euler system of compressible fluids. Explicit formulas for the coupling terms are obtained in two typical situations: for elastic hard spheres on one hand, and for collisions corresponding to the inelastic interaction with a macroscopic dust speck on the other hand.

1. Boltzmann Equations for Multicomponent Gases

As in [1] and [2], we consider a binary mixture consisting of microscopic gas molecules and much bigger solid dust particles or liquid droplets. For the sake of simplicity, we assume from now on that the dust particles or droplets are identical and that the gas is monatomic. We denote by $F \equiv F(t,x,v) \geq 0$ the distribution function of dust particles or droplets, and by $f \equiv f(t,x,w) \geq 0$ the distribution function of gas molecules. These distribution functions satisfy the system of Boltzmann equations

$$
\begin{aligned}
(\partial_t + v\cdot\nabla_x)F &= \mathcal{D}(F,f) + \mathcal{B}(F)\,,\\
(\partial_t + w\cdot\nabla_x)f &= \mathcal{R}(f,F) + \mathcal{C}(f)\,.
\end{aligned}
\tag{1}
$$

The terms $\mathcal{B}(F)$ and $\mathcal{C}(f)$ are the Boltzmann collision kernels for pairs of dust particles or liquid droplets and for gas molecules respectively. The terms $\mathcal{D}(F,f)$ and $\mathcal{R}(f,F)$ are Boltzmann type collision kernels describing the deflection of dust particles or liquid droplets subject to the impingement of gas molecules, and the slowing down of gas molecules by collisions with dust particles or liquid droplets respectively.

Collisions between molecules are assumed to be elastic, and satisfy therefore the usual local conservation laws of mass, momentum and energy, while collisions between dust particles may be inelastic, so that $\mathcal{B}(F)$ satisfies only the local conservation of mass and momentum. Since collisions between gas molecules and particles preserve the nature of the colliding objects, the collision integrals $\mathcal{D}$ and $\mathcal{R}$ satisfiy the local conservation laws of particle number per species and local balance of momentum. The local balance of energy is satisfied only if all collisions are elastic.

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1.1. **Dimensionless Boltzmann systems.** We assume for simplicity that the aerosol is enclosed in a periodic box of size $L > 0$, ($x \in \mathbf{R}^3/L\mathbf{Z}^3$). The system of Boltzmann equations (1) involves an important number of physical parameters, which are listed in the table below.

Parameter	Definition
L	size of the container (periodic box)
$\mathcal{N}_p$	number of particles/L^3
$\mathcal{N}_g$	number of gas molecules/L^3
V_p	thermal speed of particles
V_g	thermal speed of gas molecules
S_{pp}	average particle/particle cross-section
S_{pg}	average particle/gas cross-section
S_{gg}	average molecular cross-section
$\eta = m_g/m_p$	mass ratio (molecules/particles)
$\mu = (m_g\mathcal{N}_g)/(m_p\mathcal{N}_p)$	mass fraction (gas/dust or droplets)
$E = V_p/V_g$	thermal speed ratio (particles/molecules)

As in [1] and [2], we define a dimensionless position variable: $\hat{x} := x/L$, together with dimensionless velocity variables for each species: $\hat{v} := v/V_p\,, \quad \hat{w} := w/V_g\,.$ We also define a time variable, which is adapted to the slowest species, $\hat{t} := tV_p/L\,.$ Finally, we define dimensionless distribution functions for each particle species:

$$\hat{F}(\hat{t},\hat{x},\hat{v}) := V_p^3 F(t,x,v)/\mathcal{N}_p\,, \qquad \hat{f}(\hat{t},\hat{x},\hat{w}) := V_g^3 f(t,x,w)/\mathcal{N}_g\,.$$

The definition of dimensionless collision integrals is more complex and involves the average collision cross sections S_{pp}, S_{pg}, S_{gg}, whose definition is recalled below.

The collision integrals $\mathcal{B}(F)$, $\mathcal{C}(f)$, $\mathcal{D}(F,f)$ and $\mathcal{R}(f,F)$ are given by expressions of the form

$$
\begin{aligned}
\mathcal{B}(F)(v) &= \iint_{\mathbf{R}^3\times\mathbf{R}^3} F(v')F(v'_*)\Pi_{pp}(v,\mathrm{d}v'\,\mathrm{d}v'_*) \\
&\quad - F(v)\int_{\mathbf{R}^3} F(v_*)|v-v_*|\Sigma_{pp}(|v-v_*|)\,\mathrm{d}v_*\,, \\
\mathcal{C}(f)(w) &= \iint_{\mathbf{R}^3\times\mathbf{R}^3} f(w')f(w'_*)\Pi_{gg}(w,\mathrm{d}w'\,\mathrm{d}w'_*) \\
&\quad - f(w)\int_{\mathbf{R}^3} f(w_*)|w-w_*|\Sigma_{gg}(|w-w_*|)\,\mathrm{d}w_*\,, \\
\mathcal{D}(F,f)(v) &= \iint_{\mathbf{R}^3\times\mathbf{R}^3} F(v')f(w')\Pi_{pg}(v,\mathrm{d}v'\,\mathrm{d}w') \\
&\quad - F(v)\int_{\mathbf{R}^3} f(w)|v-w|\Sigma_{pg}(|v-w|)\,\mathrm{d}w\,, \\
\mathcal{R}(f,F)(w) &= \iint_{\mathbf{R}^3\times\mathbf{R}^3} F(v')f(w')\Pi_{gp}(w,\mathrm{d}v'\,\mathrm{d}w') \\
&\quad - f(w)\int_{\mathbf{R}^3} F(v)|v-w|\Sigma_{pg}(|v-w|)\,\mathrm{d}v \\
&= \iint_{\mathbf{R}^3\times\mathbf{R}^3} F(v')f(w')\Pi_{gp}(w,\mathrm{d}v'\,\mathrm{d}w') \\
&\quad -f(w)\int_{\mathbf{R}^3} F(v)\int_{\mathbf{R}^3}\Pi_{gp}(W,v,w)\,dW\,dv.
\end{aligned}
\tag{2}
$$

In these expressions, $\Pi_{pp},\Pi_{gg},\Pi_{pg},\Pi_{gp}$ are nonnegative, measure-valued measurable functions defined a.e. on $\mathbf{R}^3$, while $\Sigma_{pp},\Sigma_{gg},\Sigma_{pg}$ are nonnegative measurable functions defined a.e. on $\mathbf{R}_+$. This setting is the same as in [1], and is taken from chapter 1 in [8] (see in particular formula (3.6) there).

We refer to [1] and [2] for the relation between the quantities Π and Σ.

The dimensionless quantities associated to Σ_{pp},Σ_{gg} and Σ_{pg} are $(i,j=p,g)$

$$
\begin{aligned}
\hat{\Sigma}_{ii}(|\hat{z}|) &= \Sigma_{ii}(V_i|\hat{z}|)/S_{ii}\,, \\
\hat{\Sigma}_{ij}(|\hat{z}|) &= \Sigma_{ij}(V_j|\hat{z}|)/S_{ij}\,.
\end{aligned}
$$

Likewise

$$
\begin{aligned}
\hat{\Pi}_{pp}(\hat{v},\mathrm{d}\hat{v}'\,\mathrm{d}\hat{v}'_*) &= \Pi_{pp}(v,\mathrm{d}v'\,\mathrm{d}v'_*)/S_{pp}V_p^4\,, \\
\hat{\Pi}_{gg}(\hat{w},\mathrm{d}\hat{w}'\,\mathrm{d}\hat{w}'_*) &= \Pi_{gg}(w,\mathrm{d}w'\,\mathrm{d}w'_*)/S_{gg}V_g^4\,, \\
\hat{\Pi}_{pg}(\hat{v},\mathrm{d}\hat{v}'\,\mathrm{d}\hat{w}') &= \Pi_{pg}(v,\mathrm{d}v'\,\mathrm{d}w')/S_{pg}V_g^4\,, \\
\hat{\Pi}_{gp}(\hat{w},\mathrm{d}\hat{v}'\,\mathrm{d}\hat{w}') &= \Pi_{gp}(w,\mathrm{d}v'\,\mathrm{d}w')/S_{pg}V_gV_p^3\,.
\end{aligned}
$$

With the dimensionless quantities so defined, we arrive as in [1] and [2] at the following dimensionless form of the multicomponent Boltzmann system:

$$
\left\{
\begin{aligned}
&\partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \mathcal{N}_gS_{pg}L\frac{V_g}{V_p}\hat{\mathcal{D}}(\hat{F},\hat{f}) + \mathcal{N}_pS_{pp}L\hat{\mathcal{B}}(\hat{F})\,, \\
&\partial_{\hat{t}}\hat{f} + \frac{V_g}{V_p}\hat{w}\cdot\nabla_{\hat{x}}\hat{f} = \mathcal{N}_pS_{pg}L\frac{V_g}{V_p}\hat{\mathcal{R}}(\hat{f},\hat{F}) + \mathcal{N}_gS_{gg}L\frac{V_g}{V_p}\hat{\mathcal{C}}(\hat{f})\,.
\end{aligned}
\right.
\tag{3}
$$

Throughout the present study, we shall always assume that

$$\mathcal{N}_p S_{pp} L \ll 1\,, \tag{4}$$

so that the collision integral for dust particles or droplets $\mathcal{N}_p S_{pp} L\hat{\mathcal{B}}(\hat{F})$ is considered as formally negligible (and will not appear anymore in the equations).

We now present the Euler scaling, which is significantly different from that of [1] and [2].

We assume that the thermal speed V_p of dust particles or droplets is of the same order of magnitude as the thermal speed V_g of gas molecules, so that

$$E = \frac{V_p}{V_g} = 1, \tag{5}$$

and the scaled Boltzmann system (3) becomes

$$\begin{cases} \partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \mathcal{N}_g S_{pg} L\hat{\mathcal{D}}(\hat{F},\hat{f})\,, \\ \partial_{\hat{t}}\hat{f} + \hat{w}\cdot\nabla_{\hat{x}}\hat{f} = \mathcal{N}_p S_{pg} L\hat{\mathcal{R}}(\hat{f},\hat{F}) + \mathcal{N}_g S_{gg} L\hat{\mathcal{C}}(\hat{f})\,. \end{cases} \tag{6}$$

Recalling that η is the mass ratio, we shall assume that

$$N_p\, S_{pg}\, L = 1, \qquad \frac{\mathcal{N}_g}{\mathcal{N}_p} = \frac{1}{\eta} >> 1, \qquad \mathcal{N}_g\, S_{gg}\, L =: \frac{1}{\delta} >> 1,$$

so that we end up with the scaled system:

$$\begin{cases} \partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \dfrac{1}{\eta}\hat{\mathcal{D}}(\hat{F},\hat{f})\,, \\ \partial_{\hat{t}}\hat{f} + \hat{w}\cdot\nabla_{\hat{x}}\hat{f} = \hat{\mathcal{R}}(\hat{f},\hat{F}) + \dfrac{1}{\delta}\hat{\mathcal{C}}(\hat{f})\,. \end{cases} \tag{7}$$

Note that the scaling above implies that $\mu = 1$. In the sequel, we shall let η and δ tend to 0 without assuming any relationship between those two parameters.

Henceforth, we drop hats on all dimensionless quantities and variables introduced in this section, and only dimensionless variables, distribution functions and collision integrals will be considered.

We also use V, W as variables in the positive part of the collision operators $\mathcal{D}$ and $\mathcal{R}$, in order to avoid confusions.

We define therefore the (η-dependent) dimensionless collision integrals

$$\begin{aligned} \mathcal{C}(f)(w) = &\iint_{\mathbf{R}^3\times\mathbf{R}^3} f(w')f(w'_*)\Pi_{gg}(w, \mathrm{d}w'\,\mathrm{d}w'_*) \\ &- f(w)\int_{\mathbf{R}^3} f(w_*)|w-w_*|\Sigma_{gg}(|w-w_*|)\,\mathrm{d}w_*\,, \end{aligned} \tag{8}$$

$$\begin{aligned} \mathcal{D}(F,f)(v) = &\iint_{\mathbf{R}^3\times\mathbf{R}^3} F(V)f(W)\Pi_{pg}(v, \mathrm{d}V\,\mathrm{d}W) \\ &- F(v)\int_{\mathbf{R}^3} f(w)\,|v-w|\,\Sigma_{pg}\,(|v-w|)\,\,\mathrm{d}w\,, \end{aligned} \tag{9}$$

$$\begin{aligned} \mathcal{R}(f,F)(w) = &\iint_{\mathbf{R}^3\times\mathbf{R}^3} F(V)f(W)\Pi_{gp}(w, \mathrm{d}V\,\mathrm{d}W) \\ &- f(w)\int_{\mathbf{R}^3} F(v)\,|v-w|\,\Sigma_{pg}\,(|v-w|)\,\,\mathrm{d}v\,. \end{aligned} \tag{10}$$

With the notation defined above, the scaled Boltzmann system (7) is then recast as:

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = \frac{1}{\eta}\mathcal{D}(F,f)\,, \\ \partial_t f + w \cdot \nabla_x f = \mathcal{R}(f,F) + \frac{1}{\delta}\mathcal{C}(f)\,. \end{cases} \tag{11}$$

1.2. **Explicit formulas for the collision integrals.** In the previous section, we have introduced a general setting for the various collisional processes involved in gas-particle mixtures. The explicit formulas for the examples of collision integrals considered in this work are given in the next three paragraphs.

1.2.1. *The Boltzmann collision integral for gas molecules.* The dimensionless collision integral $\mathcal{C}(f)$ is given by the formula

$$\mathcal{C}(f)(w) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} (f(w')f(w'_*) - f(w)f(w_*))c(w-w_*,\omega)\,\mathrm{d}w_*\mathrm{d}\omega, \tag{12}$$

for each measurable f defined a.e. on $\mathbf{R}^3$ and rapidly decaying at infinity, where

$$\begin{aligned} w' &\equiv w'(w,w_*,\omega) := w - (w-w_*)\cdot\omega\omega\,, \\ w'_* &\equiv w'_*(w,w_*,\omega) := w_* + (w-w_*)\cdot\omega\omega\,, \end{aligned} \tag{13}$$

(see formulas (3.11) and (4.16) in chapter II of [4]). The collision kernel c is of the form

$$c(w-w_*,\omega) = |w-w_*|\sigma_{gg}(|w-w_*|, |\cos(\widehat{w-w_*,\omega})|), \tag{14}$$

where σ_{gg} is the dimensionless differential cross-section of gas molecules. In other words,

$$\Sigma_{gg}(|z|) = 4\pi\int_0^1 \sigma_{gg}(|z|,\mu)\,\mathrm{d}\mu\,,$$

while

$$\Pi_{gg}(w,\cdot) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} \mathrm{d}w_*\mathrm{d}\omega\,\delta_{w'(w,w_*,\omega)}\otimes\delta_{w'_*(w,w_*,\omega)}c(w-w_*,\omega). \tag{15}$$

We recall that the collision integral $\mathcal{C}$ satisfies the conservation of mass, momentum and kinetic energy — see formulas (1.16)-(1.18) in chapter II of [3].

Since our analysis is formal, we do not write down precise assumptions on σ_{gg}. We shall in fact only use the fact that the solutions to $\mathcal{C}(f) = 0$ are the maxwellian functions of w. This is a direct consequence of Boltzmann's H theorem, which holds for all classical cross sections (cf. [3]).

1.2.2. *The collision integrals $\mathcal{D}$ and $\mathcal{R}$ for elastic collisions.* For each measurable F and f defined a.e. on $\mathbf{R}^3$ and rapidly decaying at infinity, the dimensionless collision integrals $\mathcal{D}(F,f)$ and $\mathcal{R}(f,F)$ are given by the formulas

$$\begin{aligned} \mathcal{D}(F,f)(v) &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (F(v'')f(w'') - F(v)f(w))b(v-w,\omega)\,\mathrm{d}w\mathrm{d}\omega\,, \\ \mathcal{R}(f,F)(w) &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (f(w'')F(v'') - f(w)F(v))b(v-w,\omega)\,\mathrm{d}v\mathrm{d}\omega\,, \end{aligned}$$

where

$$
(16) \qquad \begin{aligned} v'' \equiv v''(v,w,\omega) &:= v - \frac{2\eta}{1+\eta}(v-w)\cdot\omega\omega\,, \\ w'' \equiv w''(v,w,\omega) &:= w - \frac{2}{1+\eta}\,(w-v)\cdot\omega\omega\,, \end{aligned}
$$

(see formula (5.10) in chapter II of [4]). The collision kernel b is that of hard spheres, that is

$$
(17) \qquad b(v-w,\omega) = |v-w|\,|\cos(\widehat{v-w,\omega})|),
$$

while

$$
(18) \qquad \begin{aligned} \Pi_{pg}(v,\cdot) &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} \mathrm{d}w\,\mathrm{d}\omega\,|(v-w)\cdot\omega|\,\delta_{v''(v,w,\omega)}\otimes\delta_{w''(v,w,\omega)}\,, \\ \Pi_{gp}(w,\cdot) &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} \mathrm{d}v\,\mathrm{d}\omega\,|(v-w)\cdot\omega|\,\delta_{v''(v,w,\omega)}\otimes\delta_{w''(v,w,\omega)}\,. \end{aligned}
$$

The reduced mass of the dust particles or droplets and gas molecules defined by formula (5.2) in chapter II of [4] is

$$
\frac{m_p m_g}{m_p+m_g} = \frac{m_g}{1+\eta} = \frac{m_p\eta}{1+\eta}\,.
$$

These formulas explain how the mass ratio η appears in the definition of v'' and w'' above.

We recall that the operators $\mathcal{D}$ and $\mathcal{R}$ defined in this subsection satisfy separately the conservation of the number of particles and molecules, and jointly the conservation of momentum (involving both operators):

$$
(19) \qquad \int_{\mathbf{R}^3} \mathcal{D}(F,f)(v)v\,\mathrm{d}v + \eta\int_{\mathbf{R}^3}\mathcal{R}(f,F)(w)w\,\mathrm{d}w = 0\,.
$$

These properties can be easily checked using the formulas

$$
(20) \qquad v'' + \eta w'' = v + \eta w\,, \quad v'' - w'' = R_\omega(v-w),
$$

where R_ω is the reflection defined by $R_\omega w = w - 2(w\cdot\omega)\omega$ for each $\omega\in\mathbf{S}^2$.

1.2.3. *An inelastic model of collision integrals $\mathcal{D}$ and $\mathcal{R}$.* Dust particles or droplets are macroscopic objects when compared to gas molecules. This suggests using the classical models of gas-surface interaction to describe the impingement of gas molecules on dust particles or droplets. Perhaps the simplest such model of collisions has been introduced by F. Charles in [5], with a detailed discussion in section 1.3 of [6] and in [7]. We briefly recall this model below.

First, the (dimensional) particle-molecule cross-section is

$$
S_{pg} = \pi(r_g + r_p)^2,
$$

where r_g is the molecular radius and r_p the radius of dust particles or droplets. Then, the dimensionless particle-molecule cross-section is

$$
\Sigma_{pg}(|v-w|) = 1\,.
$$

The formulas for S_{pg} and Σ_{pg} correspond to a binary collision between two balls of radius r_p and r_g.

Next, the measure-valued functions Π_{pg} and Π_{gp} are defined as follows:

$$\begin{aligned}\Pi_{pg}(v, \mathrm{d}V\,\mathrm{d}W) &:= K_{pg}(v, V, W)\,\mathrm{d}V\mathrm{d}W\,,\\ \Pi_{gp}(w, \mathrm{d}V\,\mathrm{d}W) &:= K_{gp}(w, V, W)\,\mathrm{d}V\mathrm{d}W\,,\end{aligned} \tag{21}$$

where,

$$\begin{aligned}K_{pg}(v, V, W) :&= \tfrac{1}{2\pi^2}\left(\tfrac{1+\eta}{\eta}\right)^4 \beta^4 \exp\left(-\tfrac{1}{2}\beta^2\left(\tfrac{1+\eta}{\eta}\right)^2\left|v - \frac{V+\eta W}{1+\eta}\right|^2\right)\\ &\times \int_{\mathbf{S}^2}(n\cdot(V-W))_+\left(n\cdot\left(\frac{V+\eta W}{1+\eta} - v\right)\right)_+ dn,\end{aligned} \tag{22}$$

$$\begin{aligned}K_{gp}(w, V, W) :&= \tfrac{1}{2\pi^2}(1+\eta)^4\beta^4 \exp\left(-\tfrac{1}{2}\beta^2(1+\eta)^2\left|w - \frac{V+\eta W}{1+\eta}\right|^2\right)\\ &\times \int_{\mathbf{S}^2}(n\cdot(V-W))_+\left(n\cdot\left(w - \frac{V+\eta W}{1+\eta}\right)\right)_+ dn.\end{aligned} \tag{23}$$

In these formulas

$$\beta = \sqrt{\frac{m_g}{k_B T_{surf}}}$$

where k_B is the Boltzmann constant and T_{surf} the surface temperature of the particles.

For this inelastic model, the collision integrals are then:

$$\begin{aligned}\mathcal{D}(F, f)(v) = &\iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathrm{d}V\mathrm{d}W F(V) f(W) \tfrac{1}{2\pi^2}\left(\tfrac{1+\eta}{\eta}\right)^4\\ &\times\beta^4 \exp\left(-\tfrac{1}{2}\beta^2\left(\tfrac{1+\eta}{\eta}\right)^2\left|v - \frac{V+\eta W}{1+\eta}\right|^2\right)\\ &\times \int_{\mathbf{S}^2}(n\cdot(V-W))_+\left(n\cdot\left(\frac{V+\eta W}{1+\eta} - v\right)\right)_+ dn,\\ &- F(v)\int_{\mathbf{R}^3} f(w)\,|v-w|\,\mathrm{d}w\,,\end{aligned} \tag{24}$$

$$\begin{aligned}\mathcal{R}(f, F)(w) = &\iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathrm{d}V\,\mathrm{d}W F(V) f(W) \tfrac{1}{2\pi^2}(1+\eta)^4\\ &\times\beta^4 \exp\left(-\tfrac{1}{2}\beta^2(1+\eta)^2\left|w - \frac{V+\eta W}{1+\eta}\right|^2\right)\\ &\times \int_{\mathbf{S}^2}(n\cdot(V-W))_+\left(n\cdot\left(w - \frac{V+\eta W}{1+\eta}\right)\right)_+ dn\\ &- f(w)\int_{\mathbf{R}^3} F(v)\,|v-w|\,\mathrm{d}v\,,\end{aligned} \tag{25}$$

or, in weak form,

$$\int_{\mathbf{R}^3}\mathcal{D}(F, f)(v)\phi(v)\,\mathrm{d}v =$$

$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} dv dW dV F(V) f(W)[\phi(v) - \phi(V)]\tfrac{1}{2\pi^2}\left(\tfrac{1+\eta}{\eta}\right)^4$$

$$\times \beta^4 \exp\left(-\tfrac{1}{2}\beta^2\left(\tfrac{1+\eta}{\eta}\right)^2\left|v-\frac{V+\eta W}{1+\eta}\right|^2\right)$$
$$\times \int_{\mathbf{S}^2}(n\cdot(V-W))_+\left(n\cdot\left(\frac{V+\eta W}{1+\eta}-v\right)\right)_+ dn$$

and

$$\int_{\mathbf{R}^3}\mathcal{R}(f,F)(w)\phi(w)\,\mathrm{d}w =$$
$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\mathrm{d}w dW dV F(V)f(W)[\phi(w)-\phi(W)]\tfrac{1}{2\pi^2}(1+\eta)^4$$
$$\times\, \beta^4 \exp\left(-\tfrac{1}{2}\beta^2(1+\eta)^2\left|w-\frac{V+\eta W}{1+\eta}\right|^2\right)$$
$$\times \int_{\mathbf{S}^2}(n\cdot(V-W))_+\left(n\cdot\left(w-\frac{V+\eta W}{1+\eta}\right)\right)_+ dn.$$

2. Passage to the limit.

We denote here by $(f^{\delta,\eta},F^{\delta,\eta})$ a solution to system (11).

Recalling that the Maxwellian functions of w are the only functions f such that $\mathcal{C}(f)=0$, we see that $f^{\delta,\eta}\to_{\delta\to 0}\mathcal{M}[n,u,\theta/m_g]$, where

$$\mathcal{M}[n,u,\theta/m_g](t,x,w)=\frac{n(t,x)}{(2\pi\theta(t,x)/m_g)^{3/2}}e^{-m_g|w-u(t,x)|^2/2\theta(t,x)},$$

the quantities n,u,θ being identified as the respective (number) density, mean velocity and temperature of the gas. In principle, these quantities still depend upon η, but we do not write explicitly this dependence.

By integrating the equation for $f^{\delta,\eta}$ in the space of velocities against m_g, $m_g\,w$ and $\frac{1}{2}\,m_g\,|w|^2$, we get the local (in time and space) conservations laws for the mass, momentum and kinetic energy of the gas.

$$\partial_t\int_{\mathbf{R}^3}\begin{pmatrix} m_g\\ m_g w\\ \frac{1}{2}m_g|w|^2\end{pmatrix} f^{\delta,\eta}\,dw+\mathrm{div}_x\int_{\mathbf{R}^3} w\begin{pmatrix} m_g\\ m_g w\\ \frac{1}{2}m_g|w|^2\end{pmatrix} f^{\delta,\eta}\,dw$$
$$=\int_{\mathbf{R}^3}\begin{pmatrix} m_g\\ m_g w\\ \frac{1}{2}m_g|w|^2\end{pmatrix}\mathcal{R}(f^{\delta,\eta},F^{\delta,\eta})(w)dw\,,$$

so that when $\delta\to 0$, still at the formal level, we get

$$\partial_t\int_{\mathbf{R}^3}\begin{pmatrix} m_g\\ m_g\,w\\ \frac{1}{2}m_g\,|w|^2\end{pmatrix}\mathcal{M}[n,u,\theta/m_g]dw+\mathrm{div}_x\int_{\mathbf{R}^3} w\begin{pmatrix} m_g\\ m_g w\\ \frac{1}{2}m_g|w|^2\end{pmatrix}\mathcal{M}[n,u,\theta/m_g]dw$$
$$=\int_{\mathbf{R}^3}\begin{pmatrix} 0\\ m_g\,w\\ \frac{1}{2}m_g\,|w|^2\end{pmatrix}\mathcal{R}(\mathcal{M}[n,u,\theta/m_g],F)dw.$$

The computation of the two first terms is classical and leads to:

$$\int_{\mathbf{R}^3} \begin{pmatrix} m_g \\ m_g\, w \\ \frac{1}{2} m_g\, |w|^2 \end{pmatrix} \mathcal{M}[n,u,\theta/m_g] dw = \begin{pmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho |u|^2 + \frac{3}{2} n\theta \end{pmatrix},$$

and

$$\int_{\mathbf{R}^3} w \begin{pmatrix} m_g \\ m_g\, w \\ \frac{1}{2} m_g\, |w|^2 \end{pmatrix} \mathcal{M}[n,u,\theta/m_g] dw = \begin{pmatrix} \rho\, u \\ \rho u^{\otimes 2} + n\theta I \\ \frac{1}{2}\rho u |u|^2 + \frac{5}{2} n u\theta \end{pmatrix},$$

where $\rho := m_g\, n$.

Also, starting from the first equation in (11), we see that

$$\partial_t F^{\delta,\eta} \, + \, v \cdot \nabla_x F^{\delta,\eta} \, = \frac{1}{\eta} \mathcal{D}(F^{\delta,\eta}, f^{\delta,\eta}) \,,$$

so that letting $\delta \to 0$, we end up (at the formal level) with the equation

$$\partial_t F \, + \, v \cdot \nabla_x F \, = \frac{1}{\eta} \mathcal{D}(F, \mathcal{M}[n,u,\theta/m_g]) \,,$$

where once again the quantities F, n, u, θ still depend upon η (but we do not write explicitly this dependence).

In order to go further in the computation, we need to let $\eta \to 0$ in the term

$$\int_{\mathbf{R}^3} \begin{pmatrix} m_g \\ m_g\, w \\ \frac{1}{2} m_g\, |w|^2 \end{pmatrix} \mathcal{M}[n,u,\theta/m_g] dw = \begin{pmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho |u|^2 + \frac{3}{2} n\theta \end{pmatrix},$$

and in the term

$$\frac{1}{\eta} \int_{\mathbf{R}^3} \phi(v) \mathcal{D}(F, \mathcal{M}[n,u,\theta/m_g])(v) \mathrm{d}v$$

(or in a weak version of this term).

The result will however depend upon the type of collisions that we will consider.

2.1. **Computation for the elastic model.** We now specifically look to the case of elastic hard spheres collisions. We start with the equation for the particles.

2.1.1. *Equation for the particles.* We compute (for a test function ϕ)

$$\frac{1}{\eta} \int_{\mathbf{R}^3} \phi(v) \mathcal{D}(F, \mathcal{M}[n,u,\theta/m_g])(v) \mathrm{d}v$$
$$= \frac{1}{\eta} \iiint_{\mathbf{R}^3} [\phi(v'') - \phi(v)] \mathcal{M}[n,u,\theta/m_g](w) F(v) |(w-v)\cdot\omega| dw dv d\omega,$$

so that

$$\frac{1}{\eta} \int_{\mathbf{R}^3} \phi(v) \mathcal{D}(F, \mathcal{M}[n,u,\theta/m_g])(v) \mathrm{d}v$$
$$= -2 \iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathcal{M}[n,u,\theta/m_g](w) F(v) \nabla_v \phi(v) \cdot \int_{\mathbf{S}^2} \omega (v-w)\cdot\omega |(w-v)\cdot\omega| d\omega dw dv + O(\eta)$$
$$= -2\pi \iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathcal{M}[n,u,\theta/m_g](w) F(v) \nabla_v \phi(v) \cdot (v-w)|w-v| dw dv + O(\eta)$$
$$= -2\pi n \frac{\theta}{m_g} \int_{\mathbf{R}^3} F(v) \nabla_v \phi(v) \cdot q\left(\frac{v-u}{\sqrt{\theta/m_g}}\right) dv + O(\eta),$$

with

$$q(a) = (2\pi)^{-3/2} \int_{\mathbf{R}^3} e^{-\frac{|w|^2}{2}} |a-w|\,(a-w)\,dw.$$

Finally, at the formal level (that is, working with F, n, u, θ as if they were not depending on η),

$$\lim_{\eta\to 0} \frac{1}{\eta} \int_{\mathbf{R}^3} \phi(v)\mathcal{D}(F, \mathcal{M}[n,u,\theta/m_g])(v)\mathrm{d}v =$$

$$2\pi n\sqrt{\frac{\theta}{m_g}} \int_{\mathbf{R}^3} \phi(v)\,\nabla_v \cdot \left[F(v)\,\bar{q}\left(\frac{v-u}{\sqrt{\theta/m_g}}\right)(v-u)\right]\mathrm{d}v,$$

where $\bar{q}$ is defined in Lemma A.3.

We now turn to the equation for the gas molecules.

2.1.2. *Equation for the gas molecules.* We observe that

$$\int_{\mathbf{R}^3} \begin{pmatrix} m_g\,w \\ \frac{1}{2} m_g\,|w|^2 \end{pmatrix} \mathcal{R}(\mathcal{M}[n,u,\theta/m_g], F)(w)dw$$

$$= m_g \iiint_{\mathbf{R}^3} \begin{pmatrix} w''-w \\ \frac{1}{2}|w''|^2 - \frac{1}{2}|w|^2 \end{pmatrix} \mathcal{M}[n,u,\theta/m_g](w)F(v)|(w-v)\cdot\omega|dwdvd\omega,$$

where w'' is defined in (16).

First, we see that

$$\iiint_{\mathbf{R}^3} (w''-w)\mathcal{M}[n,u,\theta/m_g](w)F(v)|(w-v)\cdot\omega|dwdvd\omega$$

$$= -2\frac{1}{1+\eta} \iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathcal{M}[n,u,\theta/m_g](w)F(v) \int_{\mathbf{S}^2} \omega(w-v)\cdot\omega|(w-v)\cdot\omega|d\omega dwdv$$

$$= -2\pi\frac{1}{1+\eta} \iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathcal{M}[n,u,\theta/m_g](w)F(v)(w-v)|w-v|dwdv$$

$$= 2\pi n\frac{\theta}{m_g} \int_{\mathbf{R}^3} F(v)\,q\left(\frac{v-u}{\sqrt{\theta/m_g}}\right)dv + O(\eta).$$

Then, we also see that

$$|w''|^2 - |w|^2 = -\frac{4}{(1+\eta)^2}((v+\eta w)\cdot\omega)(w-v)\cdot\omega$$

$$= -4(v\cdot\omega)(w-v)\cdot\omega + O(\eta).$$

As a consequence

$$\iiint_{\mathbf{R}^3} (|w''|^2 - |w|^2)\mathcal{M}[n,u,\theta/m_g](w)F(v)|(w-v)\cdot\omega|dwdvd\omega$$

$$= -4\iint_{\mathbf{R}^3} \mathcal{M}[n,u,\theta/m_g](w)F(v)\left(\int_{\mathbf{S}^2} (v\cdot\omega)(w-v)\cdot\omega|(w-v)\cdot\omega|d\omega\right)dwdv + O(\eta)$$

$$= 4\pi n\frac{\theta}{m_g} \int_{\mathbf{R}^3} F(v)\,v\cdot q\left(\frac{v-u}{\sqrt{\theta/m_g}}\right)dv + O(\eta)$$

$$= 4\pi n\sqrt{\frac{\theta}{m_g}} \int_{\mathbf{R}^3} F(v)\,v\cdot(v-u)\bar{q}\left(\frac{v-u}{\sqrt{\theta/m_g}}\right)dv + O(\eta).$$

Working once again at the formal level, we let $\eta \to 0$ and end up with an Euler-Vlasov system:

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0\,,$$
$$\partial_t(\rho u) + \mathrm{div}_x(\rho u^{\otimes 2} + n\theta\, I) = 2\pi n\sqrt{\theta m_g}\int_{\mathbf{R}^3} F(v_*)\bar{q}\left(\frac{v_* - u}{\sqrt{\theta/m_g}}\right)(v_* - u)\,dv_*\,,$$
$$\partial_t(\tfrac{1}{2}\rho|u|^2 + \tfrac{3}{2}n\theta) + \mathrm{div}_x(u(\tfrac{1}{2}\rho|u|^2 + \tfrac{5}{2}n\theta))$$
$$= 2\pi n\sqrt{\theta m_g}\int_{\mathbf{R}^3} F(v_*)\bar{q}\left(\frac{v_* - u}{\sqrt{\theta/m_g}}\right)(v_* - u)\cdot v_*\,dv_*\,,$$
$$\partial_t F + v\cdot\nabla_x F = 2\pi n\sqrt{\frac{\theta}{m_g}}\,\mathrm{div}_v\left(F(v)\,\bar{q}\left(\frac{v-u}{\sqrt{\theta/m_g}}\right)(v-u)\right).$$

We recall that $\bar{q}$ is defined in Lemma A.3. It can be expressed in terms of usual functions (including the Erf function), as shown in Lemma A.3 and Lemma A.4.

2.2. **Computation for the inelastic model.**

The computations of the previous subsection can be reproduced in the case of the inelastic model defined in paragraph 1.2.3.

As previously, we start with the computation of the equation for the particles.

2.2.1. *Equation for the particles.* Defining $M(v) := \mathcal{M}[n, u, \theta/m_g](v)$, we look for

$$\lim_{\eta\to 0}\frac{1}{\eta}\int_{\mathbf{R}^3}\mathcal{D}(F,M)(v)\phi(v)\,\mathrm{d}v = \lim_{\eta\to 0}\frac{1}{\eta}$$
$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} dv dW dV F(V)M(W)[\phi(v)-\phi(V)]\tfrac{1}{2\pi^2}\left(\tfrac{1+\eta}{\eta}\right)^4$$
$$\times\,\beta^4\exp\left(-\tfrac{1}{2}\beta^2\left(\tfrac{1+\eta}{\eta}\right)^2\left|v - \frac{V+\eta W}{1+\eta}\right|^2\right)$$
$$\times\int_{\mathbf{S}^2}(n\cdot(V-W))_+\left(n\cdot\left(\frac{V+\eta W}{1+\eta} - v\right)\right)_+ dn.$$

Defining $z = \frac{1}{\eta}(v - \frac{V+\eta W}{1+\eta})$ and letting $\eta \to 0$, the previous formula becomes (at the formal level)

$$\lim_{\eta\to 0}\frac{1}{\eta}\int_{\mathbf{R}^3}\mathcal{D}(F,M)(v)\phi(v)\,\mathrm{d}v =$$
$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} dz dW dV F(V)M(W)[\nabla_V\phi(V)]\cdot[z+W-V]\tfrac{1}{2\pi^2}\beta^4\exp\left(-\tfrac{1}{2}\beta^2 z^2\right)$$
$$\times\int_{\mathbf{S}^2}(n\cdot(V-W))_+(-n\cdot z)_+\,dn.$$

We now define $\cos\delta = \frac{(W-V)\cdot z}{|W-V||z|}$ and, using Lemma A.1 of the Appendix and spherical coordinates with polar axis $\frac{(V-W)}{|V-W|}$ (so that $z = -r\cos\delta\frac{(V-W)}{|V-W|} + r\sin\delta\cos\phi\hat{i} + r\sin\delta\sin\phi\hat{j}$), we get (after integrating in ϕ in $(0, 2\pi)$):

$$\lim_{\eta\to 0}\frac{1}{\eta}\int_{\mathbf{R}^3}\mathcal{D}(F,M)(v)\phi(v)\,\mathrm{d}v =$$
$$\tfrac{1}{\pi}\beta^4\int_{\mathbf{R}^3} dV F(V)[\nabla_V\phi(V)]\cdot\int_{\mathbf{R}^3} dW M(W)\int_0^\infty dr r^2\int_0^\pi d\delta\sin\delta\exp\left(-\tfrac{1}{2}\beta^2 r^2\right)$$

$$\times\frac{2}{3}[r\frac{(W-V)}{|W-V|}\cos\delta+W-V]\,|V-W|r[\sin\delta+(\pi-\delta)\cos\delta]$$
$$=\tfrac{1}{\pi}\beta^4\int_{\mathbf{R}^3}dVF(V)[\nabla_V\phi(V)]\cdot\int_{\mathbf{R}^3}dWM(W)$$
$$\left\{\int_0^\infty drr^4\exp\left(-\tfrac12\beta^2r^2\right)\frac{2}{3}(W-V)\int_0^\pi d\delta\sin\delta\cos\delta[\sin\delta+(\pi-\delta)\cos\delta]\right.$$
$$\left.+\int_0^\infty drr^3\exp\left(-\tfrac12\beta^2r^2\right)\frac{2}{3}(W-V)|W-V|\int_0^\pi d\delta\sin\delta[\sin\delta+(\pi-\delta)\cos\delta]\right\}.$$

Thanks to the computations of Lemma A.2 in the Appendix, we get the following formal limit:

$$\lim_{\eta\to0}\frac{1}{\eta}\int_{\mathbf{R}^3}\mathcal{D}(F,M)(v)\phi(v)\,\mathrm{d}v=$$
$$\frac{\sqrt{2\pi}}{3\beta}\int_{\mathbf{R}^3}dVF(V)[\nabla_V\phi(V)]\cdot\int_{\mathbf{R}^3}dWM(W)(W-V)$$
$$+\int_{\mathbf{R}^3}dVF(V)[\nabla_V\phi(V)]\cdot\int_{\mathbf{R}^3}dWM(W)(W-V)|W-V|$$
$$=-\frac{\sqrt{2\pi}}{3\beta}n\sqrt{\frac{\theta}{m_g}}\int_{\mathbf{R}^3}dVF(V)[\nabla_V\phi(V)]\frac{V-u}{\sqrt{\theta/m_g}}\,dV$$
$$-n\sqrt{\frac{\theta}{m_g}}\int_{\mathbf{R}^3}dVF(V)[\nabla_V\phi(V)]\bar q\left(\frac{V-u}{\sqrt{\theta/m_g}}\right)\cdot(V-u),$$

where we recall that $\bar q$ is defined in Lemma A.3 of the Appendix.

We finally compute the equation for the gas molecules, in the inelastic case.

2.2.2. *Equation for the gas molecules.*

We observe first that

$$\int_{\mathbf{R}^3}\mathcal{R}(f,F)(w)\begin{pmatrix}m_gw\\ \tfrac12m_g|w|^2\end{pmatrix}\mathrm{d}w$$
$$=\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\mathrm{d}wdWdVF(V)f(W)\begin{pmatrix}m_g(w-W)\\ \tfrac12m_g(|w|^2-|W|^2)\end{pmatrix}$$
$$\times\tfrac{1}{2\pi^2}(1+\eta)^4\beta^4\exp\left(-\tfrac12\beta^2(1+\eta)^2\left|w-\frac{V+\eta W}{1+\eta}\right|^2\right)$$
$$\times\int_{\mathbf{S}^2}(n\cdot(V-W))_+\left(n\cdot\left(w-\frac{V+\eta W}{1+\eta}\right)\right)_+dn.$$

We get (at the formal level) that

$$\lim_{\eta\to0}\int_{\mathbf{R}^3}\mathcal{R}(f,F)(w)\begin{pmatrix}m_gw\\ \tfrac12m_g|w|^2\end{pmatrix}\mathrm{d}w$$
$$=\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\mathrm{d}wdWdVF(V)M(W)\begin{pmatrix}m_g(w-W)\\ \tfrac12m_g(|w|^2-|W|^2)\end{pmatrix}$$
$$\times\tfrac{1}{2\pi^2}\beta^4\exp\left(-\tfrac12\beta^2\left|w-V\right|^2\right)\int_{\mathbf{S}^2}(n\cdot(V-W))_+\,(n\cdot(w-V))_+\,dn$$
$$=\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\mathrm{d}zdWdVF(V)M(W)\begin{pmatrix}m_g(z+V-W)\\ \tfrac12m_g(|z+V|^2-|W|^2)\end{pmatrix}$$

$$\times \tfrac{1}{2\pi^2}\beta^4 \exp\left(-\tfrac{1}{2}\beta^2|z|^2\right) \int_{\mathbf{S}^2} (n\cdot(V-W))_+ \,(n\cdot z)_+ \,dn.$$

Then, using Lemmas A.1 and A.2 of the Appendix, we see that

$$\lim_{\eta\to 0}\int_{\mathbf{R}^3} \mathcal{R}(f,F)(w) \begin{pmatrix} m_g w \\ \tfrac{1}{2}m_g|w|^2 \end{pmatrix} \mathrm{d}w$$

$$= \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} dW dV F(V) M(W)$$

$$\times \int_0^\infty \int_0^\pi \begin{pmatrix} m_g(r\cos\delta \frac{V-W}{|V-W|} + V - W) \\ \tfrac{1}{2}m_g(r^2+|V|^2-|W|^2+2r\cos\delta\frac{V-W}{|V-W|}\cdot V) \end{pmatrix}$$

$$\times \tfrac{1}{\pi}\beta^4 \exp(-\tfrac{1}{2}\beta^2 r^2)\, r^3\, |V-W|\, \frac{2}{3}\left[(\pi-\delta)\,\cos\delta + \sin\delta\right] \sin\delta\, d\delta dr$$

$$= \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} dW dV F(V) M(W)\, \frac{2\beta^4}{3\pi}$$

$$\times \begin{pmatrix} m_g(\frac{\pi}{3}\sqrt{\frac{\pi}{2}}\frac{3}{\beta^5}(V-W) + \frac{3\pi}{4}\frac{2}{\beta^4}|V-W|\,(V-W)) \\ \tfrac{1}{2}m_g(\frac{3\pi}{4}\frac{8}{\beta^6}|V-W| + \frac{2\pi}{3}\sqrt{\frac{\pi}{2}}\frac{3}{\beta^5}(V-W)\cdot V + \frac{3\pi}{4}\frac{2}{\beta^4}|V-W|(|V|^2-|W|^2)) \end{pmatrix}$$

$$= \frac{n}{\beta}\frac{m_g}{(2\pi)^{3/2}} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} dQ dV F(V)\, e^{-Q^2/2}$$

$$\times \begin{pmatrix} \frac{2}{3}\sqrt{\frac{\pi}{2}}(V-u-\sqrt{\frac{\theta}{m_g}}\,Q) \\ \frac{2}{\beta}|V-u-\sqrt{\frac{\theta}{m_g}}\,Q| + \frac{4}{3}\sqrt{\frac{\pi}{2}}(V-u-\sqrt{\frac{\theta}{m_g}}\,Q)\cdot V \end{pmatrix}$$

$$\begin{pmatrix} +\beta|V-u-\sqrt{\frac{\theta}{m_g}}\,Q|\,(V-u-\sqrt{\frac{\theta}{m_g}}\,Q) \\ +\beta|V-u-\sqrt{\frac{\theta}{m_g}}\,Q|(|V|^2-|u+\sqrt{\frac{\theta}{m_g}}\,Q|^2) \end{pmatrix}$$

$$= \frac{n}{\beta}\frac{m_g}{(2\pi)^{3/2}} \int_{\mathbf{R}^3} dV F(V)$$

$$\times \begin{pmatrix} \frac{2}{3}(2\pi)^{3/2}\sqrt{\frac{\pi}{2}}(V-u) + \beta\int_{\mathbf{R}^3} e^{-Q^2/2}|V-u-\sqrt{\frac{\theta}{m_g}}\,Q|\,(V-u-\sqrt{\frac{\theta}{m_g}}\,Q)dQ \\ \int_{\mathbf{R}^3} e^{-Q^2/2}\frac{2}{\beta}|V-u-\sqrt{\frac{\theta}{m_g}}\,Q|dQ + \frac{4}{3}(2\pi)^{3/2}\sqrt{\frac{\pi}{2}}(V-u)\cdot V \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ \beta|V|^2\int_{\mathbf{R}^3} e^{-Q^2/2}|V-u-\sqrt{\frac{\theta}{m_g}}\,Q|dQ \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ -\beta\int_{\mathbf{R}^3} e^{-Q^2/2}|V-u-\sqrt{\frac{\theta}{m_g}}\,Q||u+\sqrt{\frac{\theta}{m_g}}\,Q|^2 dQ \end{pmatrix}$$

$$= n\int_{\mathbf{R}^3} dV F(V) \begin{pmatrix} \frac{2}{3}\sqrt{\frac{\pi}{2}}\frac{m_g}{\beta}(V-u) + \sqrt{m_g\theta}\,\bar{q}\left(\left|\frac{V-u}{\sqrt{\theta/m_g}}\right|\right)(V-u) \\ \left(\frac{2}{\beta^2}\sqrt{m_g\theta} + \sqrt{m_g\theta}(|V-u|^2)\right) q_0\left(\left|\frac{V-u}{\sqrt{\theta/m_g}}\right|\right) \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ \frac{4}{3\beta}\sqrt{\frac{\pi}{2}}m_g(V-u)\cdot V \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ 2\sqrt{m_g\theta}\bar{q}\left(\left|\frac{V-u}{\sqrt{\theta/m_g}}\right|\right)(V-u)\cdot u - \theta\sqrt{\frac{\theta}{m_g}}\,q_2\left(\left|\frac{V-u}{\sqrt{\theta/m_g}}\right|\right) \end{pmatrix},$$

where we recall that $\bar{q}$ is defined in Lemma A.3 of the Appendix, and, for $i = 0, 2$,

$$q_i(|a|) := \int_{\mathbf{R}^3} e^{-Q^2/2} |a - Q| Q^i \frac{dQ}{(2\pi)^{3/2}}.$$

We end up with an Euler-Vlasov system in the inelastic case, which writes therefore

$$\begin{aligned}
&\partial_t \rho + \mathrm{div}_x(\rho u) = 0\,,\\
&\partial_t(\rho u) + \mathrm{div}_x(\rho u^{\otimes 2} + n\theta\, I)\\
&\qquad = n\sqrt{\theta m_g} \int_{\mathbf{R}^3} F(v_*) \left[\frac{2}{3\beta}\sqrt{\frac{\pi}{2}}\sqrt{\frac{m_g}{\theta}} + \bar{q}\left(\left| \frac{v_* - u}{\sqrt{\theta/m_g}} \right| \right) \right] (v_* - u)\, dv_*\,,\\
&\partial_t(\tfrac{1}{2}\rho|u|^2 + \tfrac{3}{2}n\theta) + \mathrm{div}_x(u(\tfrac{1}{2}\rho|u|^2 + \tfrac{5}{2}n\theta))\\
&\qquad = n\sqrt{\theta m_g} \int_{\mathbf{R}^3} F(v_*) \left[\left(\frac{2}{\beta^2} + |v_* - u|^2 \right) q_0\left(\left| \frac{v_* - u}{\sqrt{\theta/m_g}} \right| \right) \right.\\
&\left. + \frac{4}{3\beta}\sqrt{\frac{\pi}{2}}\sqrt{\frac{m_g}{\theta}} (v_* - u)\cdot v_* + 2\bar{q}\left(\left| \frac{v_* - u}{\sqrt{\theta/m_g}} \right| \right) (v_* - u)\cdot u - \frac{\theta}{m_g} q_2\left(\left| \frac{v_* - u}{\sqrt{\theta/m_g}} \right| \right) \right] dv_*\,,\\
&\partial_t F + v\cdot\nabla_x F = n\sqrt{\frac{\theta}{m_g}}\,\mathrm{div}_v \left(F(v) \left[\frac{2}{3\beta}\sqrt{\frac{\pi}{2}}\sqrt{\frac{m_g}{\theta}} + \bar{q}\left(\frac{v - u}{\sqrt{\theta/m_g}} \right) \right] (v - u) \right),
\end{aligned}$$

where q_0, q_2 and $\bar{q}$ are defined in Lemma A.3 of the Appendix. All those functions can be expressed in terms of usual functions (including the Erf function), cf. Lemmas A.3 and A.4 of the Appendix.

3. Appendix

We detail here some of the computations which are used in the Section 2 of this paper.

Lemma A.1: For all $a, b \in S^2$,

$$\int_{S^2} (\tilde{n}\cdot a)_+ \,(\tilde{n}\cdot b)_+ \, d\tilde{n} = \frac{2}{3}\,[(\pi - \delta)\,\cos\delta + \sin\delta],$$

where $\delta \in [0, \pi]$, $\cos\delta = a\cdot b$.

Proof:

$$\begin{aligned}
\int_{S^2} (\tilde{n}\cdot a)_+ \,(\tilde{n}\cdot b)_+ \, d\tilde{n} &= \int_0^\pi \sin^3\theta\, d\theta \int_0^{2\pi} [\cos(\phi + \delta/2)]_+ \,[\cos(\phi - \delta/2)]_+ \, d\phi\\
&= \int_0^\pi (1 - \cos^2\theta)\,\sin\theta\, d\theta \int_{-\pi/2+\delta/2}^{\pi/2-\delta/2} \cos(\phi + \delta/2)\,\cos(\phi - \delta/2)\, d\phi\\
&= \int_{-1}^1 (1 - u^2)\, du \int_{-\pi/2+\delta/2}^{\pi/2-\delta/2} \frac{\cos(2\phi) + \cos\delta}{2}\, d\phi\\
&= \frac{2}{3}\,[(\pi - \delta)\,\cos\delta + \sin\delta].
\end{aligned}$$

Lemma A.2: The following results hold for all $\beta > 0$:

$$\int_0^\infty r^3\, e^{-\beta^2 \frac{r^2}{2}}\, dr = \frac{2}{\beta^4}, \quad \int_0^\infty r^4\, e^{-\beta^2 \frac{r^2}{2}}\, dr = \sqrt{\frac{\pi}{2}}\,\frac{3}{\beta^5}, \quad \int_0^\infty r^5\, e^{-\beta^2 \frac{r^2}{2}}\, dr = \frac{8}{\beta^6},$$

$$\int_0^\pi \Big[\cos\theta(\pi-\theta)+\sin\theta\Big]\,\sin\theta\, d\theta = 3\frac{\pi}{4}, \quad \int_0^\pi \Big[\cos\theta(\pi-\theta)+\sin\theta\Big]\,\cos\theta\,\sin\theta\, d\theta = \frac{\pi}{3}.$$

Proof: All those formulas can be obtained thanks to suitable integrations by parts.

Lemma A.3: For all $a \in \mathbb{R}^3$,

$$\begin{pmatrix} q_0(|a|) \\ q_2(|a|) \end{pmatrix} := \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ |y|^2 \end{pmatrix} |a-y|\, e^{-\frac{|y|^2}{2}}\, \frac{dy}{(2\pi)^{3/2}}$$
$$= \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 2|a| I_2(|a|) + \frac{2}{3}|a|^{-1} I_4(|a|) + 2J_3(|a|) + \frac{2}{3}|a|^2 J_1(|a|) \\ 2|a| I_4(|a|) + \frac{2}{3}|a|^{-1} I_6(|a|) + 2J_5(|a|) + \frac{2}{3}|a|^2 J_3(|a|) \end{pmatrix},$$

where (for $k \in \mathbb{N}$, $x \in \mathbb{R}_+$)

$$I_k(x) = \int_0^x t^k\, e^{-t^2/2}\, dt, \qquad J_k(x) = \int_x^\infty t^k\, e^{-t^2/2}\, dt.$$

Moreover

$$q(a) := \int_{\mathbb{R}^3} (a-y)\, |a-y|\, e^{-\frac{|y|^2}{2}}\, \frac{dy}{(2\pi)^{3/2}} = \bar{q}(|a|)\, a,$$

where

$$\bar{q}(|a|) = \frac{1}{\sqrt{2\pi}} \Big\{ 2|a| I_2(|a|) + \frac{4}{3}|a|^{-1} I_4(|a|) - \frac{2}{15}|a|^{-3} I_6(|a|) + \frac{8}{15}|a|^2 J_1(|a|) + \frac{8}{3} J_3(|a|) \Big\}.$$

Proof: We first notice that

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} |a-y|\, e^{-\frac{|y|^2}{2}}\, \frac{dy}{2\pi} = \int_0^\infty \int_{-1}^1 \begin{pmatrix} 1 \\ r\, u\, \frac{a}{|a|} \\ r^2 \end{pmatrix} (r^2 + |a|^2 - 2r|a|u)^{1/2}\, r^2\, du dr$$
$$= \int_0^\infty \begin{pmatrix} Y(|a|,r) \\ r\, \frac{a}{|a|}\, Z(|a|,r) \\ r^2\, Y(|a|,r) \end{pmatrix} dr,$$

where (for $r > 0$)

$$Y(|a|,r) = \frac{1}{3|a|r} \Big[(r+|a|)^3 - |r-|a||^3\Big],$$

$$Z(|a|,r) = \frac{2}{3}\,\frac{1}{(2|a|r)^2} \Big[(r^2+|a|^2)\,[(r+|a|)^3 - |r-|a||^3] - \frac{3}{5}\,[(r+|a|)^5 - |r-|a||^5]\Big].$$

We see that when $r > |a|$,

$$Y(|a|,r) = 2r + \frac{2}{3}\frac{|a|^2}{r}, \qquad Z(|a|,r) = -\frac{2}{3}|a| + \frac{2}{15}\frac{|a|^3}{r^2},$$

and when $r \le |a|$,

$$Y(|a|,r) = 2|a| + \frac{2}{3}\frac{r^2}{|a|}, \qquad Z(|a|,r) = -\frac{2}{3}r + \frac{2}{15}\frac{r^3}{|a|^2}.$$

Then we observe that using the change of variable $w = \frac{r^2+|a|^2}{2|a|r} - u$, one gets

$$\int_{-1}^{1} (r^2 + |a|^2 - 2r|a|u)^{1/2}\, du = (2|a|r)^{1/2} \int_{\frac{r^2+|a|^2}{2|a|r}-1}^{\frac{r^2+|a|^2}{2|a|r}+1} w^{1/2}\, dw,$$

$$= \frac{1}{3|a|r} \Big[(r+|a|)^3 - |r-|a||^3 \Big],$$

and

$$\int_{-1}^{1} (r^2 + |a|^2 - 2r|a|u)^{1/2}\, u\, du = (2|a|r)^{1/2} \int_{\frac{r^2+|a|^2}{2|a|r}-1}^{\frac{r^2+|a|^2}{2|a|r}+1} \left[\frac{r^2+|a|^2}{2|a|r}\, w^{1/2} - w^{3/2} \right] dw$$

$$= \frac{2}{3}\, \frac{1}{(2|a|r)^2} \Big[(r^2+|a|^2)\, [(r+|a|)^3 - |r-|a||^3] - \frac{3}{5}\, [(r+|a|)^5 - |r-|a||^5] \Big].$$

The result is obtained by cutting the integral between the part when $r > |a|$ and the part when $r \le |a|$. In particular,

$$\int_{\mathbb{R}^3} y\, |a-y|\, e^{-\frac{|y|^2}{2}}\, \frac{dy}{2\pi} = \frac{2}{15}|a|^{-3} a I_6(|a|) - \frac{2}{3}|a|^{-1} a I_4(|a|) + \frac{2}{15}|a|^2 a J_1(|a|) - \frac{2}{3} a J_3(|a|).$$

Lemma A.4: We define for $k \in \mathbb{N}$ the integrals:

$$I_k(x) = \int_0^x t^k\, e^{-t^2/2}\, dt, \qquad J_k(x) = \int_x^\infty t^k\, e^{-t^2/2}\, dt.$$

Then the following formulas hold:

$$J_1(x) = e^{-\frac{x^2}{2}}, \qquad J_3(x) = (x^2+2)\, e^{-\frac{x^2}{2}}, \qquad J_5 = (x^4 + 4x^2 + 8)\, e^{-\frac{x^2}{2}},$$

$$I_2(x) = I_0(x) - x\, e^{-\frac{x^2}{2}}, \qquad I_4(x) = 3\, I_0(x) - x^3\, e^{-\frac{x^2}{2}} - 3x\, e^{-\frac{x^2}{2}},$$

$$I_6(x) = 15 I_0(x) - x^5\, e^{-\frac{x^2}{2}} - 5x^3\, e^{-\frac{x^2}{2}} - 15x\, e^{-\frac{x^2}{2}}.$$

Proof: These formulas are directly obtained by successive integrations by parts.

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