Smoothing Effects for Classical Solutions of the Full Landau Equation

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Abstract: In this work, we consider the smoothness of the solutions to the full Landau equation. In particular, we prove that any classical solutions (such as the ones obtained by Guo in a "close to equilibrium" setting) become immediately smooth with respect to all variables. This shows that the Landau equation is a nonlinear and nonlocal analog of an hypoelliptic equation.

1 Introduction

In this paper, we study the smoothness of the solutions to the Landau equation of plasma physics:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \left\{ \int_{\mathbf{R}^3} a(v-v_*)[f(v_*)\nabla_v f(v) - \nabla_v f(v_*) f(v)] dv_* \right\}, \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{1}$$

where $f(t,x,v) \geq 0$ is the (spatially periodic) distribution function in the phase space of charged particles which at time $t \geq 0$ and point $x \in \mathbf{T}^3 = [-\pi,\pi]^3$ move with velocity $v \in \mathbf{R}^3$. The nonnegative matrix a is given by the formula

$$a_{ij}(v) = \left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right)|v|^{\gamma+2}, \qquad \gamma \in [-3,1]. \tag{2}$$

The original (and most important) Landau collision operator is obtained when $\gamma = -3$ and corresponds to the Coulomb interaction (Cf. [20] and [8]). For $\gamma \in]-3,1]$, equation (1) is a limit model (when collisions become grazing) of the Boltzmann equation (Cf. [11], [2] and [9]). Traditionally, one calls hard potentials the case when $\gamma \in]0,1]$, Maxwellian molecules the case when $\gamma = 0$, moderately soft potentials the case when $\gamma \in]-2,0[$, and very soft potentials the case when $\gamma \in]-3,-2]$. The range $\gamma \in]1,+\infty[$ does not correspond to a physical situation, so we do not consider it in this paper. However, the results presented in this work still hold in this case (the proof being the same as that of the case $\gamma \in [-2,1]$).

The smoothness of the solutions to the spatially homogeneous Landau equation (that is, the solutions of equation (1) which do not depend on x) has been investigated by Arsen'ev and Buryak (Cf. [3]) in the Coulomb case and by Desvillettes and Villani (Cf. [13]) in the case of hard potentials. In those works, it is shown that the Landau equation behaves (from the point of view of smoothness) as a nonlinear and nonlocal version of the heat equation : roughly speaking, smoothness (in the variable v) is immediately produced even if it does not exist initially. This behavior is close to that of the (spatially homogeneous) Boltzmann equation without angular cutoff, but radically different from that of the Boltzmann equation with angular cutoff (see for example [10] and the references therein for precise statements). In this last case, propagation of regularity as well as singularities (in the variable v) occurs, thanks to the properties of the positive part of Boltzmann operator (Cf. [21], [25], [5] and [22]).

We are now interested in the solutions to the full (that is, spatially inhomogeneous) Landau equation (1), which is much closer to the real physics than the spatially homogeneous one. However, there is a new and important difficulty: for general initial data, the solutions to the full Boltzmann and Landau equations which have been builded up to now are very weak. They are called renormalized for the cutoff Boltzmann equation (Cf. [15] and [21]), and renormalized with a defect measure for the non cutoff Boltzmann or Landau equation (Cf. [24], [1] and [2]). As a consequence, it is difficult to study the smoothness of their solutions, though strong compactness and strong stability properties can sometimes be proven (Cf. [21], [1] and [2]).

This obstruction has been overcome in [7] in the case of the cutoff Boltzmann equation, thanks to the use of "close to vacuum" solutions corresponding to "small" initial data (Cf. [19], [23] and the references therein for the description of such solutions). It is proven that the solution $f(t,\cdot)$ at time t has exactly the same regularity as the initial datum $f(0,\cdot)$, as a function of both variables x and v. The proof uses a combination of the properties of regularization of the positive part of Boltzmann's operator and of averaging lemmas (Cf. [17]). Note that a recent extension of this work to solutions of the Vlasov-Poisson-Boltzmann equation (Cf. [4]) exists. Note also that a corresponding theorem for solutions close to Maxwellian could certainly be obtained.

In [18], Y. Guo presented the global existence of classical solutions to (1) in the case of initial data near Maxwellians. Thanks to this breakthrough, it is now possible to improve our knowledge of the smoothness of the solutions to eq. (1); this is what we do in this work.

We now explain our method of proof and what makes it original. By using an energy method on the equation satisfied by some (higher order) derivative of the solution of the Landau equation, it is possible to prove that one derivative with respect to the variable v is gained (such an estimate exploits the "elliptic part" of the equation, and is close to the estimates obtained in [13] for the spatially homogeneous equation). Then, instead of using a technique directly based on the brackets $[v\cdot\nabla_x, \partial_{v_j}]$, we use an averaging lemma and keep track of the dependance of some fractional norm of an average (with respect to v) of f in terms of the averaging function. An interpolation enables to gain a fractional derivative (more precisely, 1/20 of a derivative) with respect to the variable x for the function itself (and not its average). This method has already been used for example in [7]. However, there is now a new difficulty, and its resolution is the main originality of this paper: since only fractional derivatives are gained, one has to perform the energy estimates (and the estimates based on the averaging lemmas) for weighted finite differences of derivatives of f. As a consequence, one has to add a new (space) variable and to perform L^2 estimates with respect to it.

For notational simplicity, we omit the integrating domains $\mathbf{T}^3$ and $\mathbf{R}^3$, which correspond to variables x and variable v respectively. For example, we write $L^2_{x,v}$ instead of $L^2_x(\mathbf{T}^3; L^2_v(\mathbf{R}^3))$. For any integer $N \geq 0$, we define the Sobolev space

$$H^N_{x,v} = \left\{ f(x,v) : \sum_{|\alpha|+|\beta|\leq N} \|\partial^\alpha_x \partial^\beta_v f\|_{L^2_{x,v}} < +\infty \right\},$$

and for $N, s \geq 0$, we define the weighted Sobolev space

$$H^{N,s}_{x,v} = \Big\{ f(x,v) : \sum_{|\alpha|+|\beta|\leq N} \|(\partial_x^\alpha \partial_v^\beta f)\,(1+|v|^2)^{\frac{s}{2}}\|_{L^2_{x,v}} < +\infty \Big\},$$

where the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ with $x = (x_1, x_2, x_3)$. The notations for β are the same. It is obvious that $H^{N,0}_{x,v} = H^N_{x,v}$. We also define $H^\infty_{x,v}$ and $H^{\infty,s}_{x,v}$ by

$$H^\infty_{x,v} = \bigcap_{N\geq 0} H^N_{x,v}, \qquad H^{\infty,s}_{x,v} = \bigcap_{N\geq 0} H^{N,s}_{x,v}.$$

The conservation of mass, momentum and energy in equation (1) can be formulated (at the formal level) as

$$\frac{d}{dt} \int_{\mathbf{T}^3 \times \mathbf{R}^3} f(t,x,v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dxdv \equiv 0. \tag{3}$$

We introduce the normalized Maxwellian $\mu = e^{-|v|^2}$, and the standard perturbation $F := F(t,x,v)$ with respect to μ as

$$f = \mu + \sqrt{\mu} F. \tag{4}$$

By assuming that $f_0(x,v)$ has the same mass, momentum and energy as the Maxwellian μ, we can rewrite the conservation laws as

$$\int_{\mathbf{T}^3 \times \mathbf{R}^3} F(t,x,v)\sqrt{\mu} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dxdv \equiv 0. \tag{5}$$

The result of [18] can be summarized as follows

Theorem 1.1 *(**Y. Guo**) Let $\gamma \in [-3,1]$ and $N \geq 8$. Assume that $F_0 := F(0,\cdot,\cdot)$ satisfies (5) and that $f_0 := \mu + \sqrt{\mu}\, F_0$ is nonnegative. There exists a constant $\kappa_0 > 0$, such that if $\|F_0\|_{H^N_{x,v}} \leq \kappa_0$, eq. (1) has a unique (global) nonnegative classical solution $f := f(t,x,v)$. Moreover, using the notation (4), there is a constant C_0 (depending on γ, N and κ_0) such that*

1. *if $\gamma \in [-2,1]$,*
$$\|F\|_{L^\infty_t([0,+\infty[;H^N_{x,v})} \leq C_0 \, \|F_0\|_{H^N_{x,v}}, \tag{6}$$

2. *if $\gamma \in [-3,-2[$,*
$$\sup_{t\in[0,+\infty[} \sum_{|\alpha|+|\beta|\leq N} \|(\partial_x^\alpha \partial_v^\beta F)\,(1+|v|)^{(\gamma+2)|\beta|}\|_{L^2_{x,v}} \leq C_0 \sum_{|\alpha|+|\beta|\leq N} \|(\partial_x^\alpha \partial_v^\beta F_0)\,(1+|v|)^{(\gamma+2)|\beta|}\|_{L^2_{x,v}}. \tag{7}$$

Note that from (4), (6) and (7), we can easily get that for any $s \geq 0$, there exist constants $C_1 > 0$ (depending on C_0 and s) and $C_2 > 0$(depending on s) such that

$$\|f\|_{L^\infty_t([0,+\infty[;H^{N,s}_{x,v})} \leq C_1 \|F_0\|_{H^N_{x,v}} + C_2. \tag{8}$$

Our main result shows that (up to some weights in the velocity variable), any (bounded below) classical solution to eq. (1) belonging to $H^8_{x,v}$ lies in fact in $C^\infty_{x,v}$ for any time $t > 0$. More precisely, we shall suppose that our solution f to eq. (1) satisfies the

Assumption A : We suppose that $f : \mathbf{R}_+ \times \mathbf{T}^3 \times \mathbf{R}^3 \to \mathbf{R}_+$ lies in $L^\infty_t([0,+\infty[; \cap_{s\geq 0} H^{8,s}_{x,v})$. Moreover, we assume that f is bounded below in the following weak sense:

$$\exists K > 0, \forall (t,x,v) \in \mathbf{R}_+ \times \mathbf{T}^3 \times \mathbf{R}^3, \qquad \inf_{\xi \in \mathbf{S}^2} \sum_{ij} [a_{ij} *_v f(t,x,\cdot)](v)\, \xi_i \xi_j \geq K\, (1+|v|^2)^{\gamma/2}. \tag{9}$$

Then, our main result reads:

Theorem 1.2 *Let $\gamma \in [-3,1]$, and $f : \mathbf{R}_+ \times \mathbf{T}^3 \times \mathbf{R}^3 \to \mathbf{R}_+$ be a (classical) solution of eq. (1) satisfying assumption A. Then, for any $\tau > 0$,*

$$f \in W^{\infty,\infty}_t([\tau, +\infty[; \cap_{s\geq 0} H^{\infty;s}_{x,v}(\mathbf{T}^3 \times \mathbf{R}^3)). \tag{10}$$

Next, we notice that this result can be applied to the solutions of the Landau equation obtained by Guo thanks to Theorem 1.1 (provided that $\kappa_0 > 0$ is small enough). In particular, this shows that the result of Theorem 1.2 is not empty. More precisely, we have the

Theorem 1.3 *Let $\gamma \in [-3,1]$. Assume that $F_0 := F(0,\cdot,\cdot)$ satisfies (5) and that $f_0 := \mu + \sqrt{\mu}\, F_0$ is nonnegative. Then, there exists a constant $\epsilon_0 \in]0, \kappa_0[$ such that if $\|F_0\|_{H^s_{x,v}} \leq \epsilon_0$, the unique classical nonnegative solution to equation (1) given by Theorem 1.1 satisfies (for any $\tau > 0$):*

$$f \in W^{\infty,\infty}_t([\tau, +\infty[; \cap_{s\geq 0} H^{\infty;s}_{x,v}(\mathbf{T}^3 \times \mathbf{R}^3)). \tag{11}$$

This result shows that the smoothing effect proven (for example in [13]) for the spatially homogeneous Landau equation extends to the full (spatially inhomogeneous) Landau equation, for all variables t, x, v. Theorem 1.2 can be understood in this way: equation (1) is a nonlocal and nonlinear version of hypoelliptic Fokker-Planck equations such as those described in [14]. This behavior was also observed (though only for a marginal gain of smoothness) for a toy model of the Boltzmann equation without angular cutoff (Cf. [12]). We finally emphasize that the smoothing property which is shown in Theorems 1.2 and 1.3 holds uniformly when the time goes to infinity.

The rest of the paper is devoted to the proof of Theorems 1.2 and 1.3. In section 2, we prove that Theorem 1.3 is a consequence of Theorem 1.2. Section 3 is devoted to the establishment of a few lemmas which shall be used systematically in the sequel. Then, Theorem 1.2 is proven in sections 4 and 5 in the case when $\gamma \in [-2,1]$. Section 4, in which one step of the main induction argument for Theorem 1.2 is proven, is itself divided in four subsections in which one "elementary" estimate is proven, and a subsection into which those estimates are combined. Section 6 deals with the case $\gamma \in [-3,-2[$.

2 A weak lower bound

We begin this section with a few definitions.

Let $b_i(v) = \sum_j \partial_{v_j} a_{ij}(v)$. In the sequel, we shall systematically write

$$\bar{a}_{ij}(t,x,v) = [a_{ij} *_v f(t,x,\cdot)](v), \quad \text{and} \quad \bar{b}_i(t,x,v) = [b_i *_v f(t,x,\cdot)](v).$$

Then, equation (1) can be rewritten as

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\bar{a} \nabla_v f + \bar{b} f), \tag{12}$$

where $\bar{a}$ (resp. $\bar{b}$) is the matrix (resp. the vector) with coefficients $(\bar{a}_{ij})_{ij}$ (resp. $(\bar{b}_i)_i$).

We then show that the lower bound (9) of assumption A is satisfied by the solution of the Landau equation given by Theorem 1.1, provided that κ_0 is small enough. Indeed, we state the

Proposition 2.1 *Let $\gamma > 0$, $N \geq 8$ and f be a nonnegative classical solution of equation (1) given by Theorem 1.1. If $||F_0||_{H^N_{x,v}} \leq \epsilon_0$ for ϵ_0 small enough, there exists a constant $K > 0$, depending on N, ϵ_0 and γ, such that for any $t \in \mathbf{R}_+$, $x \in \mathbf{T}^3$, $v \in \mathbf{R}^3$ and $\xi \in \mathbf{R}^3$,*

$$\sum_{ij} \bar{a}_{ij}(t,x,v)\xi_i\xi_j \geq K\,(1+|v|^2)^{\frac{\gamma}{2}}|\xi|^2. \tag{13}$$

As a consequence, Theorem 1.3 is implied by Theorem 1.2.

Proof of Proposition 2.1 We use the method of proof of Proposition 4 in [13]. From (4), (6), (7) and the assumption $\|F_0\|_{H^s_{x,v}} \leq \epsilon_0$, we get by Sobolev's embedding that there is a constant $S > 0$, depending on C_0 (and thus on γ, N and ϵ_0), such that

$$\|\sqrt{\mu}F\|_{L^\infty_t([0,+\infty[;L^\infty_{x,v})} \leq S\epsilon_0, \tag{14}$$

which implies that for $|v| \leq R$ (R will be chosen later),

$$f \geq e^{-R^2} - S\epsilon_0.$$

Choosing $\epsilon_0 < (S+1)^{-1}$ and $R = R_0 = \sqrt{-\log((S+1)\epsilon_0)}$, we get

$$f(t,x,v) \geq \epsilon_0, \quad \text{for} \quad |v| \leq R_0. \tag{15}$$

For $\xi \in \mathbf{R}^3$, $|\xi| = 1$ and $0 < \theta < \frac{\pi}{2}$, let us set

$$D_{\theta,\xi}(v) = \left\{ v_* \in \mathbf{R}^3 : \left| \frac{v - v_*}{|v - v_*|} \cdot \xi \right| \geq \cos\theta \right\}.$$

Note that $D_{\theta,\xi}(v)$ is the cone centered at v, of axis directed by ξ, and of angle θ.

For all $v_* \in \mathbf{R}^3 \setminus D_{\theta,\xi}(v)$,

$$\sum_{ij} a_{ij}(v-v_*)\xi_i\xi_j = |v-v_*|^{\gamma+2} \sum_{ij} \left[\delta_{ij} - \frac{(v-v_*)_i(v-v_*)_j}{|v-v_*|^2} \right] \xi_i\xi_j \geq |v-v_*|^{\gamma+2}\sin^2\theta.$$

Then from (15) we have that for all $v \in \mathbf{R}^3$ and $\theta \in]0, \frac{\pi}{2}[$,

$$\begin{aligned} \sum_{ij} \bar{a}_{ij}(t,x,v)\xi_i\xi_j &\geq \int_{\mathbf{R}^3 \setminus D_{\theta,\xi}(v)} \mathbf{1}_{|v_*| \leq R_0} f(t,x,v_*) \sum_{ij} a_{ij}(v-v_*)\xi_i\xi_j dv_* \\ &\geq \epsilon_0 \int_{B \setminus D_{\theta,\xi}(v)} |v-v_*|^{\gamma+2} dv_* \sin^2\theta, \end{aligned} \tag{16}$$

where $B = B(0, R_0)$.

When $|v| \geq 2R_0$, we have (when $\gamma + 2 \geq 0$ as well as when $\gamma + 2 < 0$) the estimate $\mathbf{1}_{|v_*| \leq R_0}|v - v_*|^{\gamma+2} \geq C|v|^{\gamma+2}$. Then, we get

$$\sum_{ij} \bar{a}_{ij}(t,x,v)\xi_i\xi_j \geq \epsilon_0\,(C|v|)^{\gamma+2}\sin^2\theta \Big(|B| - |B \cap D_{\theta,\xi}(v)| \Big).$$

According to (46) of [13], we have

$$|B \cap D_{\theta,\xi}(v)| \leq 2\pi R_0(|v| + R_0)^2 \tan^2\theta, \tag{17}$$

so that

$$\sum_{ij} \bar{a}_{ij}(t,x,v)\xi_i\xi_j \geq \epsilon_0\,(C|v|)^{\gamma+2}\sin^2\theta \left(\frac{4}{3}\pi R_0^3 - 2\pi R_0\,(C|v|)^2 \tan^2\theta \right).$$

We finally choose θ in such a way that $\tan^2\theta = \frac{R_0^2}{(C|v|)^2\cdot 3}\left(\le \frac{1}{27}\right)$. Then,

$$\begin{aligned}\sum_{ij}\bar{a}_{ij}(t,x,v)\xi_i\xi_j &\ge \epsilon_0\,(C|v|)^{\gamma+2}\cos^2\theta\left[\frac{R_0^2}{(C|v|)^2\cdot 3}\right]\frac{2}{3}\pi R_0^3\\ &\ge C\epsilon_0 R_0^5|v|^{\gamma}. \qquad (18)\end{aligned}$$

On the other hand, when $|v|\le 2R_0$, we observe that

$$\int_{B\setminus D_{\theta,\xi}(v)}|v-v_*|^{\gamma+2}dv_* \ge \int_{B\setminus D_{\theta,\xi}(v)} 1_{|v-v_*|\ge\lambda}|v-v_*|^{\gamma+2}dv_*$$

$$\ge \inf(\lambda^{\gamma+2},(3R_0)^{\gamma+2})\left|\,|B|-|B(v,\lambda)|-|D_{\theta,\xi}(v)|\,\right|.$$

According to (16), choosing λ and θ small enough, we see that

$$\sum_{ij}\bar{a}_{ij}(t,x,v)\xi_i\xi_j \ge C > 0. \qquad (19)$$

Estimates (18) and (19) together yield (13).

We conclude by noticing that thanks to estimates (8) and (13), assumption A is satisfied by the solutions of the Landau equation given by Theorem 1.1 (for κ_0 small enough). As a consequence, Theorem 1.3 is a consequence of Theorem 1.2, which concludes Proposition 2.1.

3 Estimates on the diffusion coefficients

In this section, we present an estimate for the coefficients $\bar{a}_{ij}$ and $\bar{b}_i$ which will be used repeatedly in the proof of Theorem 1.2:

Lemma 3.1 *Let $\gamma\in[-3,1]$. Then, there exists a positive constant C which depends only on γ such that for all nonnegative $f := f(t,x,v)$ for which the derivatives are defined,*
1). *for any multi-indices α,β and Ω interval included in $[0,+\infty[$, we have*

$$\begin{aligned}\|\partial_x^\alpha\partial_v^\beta\bar{a}_{ij}(t,x,\cdot)\|_{L_t^\infty(\Omega;L_x^\infty)}(v) &\le C(1+|v|^2)^{\frac{\gamma+2}{2}}\|(\partial_x^\alpha\partial_v^\beta f)(1+|v|^2)^{\frac{\gamma+4}{2}}\|_{L_t^\infty(\Omega;H^2_{x,v})},\\ &\qquad \text{if } \gamma\in[-2,1], \qquad (20)\\ \|\partial_x^\alpha\partial_v^\beta\bar{a}_{ij}(t,x,\cdot)\|_{L_t^\infty(\Omega;L_x^\infty)}(v) &\le C\|(\partial_x^\alpha\partial_v^\beta f)(1+|v|^2)\|_{L_t^\infty(\Omega;H^2_{x,v})},\\ &\qquad \text{if } \gamma\in[-3,-2[; \qquad (21)\end{aligned}$$

2). *for any multi-indices α,β and Ω interval included in $[0,+\infty[$, we have*

$$\begin{aligned}\|\partial_x^\alpha\partial_v^\beta\bar{b}_i(t,x,\cdot)\|_{L_t^\infty(\Omega;L_x^\infty)}(v) &\le C(1+|v|^2)^{\frac{\gamma+2}{2}}\sum_{|\sigma|=1}\|(\partial_x^\alpha\partial_v^{\beta+\sigma} f)(1+|v|^2)^{\frac{\gamma+4}{2}}\|_{L_t^\infty(\Omega;H^2_{x,v})},\\ &\qquad \text{if } \gamma\in[-2,1], \qquad (22)\\ \|\partial_x^\alpha\partial_v^\beta\bar{b}_i(t,x,\cdot)\|_{L_t^\infty(\Omega;L_x^\infty)}(v) &\le C\sum_{|\sigma|=1}\|(\partial_x^\alpha\partial_v^{\beta+\sigma} f)(1+|v|^2)\|_{L_t^\infty(\Omega;H^2_{x,v})},\\ &\qquad \text{if } \gamma\in[-3,-2[. \qquad (23)\end{aligned}$$

Proof of Lemma 3.1 We first prove (20) and (21). Write

$$\partial_x^\alpha \partial_v^\beta \bar{a}_{ij}(v) = \partial_x^\alpha \partial_v^\beta (a_{ij} *_v f)(v) = a_{ij} *_v (\partial_x^\alpha \partial_v^\beta f)(v).$$

By Sobolev's embedding and Minkowski's inequality, we get

$$\begin{aligned} \|\partial_x^\alpha \partial_v^\beta \bar{a}_{ij}\|_{L_t^\infty(\Omega;L_x^\infty)}(v) &= \left\| \int_{\mathbf{R}^3} a_{ij}(v-v_*)\partial_x^\alpha \partial_v^\beta f(v_*)dv_* \right\|_{L_t^\infty(\Omega;L_x^\infty)} \\ &\leq C\sup_{t\in\Omega}\left[\sum_{|\sigma|\leq 2}\int_{\mathbf{T}^3}\left(\int_{\mathbf{R}^3} a_{ij}(v-v_*)\partial_x^{\alpha+\sigma}\partial_v^\beta f(v_*)dv_*\right)^2 dx\right]^{1/2} \\ &\leq C\sup_{t\in\Omega}\int_{\mathbf{R}^3}|a_{ij}(v-v_*)|\left(\sum_{|\sigma|\leq 2}\int_{\mathbf{T}^3}|\partial_x^{\alpha+\sigma}\partial_v^\beta f(v_*)|^2 dx\right)^{1/2} dv_*. \end{aligned} \tag{24}$$

If $\gamma \in [-2, 1]$, we see that

$$|a_{ij}(v-v_*)| \leq 2|v-v_*|^{\gamma+2} \leq C(1+|v|^2)^{\frac{\gamma+2}{2}}(1+|v_*|^2)^{\frac{\gamma+2}{2}},$$

so that

$$\begin{aligned} &\|\partial_x^\alpha \partial_v^\beta \bar{a}_{ij}\|_{L_t^\infty(\Omega;L_x^\infty)}(v) \\ \leq\ & C(1+|v|^2)^{\frac{\gamma+2}{2}}\sup_{t\in\Omega}\int_{\mathbf{R}^3}(1+|v_*|^2)^{\frac{\gamma+2}{2}}\left(\sum_{|\sigma|\leq 2}\int_{\mathbf{T}^3}|\partial_x^{\alpha+\sigma}\partial_v^\beta f(v_*)|^2 dx\right)^{1/2} dv_* \\ \leq\ & C(1+|v|^2)^{\frac{\gamma+2}{2}}\|(\partial_x^\alpha\partial_v^\beta f)(1+|v|^2)^{\frac{\gamma+4}{2}}\|_{L_t^\infty(\Omega;H_{x,v}^2)}. \end{aligned} \tag{25}$$

As for the case $\gamma \in [-3, -2[$, we deduce from (24) that

$$\|\partial_x^\alpha \partial_v^\beta \bar{a}_{ij}\|_{L_t^\infty(\Omega;L_x^\infty)}(v) \leq C\sup_{t\in\Omega}\int_{\mathbf{R}^3}|v-v_*|^{\gamma+2}\left(\sum_{|\sigma|\leq 2}\int_{\mathbf{T}^3}|\partial_x^{\alpha+\sigma}\partial_v^\beta f(v_*)|^2 dx\right)^{1/2} dv_*.$$

Using Hölder's inequality, we get

$$\begin{aligned} &\|\partial_x^\alpha \partial_v^\beta \bar{a}_{ij}\|_{L_t^\infty(\Omega;L_x^\infty)}(v) \\ \leq\ & C\left(\int_{\mathbf{R}^3}|v-v_*|^{2(\gamma+2)}(1+|v_*|^2)^{-2}dv_*\right)^{1/2}\|(\partial_x^\alpha\partial_v^\beta f)(1+|v|^2)\|_{L_t^\infty(\Omega;H_{x,v}^2)}. \end{aligned} \tag{26}$$

We now estimate the term $\int_{\mathbf{R}^3}|v-v_*|^{2(\gamma+2)}(1+|v_*|^2)^{-2}dv_*$. For $|v|\leq \frac{1}{2}$, we see that

$$1+|v-v_*| \geq 1+|v_*|-|v| \geq \frac{1}{2}+|v_*|,$$

and we can get

$$\begin{aligned} \int_{\mathbf{R}^3}|v-v_*|^{2(\gamma+2)}(1+|v_*|^2)^{-2}dv_* &= \int_{\mathbf{R}^3}|v_*|^{2(\gamma+2)}(1+|v-v_*|^2)^{-2}dv_* \\ &\leq C\int_{\mathbf{R}^3}|v_*|^{2(\gamma+2)}(\frac{1}{2}+|v_*|)^{-4}dv_* \leq C. \end{aligned} \tag{27}$$

For $|v| \geq \frac{1}{2}$, we write

$$\int_{\mathbf{R}^3} |v-v_*|^{2(\gamma+2)}(1+|v_*|^2)^{-2}dv_* = \int_{|v-v_*|\geq \frac{1+|v|}{4}} |v-v_*|^{2(\gamma+2)}(1+|v_*|^2)^{-2}dv_* + \int_{|v-v_*|\leq \frac{1+|v|}{4}} |v-v_*|^{2(\gamma+2)}(1+|v_*|^2)^{-2}dv_*. \quad (28)$$

The first term in the r.h.s of (28) is bounded by

$$\left(\frac{1+|v|}{4}\right)^{2(\gamma+2)} \int_{\mathbf{R}^3} (1+|v_*|^2)^{-2}dv_* \leq C. \quad (29)$$

Noticing that if $|v-v_*| \leq \frac{1+|v|}{4}$, then $|v_*| \geq \frac{|v|}{4}$, we can bound the second term in the r.h.s of (28) by

$$\left(1+\frac{|v|}{4}\right)^{-4} \int_{|v-v_*|\leq \frac{1+|v|}{4}} |v-v_*|^{2(\gamma+2)}dv_* \leq C\left(1+\frac{|v|}{4}\right)^{-4}\left(\frac{1+|v|}{4}\right)^{2(\gamma+2)+3} \leq C. \quad (30)$$

Combining (26) through (30), we obtain (21).

Observing that $\bar{b}_i(v) = \sum_j \partial_{v_j}\bar{a}_{ij}(v)$, we get $\partial_x^\alpha \partial_v^\beta \bar{b}_i(v) = \sum_j \Big[a_{ij} *_v (\partial_x^\alpha \partial_v^\beta \partial_{v_j} f)\Big](v)$. As a consequence, we can easily get (22) and (23) by following the proofs of (20) and (21) respectively.

4 Treatment of the Case $\gamma \in [-2, 1]$

In this section, we shall consider eq. (1) in the case when $\gamma \in [-2, 1]$. We shall use (in section 5) an induction on the number of derivatives (in x and v) that can be controlled. The following proposition shows how to get one step of this induction.

Proposition 4.1 *Let $\gamma \in [-2, 1]$, $N \geq 8$ be a given integer, and let f be a nonnegative solution of eq. (1) such that assumption A holds for a given constant K. We suppose that for any $T \in]0, +\infty[$ and $s \geq 0$, $\|f\|_{L_t^\infty([0,T];H_{x,v}^{N,s})} \leq \bar{K}$ for some constant $\bar{K} \equiv \bar{K}(s,T,N,\gamma)$. Then for any $T > 0$, $t_* \in]0, T[$ and $s \geq 0$, there is a positive constant $\widetilde{C}_0$, which depends on N, s, γ, T, K, $\bar{K}$ and t_*, such that*

$$\|f\|_{L_t^\infty([t_*,T];H_{x,v}^{N+1,s})} \leq \widetilde{C}_0. \quad (31)$$

Since the proof of Proposition 4.1 is a bit long, we shall divide it into five parts. The first one is devoted to the study of the smoothness of f with respect to v.

4.1 Gain of one derivative in v starting from a Sobolev space whose index is an integer

We prove in this subsection the following

Lemma 4.1 *Let $\gamma \in [-2, 1]$, $N \geq 8$ be a given integer, and let f be a smooth nonnegative solution of equation (1) such that assumption A holds for a given constant K. We suppose that for any $T > 0$, $s \geq 0$,*

$$||f(0,\cdot,\cdot)||_{H_{x,v}^{N,s}} \leq K_0,$$

$$||f||_{L_t^\infty([0,T];H_{x,v}^{N-1,s})} \leq K_0 \quad \text{and} \quad ||f||_{L_t^2([0,T];H_{x,v}^{N,s})} \leq K_0,$$

where $K_0 \equiv K_0(s,T,\gamma,N,K)$ is some constant.

Then, there exists a constant $\widetilde{C}_1$, which depends on N, s, γ, T, K_0 and K, such that

$$\sup_{t\in[0,T]} \int_{\mathbf{T}^3\times\mathbf{R}^3} |h|^2 dxdv + \int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} |\nabla_v h|^2 dtdxdv \le \widetilde{C}_1, \tag{32}$$

where

$$h = (\partial_x^\alpha \partial_v^\beta f)\,(1+|v|^2)^{s/2}, \tag{33}$$

with $|\alpha| + |\beta| \le N$.

Proof of Lemma 4.1 If α and β are two multi-indices, we say that $\beta \le \alpha$ when $\beta_j \le \alpha_j$ for $1 \le j \le 3$. We also denote $\alpha! = \alpha_1!\alpha_2!\alpha_3!$ and, if $\beta \le \alpha$,

$$C_\beta^\alpha = \frac{\alpha!}{\beta!(\alpha-\beta)!} = C_{\beta_1}^{\alpha_1} C_{\beta_2}^{\alpha_2} C_{\beta_3}^{\alpha_3}.$$

We finally denote by δ_i the multi-index whose ith component is 1, and the others are 0.

Since equation (1) is equivalent to (12), we know from Leibniz's formula that h satisfies the following equation (using Einstein's convention for repeated indices and denoting $g = \partial_x^\alpha \partial_v^\beta f$):

$$\partial_t h + v\cdot\nabla_x h = I + II + III, \tag{34}$$

where

$$I = \begin{cases} 0, & \text{for } |\beta| = 0, \\ -\beta_i(\partial_x^{\alpha+\delta_i}\partial_v^{\beta-\delta_i} f)(1+|v|^2)^{s/2}, & \text{for } |\beta| \ge 1, \end{cases}$$

$$\begin{aligned} II \;=\;& \partial_{v_i}(\bar a_{ij}\partial_{v_j} h) - s\bar a_{ij}(\partial_{v_j} g)(1+|v|^2)^{s/2-1} v_i - \partial_{v_i}\Big[s\bar a_{ij} g (1+|v|^2)^{s/2-1} v_j\Big] \\ & -\partial_{v_i}(\bar b_i h) + s\bar b_i g (1+|v|^2)^{s/2-1} v_i, \end{aligned}$$

$$\begin{aligned} III \;=\;& \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ |\alpha_1|\ge 1}} C_{\alpha_1}^\alpha \Big\{ \partial_{v_i}\Big[(\partial_x^{\alpha_1}\bar a_{ij})(\partial_x^{\alpha_2}\partial_{v_j} f)(1+|v|^2)^{s/2}\Big] - s(\partial_x^{\alpha_1}\bar a_{ij})(\partial_x^{\alpha_2}\partial_{v_j} f)(1+|v|^2)^{s/2-1} v_i \\ & -\partial_{v_i}\Big[(\partial_x^{\alpha_1}\bar b_i)(\partial_x^{\alpha_2} f)(1+|v|^2)^{s/2}\Big] + s(\partial_x^{\alpha_1}\bar b_i)(\partial_x^{\alpha_2} f)(1+|v|^2)^{s/2-1} v_i \Big\}, \text{ for } |\beta| = 0, \end{aligned}$$

$$\begin{aligned} III \;=\;& \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta \\ |\alpha_1|+|\beta_1|\ge 1}} C_{\alpha_1}^\alpha C_{\beta_1}^\beta \Big\{ \partial_{v_i}\Big[(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar a_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(1+|v|^2)^{s/2}\Big] \\ & -s(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar a_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(1+|v|^2)^{s/2-1} v_i - \partial_{v_i}\Big[(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar b_i)(\partial_x^{\alpha_2}\partial_v^{\beta_2} f)(1+|v|^2)^{s/2}\Big] \\ & +s(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar b_i)(\partial_x^{\alpha_2}\partial_v^{\beta_2} f)(1+|v|^2)^{s/2-1} v_i \Big\}, \text{ for } |\beta| \ge 1. \end{aligned}$$

We only consider the case $|\beta| \ge 1$, because the estimates for the case $|\beta| = 0$ are similar (and easier). Multiplying equation (34) by h, and then integrating on (t, x, v), we shall estimate the resulting equation term by term.

Note that the "main term" implying the coercivity (and thus leading to the gain of one derivative with respect to v) is II_1 (that is, the first term in II), and more precisely (with the notations prescribed below), the term A_1.

We see that

$$\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}(\partial_t h)hdtdxdv = \frac{1}{2}\left(\|h(T)\|^2_{L^2_{x,v}} - \|h(0)\|^2_{L^2_{x,v}}\right) \geq \frac{1}{2}\|h(T)\|^2_{L^2_{x,v}} - \frac{1}{2}K_0^2. \tag{35}$$

Since f is a spatially periodic function, we also see that

$$\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}(v\cdot\nabla_x h)hdtdxdv = \frac{1}{2}\int_{[0,T]\times\mathbf{R}^3} v\cdot\left(\int_{\mathbf{T}^3}\nabla_x(h^2)dx\right)dtdv = 0. \tag{36}$$

For the term containing I, Hölder's inequality entails that

$$\left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} Ihdtdxdv\right| \leq C\|h\|_{L^2_t([0,T];L^2_{x,v})}\|f\|_{L^2_t([0,T];H^{N,s}_{x,v})} \leq C\,K_0^2. \tag{37}$$

For the term containing II, we write $II = \sum_{i=1}^5 II_i$, and then estimate term by term. By integration by parts,

$$\begin{aligned}\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} II_1 hdtdxdv &= -\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}\bar{a}_{ij}(\partial_{v_i}h)(\partial_{v_j}h)dtdxdv\\ &= -\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}\bar{a}_{ij}(\partial_{v_i}g)(\partial_{v_j}g)(1+|v|^2)^s dtdxdv\\ &\quad -2s\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}\bar{a}_{ij}(\partial_{v_i}g)g(1+|v|^2)^{s-1}v_j dtdxdv\\ &\quad -s^2\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}\bar{a}_{ij}|g|^2(1+|v|^2)^{s-2}v_iv_j dtdxdv.\end{aligned} \tag{38}$$

Denote the r.h.s of the above identity as $A_1 + A_2 + A_3$. Thanks to Lemma 2.1, we get

$$\begin{aligned}A_1 &\leq -K\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}(1+|v|^2)^{\frac{\gamma}{2}}\left|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}}\right|^2 dtdxdv\\ &= -K\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}\left|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\right|^2 dtdxdv.\end{aligned} \tag{39}$$

Using then Lemma 3.1, we see that

$$\begin{aligned}|A_2| &\leq C\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}|\nabla_v g||g|(1+|v|^2)^{s+\frac{\gamma+1}{2}}dtdxdv\|f\|_{L^\infty_t([0,T];H^{2,\gamma+4}_{x,v})}\\ &\leq CK_0\left(\epsilon\|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} + C_\epsilon\|f\|^2_{L^2_t([0,T];H^{N,s+1+\frac{\gamma}{2}}_{x,v})}\right)\\ &\leq CK_0\left(\epsilon\|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} + C_\epsilon K_0^2\right),\end{aligned} \tag{40}$$

and

$$|A_3| \leq CK_0\|f\|^2_{L^2_t([0,T];H^{N,s+\frac{\gamma}{2}}_{x,v})} \leq CK_0^3. \tag{41}$$

Again by integration by parts, we obtain that

$$\begin{aligned}
\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} II_2 h dtdxdv &= s\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \partial_{v_j}\left(\bar{a}_{ij}h(1+|v|^2)^{s/2-1}v_i\right) g dtdxdv \\
&= s\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \bar{a}_{ij}(\partial_{v_j}h) g(1+|v|^2)^{s/2-1}v_i dtdxdv \\
&\quad + s\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \bar{a}_{ii} hg(1+|v|^2)^{s/2-1} dtdxdv \\
&\quad + s(s-2)\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \bar{a}_{ij} hg(1+|v|^2)^{s/2-2}v_iv_j dtdxdv \\
&\quad + s\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \bar{b}_i hg(1+|v|^2)^{s/2-1}v_i dtdxdv.
\end{aligned}$$

Thanks to Lemma 3.1, we get

$$\begin{aligned}
&\left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} II_2 h dtdxdv\right| \\
\leq\ & CK_0\left(\epsilon\|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} + C_\epsilon\|f\|^2_{L^2_t([0,T];H^{N,s+1+\frac{\gamma}{2}}_{x,v})}\right) \\
\leq\ & CK_0\left(\epsilon\|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} + C_\epsilon K_0^2\right). \qquad (42)
\end{aligned}$$

For the term containing II_3, we write

$$\begin{aligned}
\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} II_3 h dtdxdv &= s\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \bar{a}_{ij}(\partial_{v_i}h) g(1+|v|^2)^{s/2-1}v_j dtdxdv \\
&= s\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \bar{a}_{ij}(\partial_{v_i}g) g(1+|v|^2)^{s-1}v_j dtdxdv \\
&\quad + s^2\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \bar{a}_{ij}|g|^2(1+|v|^2)^{s-2}v_iv_j dtdxdv.
\end{aligned}$$

Again from Lemma 3.1, we get that

$$\begin{aligned}
&\left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} II_3 h dtdxdv\right| \\
\leq\ & CK_0\left(\epsilon\|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} + C_\epsilon\|f\|^2_{L^2_t([0,T];H^{N,s+1+\frac{\gamma}{2}}_{x,v})}\right) \\
\leq\ & CK_0\left(\epsilon\|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} + C_\epsilon K_0^2\right). \qquad (43)
\end{aligned}$$

Since

$$\begin{aligned}
\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} II_4 h dtdxdv &= \int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \bar{b}_i(\partial_{v_i}h) h dtdxdv \\
&= -\frac{1}{2}\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} (\partial_{v_i}\bar{b}_i)|h|^2 dtdxdv,
\end{aligned}$$

we know thanks to Lemma 3.1 that

$$\left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} II_4 h dtdxdv\right| \leq CK_0\|f\|^2_{L^2_t([0,T];H^{N,s+1+\frac{\gamma}{2}}_{x,v})} \leq CK_0^3. \qquad (44)$$

Finally, it is easy to see that

$$\left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} II_5 h dt dx dv\right| \leq CK_0 \|f\|^2_{L^2_t([0,T];H^{N,s+1+\frac{\gamma}{2}}_{x,v})} \leq CK_0^3. \tag{45}$$

We now turn to the terms containing the mixed derivatives. We write $III = \sum_{i=1}^4 III_i$ (we recall that we treat only the case $|\beta| \geq 1$). By integration by parts,

$$\begin{aligned}
& \int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \partial_{v_i}\left[(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(1+|v|^2)^{s/2}\right] h dt dx dv \\
= & -\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} (\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(\partial_{v_i} g)(1+|v|^2)^s dt dx dv \\
& -s\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} (\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f) g (1+|v|^2)^{s-1} v_i dt dx dv.
\end{aligned} \tag{46}$$

For the first term on the r.h.s of the above equality, we first treat the case when $|\alpha_1|+|\beta_1| \leq \left[\frac{N}{2}\right]+1$. Then (since $N \geq 8$), $|\alpha_1|+|\beta_1|+2 \leq N-1$, and Lemma 3.1 implies that

$$\begin{aligned}
\|\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij}\|_{L^\infty_t([0,T];L^\infty_x)} & \leq C(1+|v|^2)^{\frac{\gamma+2}{2}} \|f\|_{L^\infty_t([0,T];H^{N-1,\gamma+4}_{x,v})} \\
& \leq C(1+|v|^2)^{\frac{\gamma+2}{2}} K_0,
\end{aligned}$$

so that

$$\begin{aligned}
& \left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} (\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(\partial_{v_i} g)(1+|v|^2)^s dt dx dv\right| \\
\leq\; & CK_0\left(\epsilon\|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} + C_\epsilon\|f\|^2_{L^2_t([0,T];H^{N,s+2+\frac{\gamma}{2}}_{x,v})}\right) \\
\leq\; & CK_0\left(\epsilon\|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} + C_\epsilon K_0^2\right).
\end{aligned}$$

We now turn to the case when $|\alpha_1|+|\beta_1| \geq \left[\frac{N}{2}\right]+2$. If N is even, then $|\alpha_2|+|\beta_2| \leq \left[\frac{N}{2}\right]-2$, and $|\alpha_2|+|\beta_2|+5 \leq N-1$. If N is odd, then $|\alpha_2|+|\beta_2| \leq \left[\frac{N}{2}\right]-1$, and we also get $|\alpha_2|+|\beta_2|+5 \leq N-1$. Using Sobolev embedding, we see that

$$\begin{aligned}
\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}+2}\|_{L^\infty_t([0,T];L^\infty_{x,v})} & \leq C\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}+2}\|_{L^\infty_t([0,T];H^4_{x,v})} \\
& \leq C\|f\|_{L^\infty_t([0,T];H^{N-1,s+4+\frac{\gamma}{2}}_{x,v})} \leq CK_0.
\end{aligned}$$

We notice that $|a(v)| \leq C|v|^{\gamma+2}$. Then, by Hölder's and Minkowski's inequalities, we get that

$$
\begin{aligned}
&\left| \int_{[0,T]\times \mathbf{T}^3 \times \mathbf{R}^3} (\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(\partial_{v_i} g)(1+|v|^2)^s dtdxdv \right| \\
\leq\ & CK_0 \int_{[0,T]\times \mathbf{T}^3 \times \mathbf{R}^3} \left(\int_{\mathbf{R}^3} |v-v_*|^{\gamma+2} |\partial_x^{\alpha_1}\partial_v^{\beta_1} f(v_*)| dv_* \right) |\nabla_v g| (1+|v|^2)^{\frac{s}{2}-\frac{\gamma}{4}-2} dtdvdx \\
\leq\ & CK_0 \int_{[0,T]\times \mathbf{T}^3} \left(\int_{\mathbf{R}^3} \left| (\partial_x^{\alpha_1}\partial_v^{\beta_1} f)(1+|v|^2)^{\frac{\gamma+2}{2}} \right| dv \right) \left(\int_{\mathbf{R}^3} |\nabla_v g| (1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}-1} dv \right) dtdx \\
\leq\ & CK_0 \int_0^T \left[\int_{\mathbf{R}^3} (1+|v|^2)^{-1} \left(\int_{\mathbf{T}^3} \left| (\partial_x^{\alpha_1}\partial_v^{\beta_1} f)(1+|v|^2)^{\frac{\gamma+4}{2}} \right|^2 dx \right)^{1/2} dv \right] \\
& \times \left[\int_{\mathbf{R}^3} (1+|v|^2)^{-1} \left(\int_{\mathbf{T}^3} \left| (\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}} \right|^2 dx \right)^{1/2} dv \right] dt \\
\leq\ & CK_0 \int_0^T \left[\int_{\mathbf{T}^3 \times \mathbf{R}^3} \left| (\partial_x^{\alpha_1}\partial_v^{\beta_1} f)\,(1+|v|^2)^{\frac{\gamma+4}{2}} \right|^2 dxdv \right]^{1/2} \\
& \times \left[\int_{\mathbf{T}^3 \times \mathbf{R}^3} \left| (\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}} \right|^2 dxdv \right]^{1/2} dt \\
\leq\ & CK_0 \left(\epsilon \|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} + C_\epsilon \|f\|^2_{L^2_t([0,T];H^{N,\gamma+4}_{x,v})} \right) \\
\leq\ & CK_0 \left(\epsilon \|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} + C_\epsilon K_0^2 \right).
\end{aligned}
$$

Similarly, we can bound the second term on the r.h.s of (46) by CK_0^3. Then,

$$
\left| \int_{[0,T]\times \mathbf{T}^3 \times \mathbf{R}^3} III_1 h dtdxdv \right| \leq CK_0 \left(\epsilon \|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} + C_\epsilon K_0^2 \right). \tag{47}
$$

The term containing III_3 can be estimated in the same way. We do not detail the computation, but simply notice that because of Lemma 3.1, the cases $|\alpha_1| + |\beta_1| \leq \left[\frac{N}{2}\right]$ and $|\alpha_1| + |\beta_1| \geq \left[\frac{N}{2}\right] + 1$ are considered separately. As a consequence,

$$
\left| \int_{[0,T]\times \mathbf{T}^3 \times \mathbf{R}^3} III_3 h dtdxdv \right| \leq CK_0 \left(\epsilon \|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} + C_\epsilon K_0^2 \right). \tag{48}
$$

Finally, one can also prove that

$$
\left| \int_{[0,T]\times \mathbf{T}^3 \times \mathbf{R}^3} (III_2 + III_4) h dtdxdv \right| \leq CK_0^3. \tag{49}
$$

Putting (35) through (49) together, we see that

$$
\frac{1}{2} ||h(T,\cdot,\cdot)||^2_{L^2_{x,v}} + (K - CK_0\epsilon)\, \|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} \leq CK_0^2(1+K_0) + C_\epsilon K_0^3. \tag{50}
$$

The proof of Lemma 4.1 is concluded by taking $\epsilon > 0$ small enough.

We now turn to the question of the smoothness with respect to variable x. We cannot hope to get it directly by an energy estimate like we did for variable v since no diffusion term in x is available. We use instead averaging lemmas, which ensure that averages in v of the solution of a kinetic equation have some smoothness (Cf. [17] for example).

4.2 Gain of $1/20$ of a derivative in x starting from a Sobolev space whose index is an integer

We begin with the

Lemma 4.2 *Let $\gamma \in [-2,1]$, $N \geq 8$ be a given integer, and let f be a smooth nonnegative solution of eq.* (1) *such that assumption A holds for a given constant K. We suppose that for any $T \in]0,+\infty[$ and $s \geq 0$, $\|f\|_{L_t^\infty([0,T];H_{x,v}^{N,s})} \leq K_1$ for some constant $K_1 \equiv K_1(s,T,\gamma,N,K)$. Then for any $T>0$ and $s \geq 0$, there is a positive constant $\widetilde{C}_2$, which depends on N, s, γ, T, K and K_1, such that*

$$\int_{[0,T]\times\mathbf{R}^3} \|h\|^2_{\dot{H}_x^{1/20}} dtdv \leq \widetilde{C}_2, \tag{51}$$

where h is defined by (33).

Proof of Lemma 4.2 Let $p(t,x,v) = h(t,x,v)(1+|v|^2)^2$. We recall that

$$\begin{aligned}
(51) \quad &\Leftrightarrow \quad \int_{[0,T]\times\mathbf{R}^3\times\mathbf{T}^3\times\mathbf{T}^3} (1+|v|^2)^{-4} |\triangle_k p(t,x,v)|^2 |k|^{-1/10-3} dtdvdxdk \leq \widetilde{C}_2 \\
&\Leftrightarrow \quad \int_{[0,T]\times\mathbf{R}^3} (1+|v|^2)^{-4} \Bigg(\sum_{m\in\mathbf{Z}^3} |m|^{1/10} |\hat{p}(t,m,v)|^2 \Bigg) dtdv \leq \widetilde{C}_2,
\end{aligned} \tag{52}$$

where $\triangle_k p(t,x,v) = p(t,x+k,v) - p(t,x,v)$ is a finite difference, and $\hat{p}(t,m,v)$ is the m-th Fourier coefficient of p with respect to the x variable.

We wish to prove (52). Let $\chi := \chi(v) \in C_c^\infty(\mathbf{R}^3)$ be a test function which satisfies $\chi(v) \geq 0$ and $\int_{\mathbf{R}^3} \chi(v)dv = 1$. We introduce the regularizing sequence $\chi_\epsilon(v) = \epsilon^{-3}\chi\left(\frac{v}{\epsilon}\right)$ and write

$$\hat{p}(t,m,v) = \Big[\hat{p}(t,m,v) - (\hat{p}(t,m,\cdot) *_v \chi_\epsilon)(v)\Big] + (\hat{p}(t,m,\cdot) *_v \chi_\epsilon)(v). \tag{53}$$

Here, ϵ will be chosen later (and will depend on m).

For the first term of the r.h.s of the above equality, we use Minkowski's inequality and get

$$\begin{aligned}
&\int_{\mathbf{R}^3} (1+|v|^2)^{-4} \Big| \hat{p}(t,m,v) - (\hat{p}(t,m,\cdot) *_v \chi_\epsilon)(v) \Big|^2 dv \\
\leq \quad &\int_{\mathbf{R}^3} \left| \int_{\mathbf{R}^3} [\hat{p}(t,m,v) - \hat{p}(t,m,v-u)]\, \chi_\epsilon(u)du \right|^2 dv \\
\leq \quad &\left(\int_{\mathbf{R}^3} \left(\int_{\mathbf{R}^3} |\hat{p}(t,m,v) - \hat{p}(t,m,v-u)|^2 dv \right)^{1/2} \chi_\epsilon(u)du \right)^2 \\
\leq \quad &C \left(\int_{\mathbf{R}^3} \chi_\epsilon(u)|u|du \right)^2 \int_{\mathbf{R}^3} |\nabla_v \hat{p}(t,m,v)|^2 dv \leq C\epsilon^2 \int_{\mathbf{R}^3} |\nabla_v \hat{p}(t,m,v)|^2 dv
\end{aligned}$$

so that

$$\begin{aligned}
&\int_{[0,T]\times\mathbf{R}^3} (1+|v|^2)^{-4} \Bigg(\sum_{m\in\mathbf{Z}^3} |m|^{1/10} \Big| \hat{p}(t,m,v) - [\hat{p}(t,m,\cdot) *_v \chi_\epsilon](v) \Big|^2 \Bigg) dtdv \\
\leq \quad &C \int_{[0,T]\times\mathbf{R}^3} \sum_{m\in\mathbf{Z}^3} |m|^{1/10} \epsilon^2 |\nabla_v \hat{p}(t,m,v)|^2 dtdv.
\end{aligned} \tag{54}$$

Remembering that $p = h(1+|v|^2)^2$, we see that p is the solution of equation (34) with s replaced by $s+4$. Then, we can write the equation satisfied by p under the form

$$\partial_t p + v \cdot \nabla_x p = p_1 + \nabla_v \cdot p_2. \tag{55}$$

Here, $\nabla_v \cdot p_2$ is the sum of the terms II_1, II_3, II_4, III_1 and III_3, while p_1 is the sum of the remaining terms.

We claim that $p_1, p_2 \in L^2_t([0,T]; L^2_{x,v})$. We only present here the estimates for the terms I, II_1, II_5, III_1 and III_3 in the case $|\beta| \geq 1$, the others being similar.

It is obvious that

$$\|I\|_{L^2_t([0,T];L^2_{x,v})} \leq C\|f\|_{L^2_t([0,T];H^{N,s+4}_{x,v})} \leq CK_1 T^{1/2}. \tag{56}$$

As for the term II_1, we know from (20) that

$$\|\bar{a}_{ij}\|_{L^\infty_t([0,T];L^\infty_x)} \leq CK_1(1+|v|^2)^{\frac{\gamma+2}{2}},$$

which implies that

$$\|\bar{a}_{ij}\partial_{v_i} p\|_{L^2_t([0,T];L^2_{x,v})} \leq CK_1\|(\nabla_v p)(1+|v|^2)^{\frac{\gamma+2}{2}}\|_{L^2_t([0,T];L^2_{x,v})} \leq C\sqrt{\widetilde{C}_1}\, K_1, \tag{57}$$

where the last inequality holds thanks to Lemma 4.1 (applied to $p(1+|v|^2)^{(\gamma+2)/2}$).

If we use (22) instead of (20), we can analogously obtain that

$$\|\bar{b}_i g(1+|v|^2)^{s/2+1} v_i\|_{L^2_t([0,T];L^2_{x,v})} \leq CK_1^2 T^{1/2}, \tag{58}$$

and as a consequence

$$\|II_5\|_{L^2_t([0,T];L^2_{x,v})} \leq CK_1^2 T^{1/2}. \tag{59}$$

As for the term III_1, when $|\alpha_1| + |\beta_1| \leq \left[\frac{N}{2}\right] + 1$, we know from (20) that

$$\|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(1+|v|^2)^{-\frac{\gamma+2}{2}}\|_{L^\infty_t([0,T];L^\infty_x)} \leq CK_1.$$

This implies that

$$\begin{aligned} & \|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(1+|v|^2)^{\frac{s+4}{2}}\|_{L^2_t([0,T];L^2_{x,v})} \\ \leq\ & CK_0\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(1+|v|^2)^{\frac{s+\gamma+6}{2}}\|_{L^2_t([0,T];L^2_{x,v})} \leq CK_1^2 T^{1/2}. \end{aligned} \tag{60}$$

When $|\alpha_1| + |\beta_1| \geq \left[\frac{N}{2}\right] + 2$, we know that

$$\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(1+|v|^2)^{\frac{s+\gamma+8}{2}}\|_{L^\infty_t([0,T];L^\infty_{x,v})} \leq C\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(1+|v|^2)^{\frac{s+\gamma+8}{2}}\|_{L^\infty_t([0,T];H^4_{x,v})} \leq CK_1.$$

Noticing that $|a(v-v_*)| \leq C|v-v_*|^{\gamma+2} \leq C(1+|v|^2)^{\frac{\gamma+2}{2}}(1+|v_*|^2)^{\frac{\gamma+2}{2}}$, we get

$$\begin{aligned} & \|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a})(1+|v|^2)^{-\frac{\gamma+4}{2}}\|^2_{L^2_t([0,T];L^2_{x,v})} \\ \leq\ & C\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}(1+|v|^2)^{-2}\left(\int_{\mathbf{R}^3}|\partial_x^{\alpha_1}\partial_v^{\beta_1} f(v_*)|(1+|v_*|^2)^{\frac{\gamma+2}{2}}dv_*\right)^2 dtdxdv \\ \leq\ & C\int_{[0,T]\times\mathbf{T}^3}\left(\int_{\mathbf{R}^3}|\partial_x^{\alpha_1}\partial_v^{\beta_1} f(v)|(1+|v|^2)^{\frac{\gamma+2}{2}}dv\right)^2 dtdx. \end{aligned}$$

Using Minkowski's and Hölder's inequality, we get

$$\begin{aligned} & \|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a})(1+|v|^2)^{-\frac{\gamma+4}{2}}\|^2_{L^2_t([0,T];L^2_{x,v})} \\ \leq\ & C\int_0^T\left(\int_{\mathbf{R}^3}\left(\int_{\mathbf{T}^3}|\partial_x^{\alpha_1}\partial_v^{\beta_1} f(v)|^2(1+|v|^2)^{\gamma+4}dx\right)^{1/2}(1+|v|^2)^{-1}dv\right)^2 dt \\ \leq\ & C\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}|\partial_x^{\alpha_1}\partial_v^{\beta_1} f(v)|^2(1+|v|^2)^{\gamma+4}dtdxdv \leq CK_1^2 T, \end{aligned}$$

which yields for the term III_1 :

$$\begin{aligned} & \|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f)(1+|v|^2)^{\frac{s+4}{2}}\|_{L^2_t([0,T];L^2_{x,v})} \\ \leq & \; CK_1\|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a})(1+|v|^2)^{-\frac{\gamma+4}{2}}\|_{L^2_t([0,T];L^2_{x,v})} \leq CK_1^2T^{1/2}. \end{aligned} \tag{61}$$

For the term III_3, we proceed like for the corresponding term in the proof of Lemma 4.1. We use (22) instead of (20) and consider the case $|\alpha_1|+|\beta_1| \leq [\frac{N}{2}]$ and $|\alpha_1|+|\beta_1| \geq [\frac{N}{2}]+1$ separately, in order to get

$$\|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{b}_i)(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f)(1+|v|^2)^{\frac{s+4}{2}}\|_{L^2_t([0,T];L^2_{x,v})} \leq CK_1^2T^{1/2}. \tag{62}$$

Finally, we see indeed that $p_1, p_2 \in L^2_t([0,T];L^2_{x,v})$.

Now according to (2.16) in Theorem 2.1 (averaging lemma) of [6] (this is a variant of a lemma first proven in [16]), we can prove that

$$\begin{aligned} |m|^{1/2}\int_0^T \Big|(\hat{p}(t,m,\cdot) *_v \chi_\epsilon)(v)\Big|^2 dt \leq & \; C\Big(\|\chi_\epsilon(v-u)(1+|u|^2)\|_{L^\infty_u} + \|\nabla\chi_\epsilon(v-u)(1+|u|^2)\|_{L^\infty_u}\Big)^2 \\ & \times \Big(\|\hat{p}(0,m,\cdot)\|^2_{L^2_v} + \|\hat{p}(\cdot,m,\cdot)\|^2_{L^2_t([0,T];L^2_v)} \\ & + \|\hat{p}_1(\cdot,m,\cdot)\|^2_{L^2_t([0,T];L^2_v)} + \|\hat{p}_2(\cdot,m,\cdot)\|^2_{L^2_t([0,T];L^2_v)}\Big). \end{aligned}$$

Since $\|\chi_\epsilon(v-u)(1+|u|^2)\|_{L^\infty_u} \leq C\epsilon^{-3}(1+|v|^2)$, we see that

$$\begin{aligned} & \int_{[0,T]\times\mathbf{R}^3}(1+|v|^2)^{-4}\sum_{m\in\mathbf{Z}^3}|m|^{1/10}\Big|(\hat{p}(t,m,\cdot) *_v \chi_\epsilon)(v)\Big|^2 dtdv \\ \leq & \; C\sum_{m\in\mathbf{Z}^3}|m|^{\frac{1}{10}-\frac{1}{2}}(\epsilon^{-6}+\epsilon^{-8})\Big(\|\hat{p}(0,m,\cdot)\|^2_{L^2_v} \\ & + \|\hat{p}(\cdot,m,\cdot)\|^2_{L^2_t([0,T];L^2_v)} + \|\hat{p}_1(\cdot,m,\cdot)\|^2_{L^2_t([0,T];L^2_v)} + \|\hat{p}_2(\cdot,m,\cdot)\|^2_{L^2_t([0,T];L^2_v)}\Big). \end{aligned} \tag{63}$$

If we choose $\epsilon := \epsilon(m) = |m|^{-1/20}$, we can bound (54) by using Lemma 4.1. Then, we can bound (63) remembering that $p(0,\cdot,\cdot) \in L^2_{x,v}$ and $p, p_1, p_2 \in L^2_t([0,T];L^2_{x,v})$. This leads to estimate (52) and ends the proof of Lemma 4.2.

Roughly speaking, Lemma 4.2 (together with Lemma 4.1) shows that when f is a solution of eq. (1) satisfying assumption A and such that $f \in L^\infty_t(H^N_{x,v})$, then $f \in L^2_t(H^{N+1/20}_{x,v})$ (up to some weights).

The two next steps consist in proving that the same gain of 1/20 of derivatives (with respect to x and v) holds when N is replaced by $N+\delta$ with δ not integer, so that at the end, $f \in L^2_t(H^{N+\delta+1/20}_{x,v})$ (with weights). In these two steps, one has to write down the equation satisfied by a finite difference (in x) of some mixed derivative of f. The two next lemmas are therefore somewhat more technical than lemmas 4.1, 4.2, and the treatment of this difficulty is the main novelty of this paper.

4.3 Gain of one derivative in v starting from a Sobolev space whose index is not an integer

Lemma 4.3 *Let $\gamma \in [-2,1]$, $N \geq 8$ be a given integer, $\delta \in]0, 19/20]$ and let f be a smooth nonnegative solution of eq. (1) such that assumption A holds for a given constant K. We suppose that for any $T \in]0,+\infty[$ and $s \geq 0$, there exists $K_2 \equiv K_2(\gamma, N, \delta, K, T, s)$ such that*

$$||f(0,\cdot,\cdot)||_{H^{N+\delta,s}_{x,v}} \leq K_2, \qquad ||f||_{L^\infty_t([0,T];H^{N,s}_{x,v})} \leq K_2, \qquad ||f||_{L^2_t([0,T];H^{N+\delta,s}_{x,v})} \leq K_2. \tag{64}$$

Then for any $T>0$, and $s\geq 0$, there is a positive constant $\widetilde{C}_3$, which depends on N, s, γ, T, K, K_2 and δ, such that

$$\int_{[0,T]\times \mathbf{T}^3\times \mathbf{R}^3\times \mathbf{T}^3} |\nabla_v g_{\delta,k}|^2 dtdxdvdk \leq \widetilde{C}_3, \tag{65}$$

where $g_{\delta,k}(t,x,v)=\triangle_k h(t,x,v)|k|^{-\delta-\frac{3}{2}}$, and h is defined by (33).

Proof of Lemma 4.3 The main difference between this proof and that of Lemma 4.1 is that we now have an additional integration with respect to k.

By assumption, $\|g_{\delta,k}(0)\|_{L^2_{x,v,k}}\leq K_2$. We now write down the equation satisfied by $g_{\delta,k}$:

$$\partial_t g_{\delta,k}+v\cdot\nabla_x g_{\delta,k}=IV+V+VI, \tag{66}$$

where

$$IV=\begin{cases} 0, & \text{for } |\beta|=0,\\ -\beta_i(\partial_x^{\alpha+\delta_i}\partial_v^{\beta-\delta_i}\triangle_k f)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{s/2}, & \text{for } |\beta|\geq 1,\end{cases}$$

$$\begin{aligned} V \;=\;& \partial_{v_i}\left[\triangle_k(\bar{a}_{ij}\partial_{v_j}h)|k|^{-\delta-\frac{3}{2}}\right]-s\triangle_k(\bar{a}_{ij}\partial_{v_j}g)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{s/2-1}v_i\\ &-\partial_{v_i}\left[s\triangle_k(\bar{a}_{ij}g)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{s/2-1}v_j\right]-\partial_{v_i}\left[\triangle_k(\bar{b}_i h)|k|^{-\delta-\frac{3}{2}}\right]\\ &+s\triangle_k(\bar{b}_i g)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{s/2-1}v_i,\end{aligned}$$

$$\begin{aligned} VI \;=\;& \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ |\alpha_1|\geq 1}} C^{\alpha}_{\alpha_1}\Bigg\{\partial_{v_i}\left[\triangle_k((\partial_x^{\alpha_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_{v_j}f))|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{s/2}\right]\\ &-s\triangle_k\Big((\partial_x^{\alpha_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_{v_j}f)\Big)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{s/2-1}v_i\\ &-\partial_{v_i}\left[\triangle_k\Big((\partial_x^{\alpha_1}\bar{b}_i)(\partial_x^{\alpha_2}f)\Big)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{s/2}\right]\\ &+s\triangle_k\Big((\partial_x^{\alpha_1}\bar{b}_i)(\partial_x^{\alpha_2}f)\Big)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{s/2-1}v_i\Bigg\},\ \text{for } |\beta|=0,\end{aligned}$$

$$\begin{aligned} VI \;=\;& \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta\\ |\alpha_1|+|\beta_1|\geq 1}} C^{\alpha}_{\alpha_1}C^{\beta}_{\beta_1}\Bigg\{\partial_{v_i}\left[\triangle_k\Big((\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f)\Big)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{s/2}\right]\\ &-s\triangle_k\Big((\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f)\Big)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{s/2-1}v_i\\ &-\partial_{v_i}\left[\triangle_k\Big((\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{b}_i)(\partial_x^{\alpha_2}\partial_v^{\beta_2}f)\Big)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{s/2}\right]\\ &+s\,\triangle_k\Big((\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{b}_i)(\partial_x^{\alpha_2}\partial_v^{\beta_2}f)\Big)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{s/2-1}v_i\Bigg\},\ \text{for } |\beta|\geq 1.\end{aligned}$$

We still only consider the case $|\beta|\geq 1$. We multiply equation (66) by $g_{\delta,k}$, and then integrate on (t,x,v,k) in the domain $[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3$. We only estimate the terms containing the first and fourth term of V and the first term of VI, denoted by V_1, V_4 and VI_1 respectively, since the estimates for other terms are similar. Note that the "main term" implying the coercivity (and thus leading to the gain of one derivative with respect to v) is V_1, and more precisely (with the notations prescribed below), the term $B_{1,1}$.

For any function $f(x)$ and $g(x)$,

$$\triangle_k(f(x)g(x)) = f(x+k)\triangle_k g(x) + g(x)\triangle_k f(x), \tag{67}$$

so that (writing $\bar{a}_{ij}(x+k)$ instead of $\bar{a}_{ij}(t,x+k,v)$, etc.), we can write

$$V_1 = \partial_{v_i}\left[\triangle_k(\bar{a}_{ij}\partial_{v_j}h)|k|^{-\delta-\frac{3}{2}}\right] = \partial_{v_i}\left[\bar{a}_{ij}(x+k)(\partial_{v_j}g_{\delta,k})\right] + \partial_{v_i}\left[(\triangle_k\bar{a}_{ij})(\partial_{v_j}h)|k|^{-\delta-\frac{3}{2}}\right].$$

We denote the r.h.s of the above equality by B_1+B_2. After integration by parts,

$$\begin{aligned}
&\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} B_1 g_{\delta,k}\,dtdxdvdk\\
= & -\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\bar{a}_{ij}(x+k)(\partial_{v_i}g_{\delta,k})(\partial_{v_j}g_{\delta,k})\,dtdxdvdk\\
= & -\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\bar{a}_{ij}(x+k)[\partial_{v_i}(\triangle_k g)][\partial_{v_j}(\triangle_k g)](1+|v|^2)^s|k|^{-2\delta-3}\,dtdxdvdk\\
& -2s\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\bar{a}_{ij}(x+k)[\partial_{v_i}(\triangle_k g)](\triangle_k g)(1+|v|^2)^{s-1}v_j|k|^{-2\delta-3}\,dtdxdvdk\\
& -s^2\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\bar{a}_{ij}(x+k)|\triangle_k g|^2(1+|v|^2)^{s-2}v_iv_j|k|^{-2\delta-3}\,dtdxdvdk.
\end{aligned} \tag{68}$$

We write the r.h.s of the above equality as $B_{1,1}+B_{1,2}+B_{1,3}$. Since f is a spatially periodic function, we obtain from Lemma 2.1 that

$$\begin{aligned}
B_{1,1} &\leq -K\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}(1+|v|^2)^{\frac{\gamma}{2}}\left|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}}|k|^{-\delta-\frac{3}{2}}\right|^2 dtdxdvdk\\
&= -K\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\left|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\right|^2 dtdxdvdk.
\end{aligned} \tag{69}$$

Thanks to Lemma 3.1, we know that

$$\|\bar{a}_{ij}(x+k)\|_{L_t^\infty([0,T];L_{x,k}^\infty)} \leq CK_2(1+|v|^2)^{\frac{\gamma+2}{2}}.$$

Using Hölder's inequality and the hypothesis of our lemma, we get

$$\begin{aligned}
|B_{1,2}| &\leq CK_2\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\left(|\nabla_v(\triangle_k g)|(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\right)\\
&\quad\times\left((\triangle_k g)(1+|v|^2)^{\frac{s+1}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\right)dtdxdvdk\\
&\leq CK_2\left(\epsilon\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L_t^2([0,T];L^2_{x,v,k})} + C_\epsilon K_2^2\right).
\end{aligned} \tag{70}$$

Similarly, we can obtain

$$|B_{1,3}| \leq CK_2^3. \tag{71}$$

As for the term containing B_2, we treat it by integration by parts,

$$\begin{aligned}
&\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} B_2 g_{\delta,k} dtdxdvdk \\
= &-\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} (\triangle_k\bar{a}_{ij})(\partial_{v_i} g_{\delta,k})(\partial_{v_j} h)|k|^{-\delta-\frac{3}{2}} dtdxdvdk \\
= &-\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} (\triangle_k\bar{a}_{ij})(\partial_{v_i}(\triangle_k g))(\partial_{v_j} g)(1+|v|^2)^s|k|^{-2\delta-3} dtdxdvdk \\
&-s\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} (\triangle_k\bar{a}_{ij})(\partial_{v_i}(\triangle_k g))g(1+|v|^2)^{s-1}v_j|k|^{-2\delta-3} dtdxdvdk \\
&-s\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} (\triangle_k\bar{a}_{ij})(\triangle_k g)(\partial_{v_j} g)(1+|v|^2)^{s-1}v_i|k|^{-2\delta-3} dtdxdvdk \\
&-s^2\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} (\triangle_k\bar{a}_{ij})(\triangle_k g)g(1+|v|^2)^{s-2}v_iv_j|k|^{-2\delta-3} dtdxdvdk. \qquad (72)
\end{aligned}$$

We denote the r.h.s of the above equality by $\sum_{i=1}^4 B_{2,i}$. Noticing that $\triangle_k\bar{a}_{ij} = a_{ij} * \triangle_k f$, we get thanks to Lemma 3.1 the following inequality :

$$\|\triangle_k\bar{a}_{ij}\|_{L^\infty_x} \le C(1+|v|^2)^{\frac{\gamma+2}{2}}\|(\triangle_k f)(1+|v|^2)^{\frac{\gamma+4}{2}}\|_{H^2_{x,v}}.$$

Then,

$$\begin{aligned}
|B_{2,1}| \;\le\;& C\int_{[0,T]\times\mathbf{T}^3} \|(\triangle_k f)(1+|v|^2)^{\frac{\gamma+4}{2}}|k|^{-\delta-\frac{3}{2}}\|_{H^2_{x,v}} \\
&\times\left(\int_{\mathbf{T}^3\times\mathbf{R}^3} |\nabla_v(\triangle_k g)||\nabla_v g|(1+|v|^2)^{s+1+\frac{\gamma}{2}}|k|^{-\delta-\frac{3}{2}} dxdv\right) dtdk \\
\le\;& C\int_0^T \|(\triangle_k f)(1+|v|^2)^{\frac{\gamma+4}{2}}|k|^{-\delta-\frac{3}{2}}\|_{L^2_k(H^2_{x,v})}\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|_{L^2_{x,v,k}} \\
&\times\|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}+1}\|_{L^2_{x,v}} dt \\
\le\;& CK_2\left(\epsilon\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([0,T];L^2_{x,v,k})} + C_\epsilon\widetilde{C}_1\right), \qquad (73)
\end{aligned}$$

where the last inequality holds thanks to Lemma 4.1. Similarly, we can get

$$\sum_{i=2}^4 |B_{2,i}| \le C\,K_2\left(\epsilon\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([0,T];L^2_{x,v,k})} + C_\epsilon(K_2+\widetilde{C}_1)\right). \qquad (74)$$

Combining (68) through (74), we get the estimate for $\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} V_1 g_{\delta,k} dtdxdvdk$.

We now briefly describe how to treat the term V_4. We know from (67) that

$$\partial_{v_i}\left[\triangle_k(\bar{b}_i h)|k|^{-\delta-\frac{3}{2}}\right] = \partial_{v_i}\left[\bar{b}_i(x+k)g_{\delta,k}\right] + \partial_{v_i}\left[(\triangle_k\bar{b}_i)h|k|^{-\delta-\frac{3}{2}}\right].$$

We find that the estimate for $\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} \partial_{v_i}[\bar{b}_i(x+k)g_{\delta,k}]g_{\delta,k} dtdxdvdk$ is similar to that for $B_{1,2}$. Moreover, using (22), we obtain for $\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} \partial_{v_i}[(\triangle_k\bar{b}_i)h|k|^{-\delta-\frac{3}{2}}]g_{\delta,k} dtdxdvdk$ an estimate similar to that of $\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} B_2 g_{\delta,k} dtdxdvdk$. Finally, we have that

$$\begin{aligned}
&\left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} V_4 g_{\delta,k} dtdxdvdk\right| \\
\le\;& CK_2\left(\epsilon\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([0,T];L^2_{x,v,k})} + C_\epsilon(K_2^2+\widetilde{C}_1)\right). \qquad (75)
\end{aligned}$$

As for the term containing VI_1, we write

$$
\begin{aligned}
&\partial_{v_i}\left[\triangle_k((\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f))(1+|v|^2)^{s/2}|k|^{-\delta-\frac{3}{2}}\right]\\
=\ &\partial_{v_i}\left[(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij}(x+k))(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(1+|v|^2)^{s/2}|k|^{-\delta-\frac{3}{2}}\right]\\
&+\partial_{v_i}\left[(\partial_x^{\alpha_1}\partial_v^{\beta_1}\triangle_k\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f)(1+|v|^2)^{s/2}|k|^{-\delta-\frac{3}{2}}\right],
\end{aligned}
$$

and we denote it by D_1+D_2. By integration by parts,

$$
\begin{aligned}
&\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} D_1 g_{\delta,k}\,dtdxdvdk\\
=\ &-\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij}(x+k))(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(\partial_{v_i}g_{\delta,k})(1+|v|^2)^{s/2}|k|^{-\delta-\frac{3}{2}}dtdxdvdk\\
=\ &-\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij}(x+k))(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(\partial_{v_i}(\triangle_k g))(1+|v|^2)^{s}|k|^{-2\delta-3}dtdxdvdk\\
&-s\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij}(x+k))(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(\triangle_k g)\\
&\times(1+|v|^2)^{s-1}v_i|k|^{-2\delta-3}dtdxdvdk. \qquad (76)
\end{aligned}
$$

We write the r.h.s of the above equality as $D_{1,1}+D_{1,2}$. When $|\alpha_1|+|\beta_1|\le\left[\frac{N}{2}\right]+1$, Lemma 3.1 ensures that

$$
\|\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij}(x+k)\|_{L_t^\infty([0,T];L_{x,k}^\infty)}\le CK_2(1+|v|^2)^{\frac{\gamma+2}{2}}.
$$

Then, thanks to the hypothesis of our lemma, we get

$$
\begin{aligned}
|D_{1,1}| \ &\le\ CK_2\Big(\epsilon\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L_t^2([0,T];L^2_{x,v,k})}\\
&\qquad +C_\epsilon\|[\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}+1}|k|^{-\delta-\frac{3}{2}}\|^2_{L_t^2([0,T];L^2_{x,v,k})}\Big)\\
&\le\ CK_2\Big(\epsilon\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L_t^2([0,T];L^2_{x,v,k})}+C_\epsilon K_2^2\Big). \qquad (77)
\end{aligned}
$$

When $|\alpha_1|+|\beta_1|\ge\left[\frac{N}{2}\right]+2$, by Sobolev's embedding, we know that

$$
\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}+2}|k|^{-\delta-\frac{3}{2}}\|_{L^\infty_{x,v}}\le C\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}+2}|k|^{-\delta-\frac{3}{2}}\|_{H^4_{x,v}}.
$$

From the definitions of a and $\bar{a}$, we obtain (we do not write down the time variable for the sake of simplicity)

$$
\begin{aligned}
|\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij}(x+k)| \ &\le\ \int_{\mathbf{R}^3}|a(v-v_*)||\partial_x^{\alpha_1}\partial_v^{\beta_1}f(x+k,v_*)|dv_*\\
&\le\ (1+|v|^2)^{\frac{\gamma+2}{2}}\int_{\mathbf{R}^3}|\partial_x^{\alpha_1}\partial_v^{\beta_1}f(x+k,v)|(1+|v|^2)^{\frac{\gamma+2}{2}}dv.
\end{aligned}
$$

So using Cauchy-Schwarz's inequality first, and then Minkowski's inequality, we can get

$$
\begin{aligned}
|D_{1,1}| &\leq C\int_{[0,T]\times\mathbf{T}^3}\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}+2}|k|^{-\delta-\frac{3}{2}}\|_{H^4_{x,v}}\\
&\quad\times\int_{\mathbf{T}^3}\left(\int_{\mathbf{R}^3}|\partial_x^{\alpha_1}\partial_v^{\beta_1}f(x+k,v)|(1+|v|^2)^{\frac{\gamma+2}{2}}dv\right)\\
&\quad\times\left(\int_{\mathbf{R}^3}|\nabla_v(\triangle_k g)|(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}-1}|k|^{-\delta-\frac{3}{2}}dv\right)dxdtdk\\
&\leq C\int_{[0,T]\times\mathbf{T}^3}\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}+2}|k|^{-\delta-\frac{3}{2}}\|_{H^4_{x,v}}\\
&\quad\times\left[\int_{\mathbf{R}^3}(1+|v|^2)^{-1}\left(\int_{\mathbf{T}^3}\left|(\partial_x^{\alpha_1}\partial_v^{\beta_1}f(x+k,v))(1+|v|^2)^{\frac{\gamma+4}{2}}\right|^2dx\right)^{1/2}dv\right]\\
&\quad\times\left[\int_{\mathbf{R}^3}(1+|v|^2)^{-1}\left(\int_{\mathbf{T}^3}\left|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\right|^2dx\right)^{1/2}dv\right]dtdk.
\end{aligned}
$$

Using Cauchy-Schwarz's inequality again, we finally obtain

$$
\begin{aligned}
|D_{1,1}| &\leq C\int_0^T\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}+2}|k|^{-\delta-\frac{3}{2}}\|_{L^2_k(H^4_{x,v})}\\
&\quad\times\|(\partial_x^{\alpha_1}\partial_v^{\beta_1}f)(x+k,v)(1+|v|^2)^{\frac{\gamma+4}{2}}\|_{L^2_{x,v}}\\
&\quad\times\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|_{L^2_{x,v,k}}dt\\
&\leq CK_2\left(\epsilon\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([0,T];L^2_{x,v,k})}+C_\epsilon K_2^2\right). \qquad (78)
\end{aligned}
$$

Estimates (77) and (78) together yield

$$
|D_{1,1}|\leq CK_2\left(\epsilon\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([0,T];L^2_{x,v,k})}+C_\epsilon K_2^2\right). \qquad (79)
$$

Similarly, we can get

$$
|D_{1,2}|\leq C\,K_2^3. \qquad (80)
$$

As for D_2, by integration by parts,

$$
\begin{aligned}
&\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}D_2g_{\delta,k}dtdxdvdk\\
=&\ -\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}(\partial_x^{\alpha_1}\partial_v^{\beta_1}\triangle_k\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f)(\partial_{v_i}g_{\delta,k})(1+|v|^2)^{s/2}|k|^{-\delta-\frac{3}{2}}dtdxdvdk\\
=&\ -\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}(\partial_x^{\alpha_1}\partial_v^{\beta_1}\triangle_k\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f)(\partial_{v_i}(\triangle_k g))(1+|v|^2)^{s}|k|^{-2\delta-3}dtdxdvdk\\
&-s\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}(\partial_x^{\alpha_1}\partial_v^{\beta_1}\triangle_k\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f)(\triangle_k g)\\
&\times(1+|v|^2)^{s-1}v_i|k|^{-2\delta-3}dtdxdvdk. \qquad (81)
\end{aligned}
$$

We denote the r.h.s of the above equality by $D_{2,1}+D_{2,2}$. When $|\alpha_1|+|\beta_1|\leq\left[\frac{N}{2}\right]+1$, we use Lemma 3.1 to get the estimate

$$
\|\partial_x^{\alpha_1}\partial_v^{\beta_1}\triangle_k\bar{a}_{ij}\|_{L^\infty_x}\leq C(1+|v|^2)^{\frac{\gamma+2}{2}}\|(\triangle_k f)(1+|v|^2)^{\frac{\gamma+4}{2}}\|_{H^{N-1}_{x,v}}.
$$

Proceeding like in the estimate for $B_{2,1}$, we can get

$$|D_{2,1}| \leq CK_2 \left(\epsilon \|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([0,T];L^2_{x,v,k})} + C_\epsilon K_2^2\right). \tag{82}$$

When $|\alpha_1|+|\beta_1| \geq \left[\frac{N}{2}\right]+2$, we get by Sobolev's embedding,

$$\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}+2}\|_{L^\infty_{x,v}} \leq C\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}+2}\|_{H^4_{x,v}}$$

Since $\triangle_k \bar{a}_{ij} = a_{ij} *_v \triangle_k f$, we see that

$$|\partial_x^{\alpha_1}\partial_v^{\beta_1}\triangle_k \bar{a}_{ij}| \leq C(1+|v|^2)^{\frac{\gamma+2}{2}} \int_{\mathbf{R}^3} |\partial_x^{\alpha_1}\partial_v^{\beta_1}\triangle_k f(v)|(1+|v|^2)^{\frac{\gamma+2}{2}} dv,$$

and therefore

$$\begin{aligned}
|D_{2,1}| \quad \leq \quad & C\int_{[0,T]\times\mathbf{T}^3} \|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}+2}\|_{H^4_{x,v}} \\
& \times \int_{\mathbf{T}^3}\left(\int_{\mathbf{R}^3} |\partial_x^{\alpha_1}\partial_v^{\beta_1}\triangle_k f|(1+|v|^2)^{\frac{\gamma+2}{2}}|k|^{-\delta-\frac{3}{2}} dv\right) \\
& \times \left(\int_{\mathbf{R}^3} |\nabla_v(\triangle_k g)|(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}-1}|k|^{-\delta-\frac{3}{2}} dv\right) dxdtdk.
\end{aligned}$$

Estimating this quantity like we did for $D_{1,1}$, and using the hypothesis of our lemma, we end up with

$$|D_{2,1}| \leq CK_2 \left(\epsilon \|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([0,T];L^2_{x,v,k})} + C_\epsilon K_2\right). \tag{83}$$

Similarly, we can get

$$|D_{2,2}| \leq CK_2^3. \tag{84}$$

Collecting (79) - (84), we end up with

$$\begin{aligned}
& \left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} VI_1 g_{\delta,k}\, dtdxdvdk\right| \\
\leq \quad & CK_2 \left(\epsilon \|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([0,T];L^2_{x,v,k})} + C_\epsilon K_2^2)\right).
\end{aligned} \tag{85}$$

As in the end of the proof of Lemma 4.1, we can conclude the proof of Lemma 4.3 by choosing ϵ small enough.

We finally write down a lemma allowing to recover a fractional derivative in x for the solution of eq. (1) when we start from a function which lies already in a fractional Sobolev space.

4.4 Gain of a $1/20$ of a derivative in x starting from a Sobolev space whose index is not an integer

Lemma 4.4 *Let $\gamma \in [-2,1]$, $N \geq 8$ be a given integer, $\delta \in]0,19/20]$ and let f be a smooth nonnegative solution of eq. (1) such that assumption A holds for a given constant K. We suppose that for any $T \in]0,+\infty[$ and $s \geq 0$, there exists $K_3 \equiv K_3(\gamma,N,\delta,K,T,s)$ such that*

$$||f(0,\cdot,\cdot)||_{H^{N+\delta,s}_{x,v}} \leq K_3, \qquad \|f\|_{L^\infty_t([0,T];H^{N,s}_{x,v})} \leq K_3, \qquad \|f\|_{L^2_t([0,T];H^{N+\delta,s}_{x,v})} \leq K_3. \tag{86}$$

Then for any $T>0$ and $s \geq 0$, there is a positive constant $\widetilde{C}_4$ which depends on N, s, γ, T, K, K_3 and δ, such that

$$\int_{[0,T]\times\mathbf{R}^3} \|h\|^2_{\dot{H}^{\delta+1/20}_x} dtdv \leq \widetilde{C}_4, \tag{87}$$

where h is given by (33).

Proof of Lemma 4.4 Noticing that

$$\int_{\mathbf{T}^3} |\hat{g}_{\delta,k}(m)|^2 dk = C|m|^{2\delta}|\hat{h}(m)|^2,$$

we know that

$$\begin{aligned}
(87) \quad &\Leftrightarrow \quad \int_{[0,T]\times\mathbf{R}^3} \sum_{m\in\mathbf{Z}^3} |m|^{2\delta+1/10}|\hat{h}(m)|^2 dtdv \leq \widetilde{C}_4 \\
&\Leftrightarrow \quad \int_{[0,T]\times\mathbf{R}^3\times\mathbf{T}^3} \sum_{m\in\mathbf{Z}^3} |m|^{1/10}|\hat{g}_{\delta,k}(m)|^2 dtdvdk \leq \widetilde{C}_4 \\
&\Leftrightarrow \quad \int_{[0,T]\times\mathbf{R}^3\times\mathbf{T}^3} \|g_{\delta,k}\|^2_{\dot{H}^{1/10}_x} dtdvdk \leq \widetilde{C}_4. \qquad (88)
\end{aligned}$$

In order to prove estimate (88), we use the method of the proof of Lemma 4.2. We introduce $p_{\delta,k} = g_{\delta,k}(1+|v|^2)^2$, and write (for $\hat{p}_{\delta,k}$ the Fourier coefficient with respect to x of $p_{\delta,k}$) :

$$\hat{p}_{\delta,k}(t,m,v) = \Big[\hat{p}_{\delta,k}(t,m,v) - (\hat{p}_{\delta,k}(t,m,\cdot) *_v \chi_\epsilon)(v)\Big] + (\hat{p}_{\delta,k}(t,m,\cdot) *_v \chi_\epsilon)(v), \qquad (89)$$

the parameter ϵ being chosen later.

Following the proof of estimate (54), we get

$$\begin{aligned}
&\int_{[0,T]\times\mathbf{R}^3\times\mathbf{T}^3} (1+|v|^2)^{-4} \sum_{m\in\mathbf{Z}^3} |m|^{1/10}\Big|\hat{p}_{\delta,k}(t,m,v) - (\hat{p}_{\delta,k}(t,m,\cdot) *_v \chi_\epsilon)(v)\Big|^2 dtdvdk \\
\leq\ & C\int_{[0,T]\times\mathbf{R}^3\times\mathbf{T}^3} \sum_{m\in\mathbf{Z}^3} |m|^{1/10}\epsilon^2|\nabla_v\hat{p}_{\delta,k}(t,m,v)|^2 dtdvdk. \qquad (90)
\end{aligned}$$

Notice that $p_{\delta,k}$ is the solution of equation (66) where s replaced by $s+4$. As a consequence, we can write

$$\partial_t p_{\delta,k} + v\cdot\nabla_x p_{\delta,k} = p^{(1)}_{\delta,k} + \nabla_v\cdot p^{(2)}_{\delta,k}, \qquad (91)$$

where $p^{(1)}_{\delta,k}$ consists in the sum of the terms IV, V_2, V_5, VI_2 and VI_4.

We claim that $p^{(1)}_{\delta,k}, p^{(2)}_{\delta,k} \in L^2_t([0,T]; L^2_{x,v,k})$. We only present the estimate for the term VI_1 in the case $|\beta|\geq 1$ here, the others being similar. We write

$$\begin{aligned}
&\triangle_k\Big((\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f)\Big)(1+|v|^2)^{\frac{s+4}{2}}|k|^{-\delta-\frac{3}{2}} \\
=\ & (\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij}(x+k))(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(1+|v|^2)^{\frac{s+4}{2}}|k|^{-\delta-\frac{3}{2}} \\
& +(\partial_x^{\alpha_1}\partial_v^{\beta_1}\triangle_k\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f)(1+|v|^2)^{\frac{s+4}{2}}|k|^{-\delta-\frac{3}{2}},
\end{aligned}$$

and we only estimate the first term of the r.h.s of the above equality, the estimate for the second term being similar.

When $1\leq|\alpha_1|+|\beta_1|\leq\left[\frac{N}{2}\right]+1$, we know thanks to (20) that

$$\|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij}(x+k))(1+|v|^2)^{-\frac{\gamma+2}{2}}\|_{L^\infty_t([0,T];L^\infty_{x,k})} \leq C\,K_3.$$

Thanks to the hypothesis of our lemma, we see that

$$\begin{aligned}
&\|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij}(x+k))(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(1+|v|^2)^{\frac{s+4}{2}}|k|^{-\delta-\frac{3}{2}}\|_{L^2_t([0,T];L^2_{x,v,k})} \\
\leq\ & CK_3\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(1+|v|^2)^{\frac{s+\gamma+6}{2}}|k|^{-\delta-\frac{3}{2}}\|_{L^2_t([0,T];L^2_{x,v,k})} \\
\leq\ & CK_3^2. \qquad (92)
\end{aligned}$$

When $|\alpha_1|+|\beta_1|\geq \left[\frac{N}{2}\right]+2$, we know that

$$
\begin{aligned}
&\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(1+|v|^2)^{\frac{s+\gamma+8}{2}}|k|^{-\delta-\frac{3}{2}}\|_{L^\infty_{x,v}}\\
\leq\ & C\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(1+|v|^2)^{\frac{s+\gamma+8}{2}}|k|^{-\delta-\frac{3}{2}}\|_{H^4_{x,v}}.
\end{aligned}
$$

Remembering that $|a(v-v_*)|\leq C|v-v_*|^{\gamma+2}\leq C(1+|v|^2)^{\frac{\gamma+2}{2}}(1+|v_*|^2)^{\frac{\gamma+2}{2}}$, we see that

$$
\begin{aligned}
&\|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}(x+k))(1+|v|^2)^{-\frac{\gamma+4}{2}}\|^2_{L^2_{x,v}}\\
\leq\ & C\int_{\mathbf{T}^3\times\mathbf{R}^3}(1+|v|^2)^{-2}\left(\int_{\mathbf{R}^3}\left|\partial_x^{\alpha_1}\partial_v^{\beta_1}f(x+k,v_*)\right|(1+|v_*|^2)^{\frac{\gamma+2}{2}}dv_*\right)^2 dxdv\\
\leq\ & C\int_{\mathbf{T}^3}\left(\int_{\mathbf{R}^3}\left|\partial_x^{\alpha_1}\partial_v^{\beta_1}f(x+k,v)\right|(1+|v|^2)^{\frac{\gamma+2}{2}}dv\right)^2 dx.
\end{aligned}
$$

Using Minkowski's and Hölder's inequalities and proceeding like in the study of the term III_1 in the proof of Lemma 4.2, we get

$$
\begin{aligned}
&\|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}(x+k))(1+|v|^2)^{-\frac{\gamma+4}{2}}\|^2_{L^2_{x,v}}\\
\leq\ & C\left(\int_{\mathbf{R}^3}\left(\int_{\mathbf{T}^3}|\partial_x^{\alpha_1}\partial_v^{\beta_1}f(x+k,v)|^2(1+|v|^2)^{\gamma+4}dx\right)^{1/2}(1+|v|^2)^{-1}dv\right)^2\\
\leq\ & C\int_{\mathbf{T}^3\times\mathbf{R}^3}|\partial_x^{\alpha_1}\partial_v^{\beta_1}f|^2(1+|v|^2)^{\gamma+4}dxdv.
\end{aligned}
$$

Thus, using Hölder's inequality and the hypothesis of our lemma, we get

$$
\begin{aligned}
&\|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij}(x+k))(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(1+|v|^2)^{\frac{s+4}{2}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([0,T];L^2_{x,v,k})}\\
\leq\ & C\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}\triangle_k f)(1+|v|^2)^{\frac{s+\gamma+8}{2}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([0,T];L^2_k(H^4_{x,v}))}\\
&\times\|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}(x+k))(1+|v|^2)^{-\frac{\gamma+4}{2}}\|^2_{L^\infty_t([0,T];L^\infty_k(L^2_{x,v}))}\leq CK_3^4.
\end{aligned}
\tag{93}
$$

Proceeding like in the estimate for (63), we can get

$$
\begin{aligned}
&\int_{[0,T]\times\mathbf{R}^3\times\mathbf{T}^3}(1+|v|^2)^{-4}\sum_{m\in\mathbf{Z}^3}|m|^{1/10}\left|(\hat{p}_{\delta,k}(t,m,\cdot)*\chi_\epsilon)(v)\right|^2 dtdvdk\\
\leq\ & C\sum_{m\in\mathbf{Z}^3}|m|^{\frac{1}{10}-\frac{1}{2}}(\epsilon^{-6}+\epsilon^{-8})\Big(\|\hat{p}_{\delta,k}(0,m,\cdot)\|^2_{L^2_{v,k}}+\|\hat{p}_{\delta,k}(t,m,\cdot)\|^2_{L^2_t([0,T];L^2_{v,k})}\\
&+\|\hat{p}^{(1)}_{\delta,k}(t,m,\cdot)\|^2_{L^2_t([0,T];L^2_{v,k})}+\|\hat{p}^{(2)}_{\delta,k}(t,m,\cdot)\|^2_{L^2_t([0,T];L^2_{v,k})}\Big).
\end{aligned}
\tag{94}
$$

Choosing $\epsilon=|m|^{-1/20}$, we get (using (90)),

$$
\int_{[t_0,T]\times\mathbf{R}^3\times\mathbf{T}^3}(1+|v|^2)^{-4}\sum_{m\in\mathbf{Z}^3}|m|^{1/10}|\hat{p}_{\delta,k}(m)|^2 dtdvdk\leq C,
\tag{95}
$$

which implies the estimate (88). This ends the proof of Lemma 4.4.

We now finish the proof of Proposition 4.1.

4.5 Proof of Proposition 4.1

We begin by applying Lemmas 4.1 and 4.2. We obtain that $\int_0^T ||f(\theta,\cdot,\cdot)||^2_{H^{N+1/20,s}_{x,v}}\, d\theta < +\infty$. As a consequence, we can find some time $t_1 \in]0, t_*[$ such that $||f(t_1,\cdot,\cdot)||_{H^{N+1/20,s}_{x,v}} < +\infty$.

This enables to use Lemmas 4.3 and 4.4 for the function $f(\cdot - t_1,\cdot,\cdot)$ and obtain that $\int_{t_1}^T ||f(\theta,\cdot,\cdot)||^2_{H^{N+2/20,s}_{x,v}}\, d\theta < +\infty$. As a consequence, we can find some time $t_2 \in]t_1, t_*[$ such that $||f(t_2,\cdot,\cdot)||_{H^{N+2/20,s}_{x,v}} < +\infty$.

We iterate this procedure (that is, Lemmas 4.3 and 4.4) nineteen times and obtain that (for some $t_{19} \in]0, t_*[$)

$$\int_{t_{19}}^T ||f(\theta,\cdot,\cdot)||^2_{H^{N+20/20,s}_{x,v}}\, d\theta < +\infty. \tag{96}$$

As a consequence, we can find some time $t_{20} \in]t_{19}, t_*[$ such that

$$||f(t_{20},\cdot,\cdot)||_{H^{N+1,s}} < +\infty. \tag{97}$$

Taking into account estimates (96) and (97), we use then a last time Lemma 4.1 (with $N+1$ instead of N and for $f(\cdot - t_{20},\cdot,\cdot)$) and obtain that $f \in L^\infty([t_*,T]; H^{N+1,s}_{x,v})$, so that Proposition 4.1 is proven.

5 Proof of Theorem 1.2 in the case $\gamma \in [-2, 1]$

We now end up the proof of Theorem 1.2 in the case when $\gamma \in [-2, 1]$.

Using Proposition 4.1 repeatedly, we get by induction of N that for any $0 < \tau < T < +\infty$ and $s \geq 0$,

$$f \in L^\infty_t([\tau,T]; H^{\infty,s}_{x,v}). \tag{98}$$

We now prove by induction on n that $\partial_t^n f \in L^\infty([\tau,T]; H^{\infty,s}_{x,v})$. According to (98), this is true for $n = 0$. Let us assume that the induction hypothesis holds for any integer $k \leq n$. Then for all multi-indices α and β and $s \geq 0$,

$$\begin{aligned}
&\left[\partial_x^\alpha \partial_v^\beta (\partial_t^{n+1} f)\right](1+|v|^2)^{s/2} = -\left(\partial_x^\alpha \partial_v^\beta [v\cdot\nabla_x(\partial_t^n f)]\right)(1+|v|^2)^{s/2} \\
&\quad + \partial_x^\alpha \partial_v^\beta \left\{\sum_{l=0}^n C_l^n \nabla_v \cdot \left[(a *_v (\partial_t^l f))\nabla_v(\partial_t^{n-l} f) - (b *_v (\partial_t^l f))(\partial_t^{n-l} f)\right]\right\}(1+|v|^2)^{s/2} \\
= \;& -\left(\partial_x^\alpha \partial_v^\beta [v\cdot\nabla_x(\partial_t^n f)]\right)(1+|v|^2)^{s/2} + \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} \sum_{l=0}^n C^\alpha_{\alpha_1} C^\beta_{\beta_1} C^n_l \left[\left(\partial_x^{\alpha_1}\partial_v^{\beta_1}(a_{ij} *_v (\partial_t^l f))\right)\right. \\
&\times \left(\partial_x^{\alpha_2}\partial_v^{\beta_2}\partial_{v_i}\partial_{v_j}\partial_t^{n-l} f\right)(1+|v|^2)^{s/2} + \left(\partial_x^{\alpha_1}\partial_v^{\beta_1}\partial_{v_i}(a_{ij} *_v (\partial_t^l f))\right)\left(\partial_x^{\alpha_2}\partial_v^{\beta_2}\partial_{v_j}\partial_t^{n-l} f\right)(1+|v|^2)^{s/2} \\
&- \left(\partial_x^{\alpha_1}\partial_v^{\beta_1}(b_i *_v (\partial_t^l f))\right)\left(\partial_x^{\alpha_2}\partial_v^{\beta_2}\partial_{v_i}\partial_t^{n-l} f\right)(1+|v|^2)^{s/2} \\
&\left.- \left(\partial_x^{\alpha_1}\partial_v^{\beta_1}\partial_{v_i}(b_i *_v (\partial_t^l f))\right)\left(\partial_x^{\alpha_2}\partial_v^{\beta_2}\partial_t^{n-l} f\right)(1+|v|^2)^{s/2}\right].
\end{aligned}$$

We denote the r.h.s of the above equality by $E + \sum_{i=1}^4 G_i$. It is clear that $E \in L^2_{x,v}$ thanks to the induction hypothesis. Let us then consider the terms G_i. Using Lemma 3.1, we see that

$$\|\partial_x^{\alpha_1}\partial_v^{\beta_1}(a_{ij} *_v (\partial_t^l f))\|_{L^\infty_x} \leq C(1+|v|^2)^{\frac{\gamma+2}{2}} \|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\partial_t^l f)(1+|v|^2)^{\frac{\gamma+4}{2}}\|_{H^2_{x,v}},$$

and therefore

$$
\begin{aligned}
& \|G_1\|_{L^\infty_t([\tau,T];L^2_{x,v})} \\
\leq \;& C\|(\partial_x^{\alpha_1}\partial_v^{\beta_1}\partial_t^l f)(1+|v|^2)^{\frac{\gamma+4}{2}}\|_{L^\infty_t([\tau,T];H^2_{x,v})}\|(\partial_x^{\alpha_2}\partial_v^{\beta_2}\partial_{v_i}\partial_{v_j}\partial_t^{n-l} f)(1+|v|^2)^{\frac{s+\gamma+2}{2}}\|_{L^\infty_t([\tau,T];L^2_{x,v})}.
\end{aligned}
$$

From the induction hypothesis, we conclude that $G_1 \in L^2_{x,v}$. The other terms can be treated in the same way.

We now briefly explain how the estimate that we obtained up to now can be made independent on T. We observe that for any $s>0$, $N>0$, $\tau \in]0,1[$, $\theta \geq 0$, the norm

$$
\sup_{|k|\leq N}\sup_{t'\in[\tau,1]} ||\partial_t^k f(\theta+t',\cdot,\cdot)||_{H^{N,s}_{x,v}}
$$

has been estimated from above by a constant which depends only on γ, s, N, τ and the $H^{8,s'}_{x,v}$ (for any $s'>0$) norm of f and the constant K appearing in (9) for times $t \in [\theta, \theta+1]$. Since those quantities are supposed to be bounded on $[0,+\infty[$ (Cf. Assumption A), we see that for any $\tau > 0$, $f \in W^{\infty,\infty}([\tau,+\infty[;\cap_{s\geq 0} H^{\infty,s}_{x,v})$.

This ends the proof of Theorem 1.2 when $\gamma \in [-2,1]$.

Remark 5.1 *Notice that in order to have a completely rigorous proof, all the estimates above should in fact be made on a version of equation* (1) *with smooth data and then extended to the solution under consideration by a passage to the limit. This leads to no difficulty.*

6 The case $\gamma \in [-3,-2[$

In this section, we present the main differences in the proof of Theorem 1.2 when γ lies in $[-3,-2[$ instead of lying in $[-2,1]$. The differences are concentrated in the treatment of inequalities like

$$
|a(v-v_*)| \leq C|v-v_*|^{\gamma+2} \leq C(1+|v|^2)^{\frac{\gamma+2}{2}}(1+|v_*|^2)^{\frac{\gamma+2}{2}}. \tag{99}
$$

Indeed, this inequality holds when $\gamma \geq -2$ only. When $\gamma \in [-3,-2[$, estimate (99) is no longer true, and has to be replaced by inequlities used in (27) - (30) and which ensure that

$$
\int_{\mathbf{R}^3} |v-v_*|^{2(\gamma+2)}(1+|v_*|^2)^{-2} dv_* \leq C. \tag{100}
$$

Since apart from this modification, the proof is very similar to that of the case $\gamma \in [-2,1]$, we only show the estimate for a representative term. We choose for that the first term on the r.h.s of (46) in the case $|\beta| \geq 1$.

When $|\alpha_1|+|\beta_1| \leq \left[\frac{N}{2}\right]+1$, we know thanks to Lemma 3.1 that

$$
\|\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij}\|_{L^\infty_t([0,T];L^\infty_{x,v})} \leq CK_0.
$$

Then,

$$
\begin{aligned}
& \left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} (\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j} f)(\partial_{v_i} g)(1+|v|^2)^s dtdxdv\right| \\
\leq \;& C\,K_0\left(\epsilon\|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} + C_\epsilon\|f\|^2_{L^2_t([0,T];H^{N,s-\frac{\gamma}{2}}_{x,v})}\right) \\
\leq \;& C\,K_0\left(\epsilon\|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([0,T];L^2_{x,v})} + C_\epsilon K_0^2\right).
\end{aligned}
$$

When $|\alpha_1|+|\beta_1|\geq\left[\frac{N}{2}\right]+2$, using Sobolev's embedding, we get

$$\begin{aligned}
& \|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f)(1+|v|^2)^{\frac{s}{2}-\frac{\gamma}{4}+1}\|_{L_t^\infty([0,T];L^\infty_{x,v})} \\
\leq\; & C\|(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f)(1+|v|^2)^{\frac{s}{2}-\frac{\gamma}{4}+1}\|_{L_t^\infty([0,T];H^4_{x,v})}\leq CK_0.
\end{aligned}$$

Then, thanks to (100),

$$\begin{aligned}
& \left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f)(\partial_{v_i}g)(1+|v|^2)^s dtdxdv\right| \\
\leq\; & CK_0\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}\left(\int_{\mathbf{R}^3}|v-v_*|^{\gamma+2}|\partial_x^{\alpha_1}\partial_v^{\beta_1}f(v_*)|dv_*\right)|\nabla_v g|(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}-1}dtdxdv \\
\leq\; & C\,K_0\int_{[0,T]\times\mathbf{T}^3}\left(\int_{\mathbf{R}^3}|(\partial_x^{\alpha_1}\partial_v^{\beta_1}f)(1+|v|^2)|^2dv\right)^{1/2}\left(\int_{\mathbf{R}^3}|\nabla_v g|(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}-1}dv\right)dtdx.
\end{aligned}$$

Using then Cauchy-Schwarz's and Minkowski's inequality, we get

$$\begin{aligned}
& \left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}(\partial_x^{\alpha_1}\partial_v^{\beta_1}\bar{a}_{ij})(\partial_x^{\alpha_2}\partial_v^{\beta_2+\delta_j}f)(\partial_{v_i}g)(1+|v|^2)^s dtdxdv\right| \\
\leq\; & C\,K_0\int_0^T\left[\int_{\mathbf{T}^3\times\mathbf{R}^3}|(\partial_x^{\alpha_1}\partial_v^{\beta_1}f)(1+|v|^2)|^2dxdv\right]^{1/2} \\
& \times\left[\int_{\mathbf{R}^3}(1+|v|^2)^{-1}\left(\int_{\mathbf{T}^3}\left|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\right|^2dx\right)^{1/2}dv\right]dt \\
\leq\; & C\,K_0\int_0^T\left[\int_{\mathbf{T}^3\times\mathbf{R}^3}\left|(\partial_x^{\alpha_1}\partial_v^{\beta_1}f)(1+|v|^2)\right|^2dxdv\right]^{1/2} \\
& \times\left[\int_{\mathbf{T}^3\times\mathbf{R}^3}\left|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\right|^2dxdv\right]^{1/2}dt \\
\leq\; & C\,K_0\left(\epsilon\|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L_t^2([0,T];L^2_{x,v})}+C_\epsilon\|f\|^2_{L_t^2([0,T];H^{N,2}_{x,v})}\right) \\
\leq\; & C\,K_0\left(\epsilon\|(\nabla_v g)(1+|v|^2)^{\frac{s}{2}+\frac{\gamma}{4}}\|^2_{L_t^2([0,T];L^2_{x,v})}+C_\epsilon K_0^2\right).
\end{aligned}$$

The other terms appearing in the various estimates of the proof of Proposition 4.1 and Theorem 1.2 in the case when $\gamma\in[-3,-2[$ can be treated in the same way.

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