About Global Existence for Quadratic Systems of Reaction-Diffusion

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Abstract: We prove global existence in time of weak solutions to a class of quadratic reaction-diffusion systems for which a Lyapounov structure of $L \log L$-entropy type holds. The approach relies on an a priori dimension-independent L^2-estimate, valid for a wider class of systems including also some classical Lotka-Volterra systems, and which provides an L^1-bound on the nonlinearities, at least for not too degenerate diffusions. In the more degenerate case, some global existence may be stated with the use of a weaker notion of renormalized solution with defect measure, arising in the theory of kinetic equations.

Key Words: reaction-diffusion system, weak solutions, renormalized solutions, entropy methods
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1 Introduction

To introduce the purpose of this paper, let us consider the following 4×4 reaction-diffusion system (arising in reversible chemistry, Cf. [BD]) set on a regular bounded domain $\Omega \subset \mathbf{R}^N$: for $i = 1, 2, 3, 4$,

$$\begin{cases} \partial_t a_i - d_i \Delta a_i = (-1)^i [a_1 a_3 - a_2 a_4] \\ n \cdot \nabla_x a_i = 0 \text{ on } \partial\Omega \\ a_i(0) = a_{i0} \geq 0, \end{cases} \tag{1}$$

where the d_i are positive constants, $a_{i0} \in L^\infty(\Omega)$, and n denotes the outer normal to $\partial\Omega$. The existence of a positive regular solution locally in time is classical. The global existence in time of a regular solution is not so obvious, and is even an open question in higher space dimensions. However, besides the preservation of positivity, this system offers some specificities which may be used to prove at least the existence of *global weak solutions in time*.

Indeed, a main point is that the nonlinear reactive terms add up to zero. Then, according to the Remark 2.2 in [PSch], it follows (by a duality argument) that the a_i are a priori bounded in $L^2(Q_T)$ for any T where we denote $Q_T = (0, T) \times \Omega$ and for *any dimension* N. Consequently, the nonlinearities are a priori bounded in $L^1(Q_T)$ for all T. Now, it follows from the results in [Pie] that the above structure, together with L^1-bounds on the nonlinearities, provides the existence of *global weak solutions* (see below for the meaning). We recall in the Appendix the main steps of this approach.

Here, we would like to extend the use of the dimension-independent L^2-estimate just mentioned and show how it can be quite more exploited for this kind of systems and how it is robust enough to carry over to variable diffusion coefficients and even to degenerate diffusions coefficients. Let us explain our goals on the above specific system.

As it is well known, besides the property $\sum f_i(a) = 0$ (where we denote $a = (a_1, a_2, a_3, a_4)$ and $f_i(\cdot)$ is the i-th nonlinearity), it also satisfies the entropy inequality

$$\sum_i \log(a_i) f_i(a) \leq 0. \tag{2}$$

As a consequence, if we denote $z_i = a_i \log(a_i) - a_i$, one has

$$(z_1 + z_2 + z_3 + z_4)_t - \Delta_x(d_1 z_1 + d_2 z_2 + d_3 z_3 + d_4 z_4) \leq 0.$$

Using the same L^2-estimate as the one just mentioned (see Appendix and Theorem 3.1), we can prove that $z = \sum_i z_i$ is bounded in $L^2(Q_T)$ for all T. This means that, not only the right-hand side of (1) is bounded in L^1, but it is *uniformly integrable*. Therefore, if we consider a good approximation of the system for which global existence in time of classical solutions holds, the nonlinear terms are uniformly integrable. On the other hand, by compactness properties of the heat operator, the $L^1(Q_T)$-bound of the right-hand side provides $L^1(Q_T)$-compactness of the approximate solution (a^n), and, up to a subsequence, convergence a.e. of the nonlinear terms. This, together with uniform integrability, yields convergence of the right-hand side in L^1. As a consequence, we obtain global existence for (1) *using only the L^2-estimates*.

This is what we show below for a general class of systems for which a structure of type (2) exists. Moreover, we show how the main L^2-estimate may be extended to time-space dependent diffusions (and therefore to some quasi-linear problems) and even to some degenerate situations. It may also be applied to some classical quadratic Lotka-Volterra systems for which global existence of classical solutions is unresolved in high dimension (see [Leung],[FHM]).

In the last part of the paper, we provide some alternative when the nonlinearities are not bounded in L^1. This is for instance the case when the diffusions are very degenerate. We then take up ideas around *renormalized solutions* from the theory of kinetic equations (see e.g. [DiL, CIP] for Boltzmann's equation of gas dynamics), or more precisely, renormalized solutions with defect measure (Cf. [V2, AlV1, AlV2] for Landau's equation of plasma physics and Boltzmann's equation without angular cutoff). In particular, for some typical example of such a very degenerate situation, we prove convergence of approximate solutions toward renormalized global solutions with defect measure. Another situation where those renormalized solutions are useful is described : it occurs when higher powers of nonlinearities appear in the reaction term.

Finally, we present briefly in an appendix a short proof of the duality argument in the simplest case (this proof is taken from [Pie, PSch], and the a priori estimates which are obtained without using the duality argument, by an approach centered on the entropy estimate (such an approach was used for getting explicit rates of convergence toward equilibrium for reaction diffusion systems in [DF]).

2 Notations and general assumptions

All along the paper, we will use the following general notations and assumptions: we denote by $q \geq 1$ the number of equations of the system, and, for all $i = 1, ..., q$, we are given:

- $d_i \in C^1([0, +\infty) \times \overline{\Omega}), d_i \geq 0$, with $\nabla_x \sqrt{d_i} \in L^\infty(Q_T)$ (that is $\sigma = 2\sum_i \|\nabla_x \sqrt{d_i}\|_{L^\infty(Q_T)} < \infty$ for all T),
- $f : (0, +\infty) \times \Omega \times [0, +\infty)^q \to \mathbf{R}^q$ measurable and "locally Lipschitz continuous", that is: for $f = (f_1, ..., f_q)$ and $|\cdot|$ denoting the Euclidean norm in $\mathbf{R}^q$:

$$\left.\begin{array}{l} \text{there exists } k(\cdot) : [0, +\infty) \to [0, +\infty) \text{ nondecreasing such that} \\ a.e. (t, x) \in (0, +\infty) \times \Omega, \text{and } \forall r, \hat{r} \in [0, +\infty)^q : \\ |f(t, x, r) - f(t, x, \hat{r})| \leq k(\max\{|r|, |\hat{r}|\})|r - \hat{r}|, \\ \text{and } \ \forall T > 0 : \ [(t, x) \to f(t, x, 0)] \in L^\infty(Q_T), \end{array}\right\} \tag{3}$$

- and the *positivity preserving condition*

$$a.e. (t, x), \forall i, \forall r \in [0, +\infty)^q : \ f_i(t, x, r_1, ..., r_{i-1}, 0, r_{i+1}, ...r_q) \geq 0.$$

For simplicity, we will often write $f(r) = f(t, x, r)$, even when f does depend on (t, x).

We will consider the following reaction-diffusion systems: for all $i = 1, ..., q$

$$\left.\begin{array}{rcl} \partial_t a_i - \nabla_x \cdot (d_i \nabla_x a_i) & = & f_i(a), \\ n \cdot \nabla_x a_i & = & 0 \ \ [\text{ or } \forall i = 1, ..., q, \ a_i = 0 \ ] \text{ on } \ \partial\Omega. \\ a_i(0) & = & a_{i0} \geq 0. \end{array}\right\} \tag{4}$$

By *regular solution* on $(0, T)$, we mean a function $a \in C([0, T) \times \Omega)$ such that

$$\partial_t a_i, \partial_{x_k} a_i, \partial_{x_k x_l} a_i, f_i(a) \in L^2(Q_T)$$

and which satisfies the system pointwise a.e. (with the boundary condition as well).

By *weak solution*, we mean a solution "in the sense of the variation of constant formula", that is, $f(a) \in L^1(Q_T)^q$ for all T and

$$\forall t \geq 0, \ \ a(t) = S(t)a_0 + \int_0^t S(t - s)f(a(s))\, ds, \tag{5}$$

where $S(t)$ is the linear semi-group associated with the linear part of the system with the same boundary conditions (that is, $t \to S(t)a_0$ is solution of the system with $f \equiv 0$ and the initial data $a(0) = a_0$).

For the definition of renormalized solutions, we introduce truncation functions $T_k : [0, +\infty) \to [0, +\infty)$ of class C^2, nondecreasing, concave and such that

$$\forall r \in [0, k - 1], T_k(r) = r, \ \forall r \geq k + 1, T_k(r) = k, \ \forall r, \ 0 \leq T_k'(r) \leq 1. \tag{6}$$

By *renormalized solution with defect measure*, or *"renormalized supersolution"*, we mean a function $a \in L^1(Q_T)^q$ with $T_k'(a_i)f_i(a) \in L^1(Q_T)$ and $T_k'(a_i)\, d_i \nabla_x a_i \in L^2(Q_T)$ such that, for all $k > 0$ and all $i = 1, ..., q$

$$\partial_t T_k(a_i) - \nabla_x \cdot (d_i \nabla_x T_k(a_i)) \geq T_k'(a_i) f_i(a) - T_k''(a_i)\, d_i |\nabla_x a_i|^2. \tag{7}$$

If such a renormalized solution is regular enough so that $f_i(a) \in L^1(Q_T)$ and $d_i \nabla_x a_i \in L^1(Q_T)$, then we may let k tend to $+\infty$ in (7) to obtain

$$\partial_t a_i - \nabla_x \cdot (d_i \nabla_x a_i) \geq f_i(a). \tag{8}$$

Next, if on the other hand, the nonlinearity presents some kind of dissipative law like:

$$\sum_i f_i(a) \leq 0, \tag{9}$$

then we do obtain the reverse inequality in (8) for renormalized solution obtained as limits of regular solutions, and we are led to a (weak)-global solution (see Appendix and the proofs of Theorem 4.2 and Corollary 5.1 for such a two-sided approach). Note that all the renormalized solutions built in this paper correspond to cases where (9) is satisfied.

About uniform integrability. In several proofs, we will use the following fact: let $(U_n)_{n\geq 0}$ be a bounded sequence in $L^1(Q_T)$ satisfying the two properties

- (U_n) is uniformly integrable, that is: $\forall \epsilon > 0,\ \exists \delta_\epsilon > 0$ such that

$$[K \subset Q_T \text{ measurable},\ |K| \leq \delta_\epsilon] \Rightarrow [\forall n \geq 0,\ \int_K |U_n| \leq \epsilon], \tag{10}$$

- (U_n) converges a.e. to U.

Then, (U_n) actually converges in $L^1(Q_T)$ to U. Indeed, recall that, by a.e. convergence, for all $\epsilon > 0$, there exists $K \subset Q_T$ measurable such that $|K| \leq \delta_\epsilon$ and (U_n) converges uniformly to U on $Q_T \setminus K$. We then couple this with the uniform integrability. Note that (10) is satisfied as soon as $\sup_n \int_{Q_T} \Phi(|U_n|) < \infty$ where $\Phi : (0, \infty) \to (0, \infty)$ is an increasing function such that $\lim_{r\to+\infty} \Phi(r)/r = +\infty$.

Last remark: We will always consider *nonnegative solutions.*

3 The main L^2-estimate

The main result of this section is the following.

Theorem 3.1 *Assume that f satisfies the general assumptions of Section 2 and*

$$\forall r \in [0, +\infty)^q,\ a.e.(t, x),\ \sum_{i=1}^q h_i'(r_i) f_i(r) \leq \Theta(t, x) + \mu \sum_i h_i(r_i),$$

where $\Theta \in L^2_{loc}\big([0, +\infty), L^2(\Omega)\big)$, $\mu \in [0, +\infty)$ and for $i = 1, ..., q$

$$h_i : [0, +\infty) \to [0, +\infty) \text{ is convex continuous}, \in W^{1,\infty}_{loc}(0, +\infty),\ h_i(0) = 0.$$

Let a be a regular positive solution of (4). Then, setting $z_i = h_i(a_i), z = \sum z_i, z_d = \sum d_i z_i$, we have

$$\int_{Q_T} z\, z_d \leq C\left(\|z(0)\|^2_{L^2(\Omega)} + T\|\Theta\|^2_{L^2(Q_T)} \right), \tag{11}$$

where $C = C(\mu, \sigma, \max_i\{\|d_i\|_\infty\}, T, \Omega)$.

Remark: Note that

$$\min_i\{\inf_{Q_T} d_i\} \int_{Q_T} z^2 \leq \int_{Q_T} z\, z_d.$$

Therefore, in the nondegenerate case (that is: $\min_i\{\inf_{Q_T} d_i\} > 0$), z is bounded in $L^2(Q_T)$. It is interesting to notice that the product $z\, z_d$ is always bounded in $L^1(Q_T)$ independently of a lower bound for the d_i's (that is to say, even if the system is degenerate).

We may slightly improve the dependence in $z(0)$ in estimate (11) (see the Remark after the proof).

In the case of Dirichlet conditions, we may choose $C = C(\Omega)e^{2(\sigma^2+\mu)T}$ (see (17),(18) in the proof). If moreover $0 = \sigma = \mu = \Theta$, we obtain an estimate up to $T = +\infty$, namely

$$\min_i\{\inf d_i\} \int_{[0,+\infty)\times\Omega} z^2 \leq \int_{[0,+\infty)\times\Omega} z\, z_d \leq C(\Omega)\|z(0)\|^2_{L^2(\Omega)}.$$

This provides a first information for the asymptotic behavior of $a(t)$ in the globally nondegenerate case ($\min_i\{\inf d_i\} > 0$).

Proof: It is adapted from the particular case $\sigma = \mu = \Theta = 0$ (see [PSch] and also the Appendix which may be used in a first reading).

Using $h_i'' \geq 0$, we have for all i:

$$\partial_t z_i - \nabla_x \cdot (d_i \nabla_x z_i) \leq h_i'(a_i) f_i(a),$$

so that

$$\partial_t z - \nabla_x \cdot (\sum d_i \nabla_x z_i) \leq \sum h_i'(a) f_i(a) \leq \Theta + \mu z. \tag{12}$$

Let us estimate z by duality. If we multiply the above inequation by some $w \geq 0$ regular enough, with $w(T) = 0$, and satisfying the same boundary conditions as the $a_i's$, we obtain

$$-\int_\Omega w(0)z(0) - \int_{Q_T} w_t z + w\, \nabla_x \cdot (\sum d_i \nabla_x z_i) \leq \int_{Q_T} \mu z w + \Theta w$$

or also, after integration by parts

$$-\int_\Omega w(0)z(0) - \int_{Q_T} w_t z + \sum z_i \nabla_x \cdot (d_i \nabla_x w) \leq \int_{Q_T} \mu z w + \Theta w,$$

where we used

$$\int_{\partial\Omega} w \sum d_i \nabla_x z_i \cdot n - \nabla_x w \cdot n \sum d_i z_i = 0.$$

Thus

$$\int_{Q_T} \Theta w + \int_\Omega w(0)z(0) \geq -\int_{Q_T} z(w_t + A\Delta w + B\cdot\nabla_x w + \mu w), \tag{13}$$

where we set

$$A := z_d/z, \;\; B := (\sum z_i \nabla_x d_i)/z \;\text{ where }\; z \neq 0,$$

$$A := \min_i\{\inf d_i\}, \;\; B = 0 \;\text{ where }\; z = 0.$$

The dual problem: To estimate z by duality, we introduce the following dual problem where $H \in \mathcal{C}_0^\infty(Q_T)$ is an arbitrary *nonnegative* test-function and the boundary condition is the same as for the $a_i's$:

$$\left.\begin{array}{rcl} -(w_t + A\Delta_x w + B\cdot\nabla_x w + \mu w) & = & H\sqrt{A}, \\ n\cdot\nabla_x w = 0, \;\big[\text{resp. } w & = & 0\big] \text{ on } \partial\Omega, \;\; w(T) = 0. \end{array}\right\} \tag{14}$$

Thanks to the nonnegativity of the z_i, we have

$$0 \leq \min_i\{\inf d_i\} \leq A \leq \max_i\{\max d_i\}.$$

If A, B are regular enough and $\min_i\{\inf d_i\} > 0$, then up to changing t into $T-t$, (14) is a good classical parabolic problem for which a unique positive solution w exists (see e.g. [LSU]). In general, we solve (14) for regular approximations A_n, B_n and we plug $w = w_n$, its solution, into (13). It is easy to pass to the limit in (13) using the estimates that we are going to derive on w_n. In particular, they will not depend on $\min_i\{\inf d_i\}$ and neither on the regularity of A_n, B_n. Therefore, in what follows, we drop the n-indices and we make estimates on problem (14) assuming enough regularity.

With $\sigma = 2\sum_i \|\nabla_x \sqrt{d_i}\|_\infty = \sum_i \|\nabla_x d_i/\sqrt{d_i}\|_\infty$, we have

$$|B| \leq \sigma(\sum z_i\sqrt{d_i})/z = \sigma(\sum \sqrt{z_i}\sqrt{z_i}\sqrt{d_i})/z \leq \sigma z^{1/2} z_d^{1/2}/z = \sigma\sqrt{A}.$$

We deduce for the dual problem (14) that

$$-(w_t + A\Delta_x w) \leq \sqrt{A}\,(\sigma|\nabla_x w| + H) + \mu w. \tag{15}$$

Multiplying (15) by $-\Delta w$ and integrating over Ω give for all $t \in (0,T)$:

$$\begin{array}{l} -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla_x w(t)|^2 + \int_\Omega A(\Delta_x w)^2 \\ \leq \int_\Omega \sqrt{A}|\Delta_x w|(\sigma|\nabla_x w| + H) + \mu\int_\Omega |\nabla_x w|^2. \end{array} \tag{16}$$

We set $\beta(t) = \int_\Omega |\nabla_x w(t)|^2$. By Young's inequality, the right-hand side of (16) may be bounded from above by

$$\frac{1}{2}\int_\Omega A(\Delta_x w)^2 + \int_\Omega \Big(\sigma^2|\nabla_x w|^2 + H^2 + \mu|\nabla_x w|^2\Big).$$

All this may be rewritten

$$-\beta'(t) - 2(\sigma^2 + \mu)\beta(t) + \int_\Omega A(\Delta_x w)^2 \leq 2\int_\Omega H^2.$$

We deduce, setting $\rho(t) = e^{2(\sigma^2+\mu)t}$, that

$$-\frac{d}{dt}(\rho(t)\beta(t)) + \rho(t)\int_\Omega A(\Delta_x w)^2 \le \rho(t)\int_\Omega 2H^2.$$

We integrate from t to T and use $w(T) = 0$ to obtain

$$\forall t \in (0,T),\ \rho(t)\beta(t) + \int_t^T \rho(\tau)d\tau \int_\Omega A(\Delta_x w)^2 \le \int_t^T \rho(\tau)d\tau \int_\Omega 2H^2(\tau),$$

which implies

$$\int_\Omega |\nabla_x w(t)|^2 + \int_{[t,T]\times\Omega} A(\Delta_x w)^2 \le 2\rho(T)\int_{Q_T} H^2. \tag{17}$$

Recall that, for some $C = C(\Omega)$

$$C\int_\Omega |\nabla_x w(t)|^2 \ge \begin{cases} \int_\Omega w(t)^2 & \text{if } w = 0 \text{ on } \partial\Omega \\ \int_\Omega (w(t) - \frac{1}{|\Omega|}\int_\Omega w(t))^2 & \text{in all cases.} \end{cases} \tag{18}$$

We can bound the averages of $w(t)$ by going back to the equation (14) and using that

$$\begin{array}{rcl} \int_\Omega e^{\mu t} w(t) & = & -\int_{[t,T]\times\Omega} e^{\mu\tau}(w_t + \mu w) \\ & \le & e^{\mu T}\int_{Q_T} |A\Delta_x w + B\cdot\nabla_x w + H\sqrt{A}| \\ & \le & \sqrt{\max_i \|d_i\|_\infty}\, C(T,\mu,\sigma,|\Omega|)(\int_{Q_T} H^2)^{1/2}, \end{array} \tag{19}$$

so that we get in all cases

$$\sup_{t\in[0,T]} \int_\Omega w(t)^2 \le C\int_{Q_T} H^2, \quad C = C(\mu,\sigma,\max\|d_i\|_\infty, T, \Omega). \tag{20}$$

Back to the estimate of z: We now come back to the inequality (13) which writes

$$\int_{Q_T} zH\sqrt{A} \le \int_{Q_T} \Theta w + \int_\Omega w(0)z(0). \tag{21}$$

Note that $\int_\Omega w(0)z(0) \le \|w(0)\|_{L^2(\Omega)}\|z(0)\|_{L^2(\Omega)}$ and

$$\int_{Q_T} \Theta w \le \|\Theta\|_{L^2(Q_T)}\|w\|_{L^2(Q_T)} \le \|\Theta\|_{L^2(Q_T)}\sqrt{T}\sup_{t\in[0,T]}\|w(t)\|_{L^2(\Omega)}.$$

Since H is arbitrary, we deduce by duality from (21),(20) that

$$\int_{Q_T} z\, z_d = \|\sqrt{A}z\|^2_{L^2(Q_T)} \le C[\|z(0)\|^2_{L^2(\Omega)} + T\|\Theta\|^2_{L^2(Q_T)}].$$

Remarks: Note that we actually get, not only an L^2-estimate for the solution w of (14), but even "maximal regularity" in $L^2(Q_T)$ for this equation in the sense that: if $H \in L^2(Q_T)$, then $\sqrt{A}\Delta_x w, w_t$ and $\nabla_x w$ are separately in $L^2(Q_T)$. We refer to the comments in [PSch] for the same questions in L^p.

We may improve the dependence on the initial data in Theorem 3.1 by using Sobolev imbedding in (18). Indeed, we may use instead

$$\left(\int_\Omega w(0)^p\right)^{2/p} \le C\left(\int_\Omega |\nabla_x w(0)|^2 + \int_{\partial\Omega} w(0)^2\right), \tag{22}$$

for $p = +\infty$ if $N = 1$, any $p < +\infty$ if $N = 2$ and $p = 2N/(N-2)$ if $N \geq 3$.
As a consequence, using $\int_\Omega w(0)z(0) \leq \|w(0)\|_{L^p}\|z(0)\|_{L^q}$ in the proof instead of an L^2-duality, in Theorem 3.1, we may replace $\|z(0)\|_{L^2(\Omega)}$ by $\|z(0)\|_{L^q(\Omega)}$ where $q = 1$ if $N = 1$, any $q > 1$ if $N = 2$ and $q = 2N/(N+2)$ if $N \geq 3$. This allows to solve the systems with weaker assumptions on the initial data.

4 Application to quadratic systems

Let us apply the above estimates to systems similar to the one given in the introduction, that is where the *nonlinearity is at most quadratic* and where the "entropy" is controlled.

Theorem 4.1 *Besides the assumptions of the introduction, assume that:*
- *the function $k(\cdot)$ in (3) satisfies $k(r) \leq C\,(|r|+1)$,*
- *$\forall r \in (1,+\infty)^q$, a.e.$(t,x)$, $\sum_i \log(r_i) f_i(t,x,r) \leq \Theta(t,x) + \mu \sum_i r_i \log r_i$ where $\Theta \in L^2(Q_T), \mu \in [0,+\infty)$.*
- *$\exists\, d_0 \in (0,+\infty)$ such that, $\forall i = 1,...,q,\ 0 < d_0 \leq d_i$.*

Then, the system (4) has a global weak solution in any dimension for all nonnegative initial data a_0 such that $|a_0|\log(|a_0|) \in L^2(\Omega)$.

Remark: As noticed at the end of the previous Section, we may relax the condition on the initial data to $|a_0|\log(|a_0|) \in L^q(\Omega)$ for some $q < 2$ well-chosen.

Proof of Theorem 4.1: We regularize the initial data and we truncate the nonlinearities f_i by setting $f_i^n(r) := \psi_n(r) f_i(r)$ where $\psi_n(r) = \psi_1(|r|/n)$ and $\psi_1 : [0,+\infty) \to [0,1]$ is $\mathcal{C}^\infty$ and satisfies

$$\forall\, 0 \leq s \leq 1,\ \psi_1(s) = 1,\ \forall s \geq 2, \psi_1(s) = 0.$$

Then, we easily check that the function f^n satisfies also the assumptions of the introduction and of Theorem 3.1 where we set $h_i(x) = [x\log(x) - x]^+$ (note that $r_i \log r_i \leq 2h_i(r_i)$ for large r_i). Moreover f^n is bounded on Q_T for all n. Therefore, by the classical theory of existence (see e.g. [Ama85], [Rothe], [LSU] and their references), the approximate system has a unique regular global solution a^n on $(0,\infty)$ for regular approximations a_0^n of the initial data a_0.

We now apply Theorem 3.1 with h_i chosen as above. It follows that $a_i^n \log(a_i^n)$ is bounded in $L^2(Q_T)$ independently of n. Since, the nonlinearity f is at most quadratic, it follows that $f^n(a^n)$ is uniformly integrable on Q_T. By compactness of the linear operator in $L^1(Q_T)$ (see e.g. [BP]), we may assume (up to a subsequence) that a^n converges as $n \to +\infty$ in $L^1(Q_T)$ and a.e. to some a, this for all T. In particular, $f^n(a^n)$ converges a.e. to $f(a)$. But, uniform integrability on Q_T and convergence a.e. imply convergence in $L^1(Q_T)$. Consequently, we can pass to the limit in the formula

$$a^n(t) = S(t)a_0^n + \int_0^t S(t-s) f^n(a^n(s))\,ds,$$

and this proves Theorem 4.1.

Remark: The above proof is rather simple thanks to the fact that the L^2-estimate directly provides the uniform integrability of the nonlinearities. The situation is more delicate when one has only an L^1-bound on the nonlinearities. Then, as explained in the Appendix, we may apply results from [Pie].

As a new example of the usefulness of the L^2-estimate, we show here how one may prove global existence of weak solutions for quadratic Lotka-Volterra systems of the type described in [Leung] (see also [FHM]) and given as follows:

$$\left.\begin{array}{rcl} \partial_t a &=& D\Delta_x a + \mathcal{A}P(a-z) \\ \nabla_x a\cdot n &=& 0 \text{ on } \partial\Omega,\ a(0,\cdot)=a_0(\cdot) \end{array}\right\} \tag{23}$$

where the data are : $D = \operatorname{diag}\{d_1,...,d_q\}$ a diagonal matrix with positive constants d_i, $P = [p_{ij}]$ a $q\times q$ matrix and $z\in(0,+\infty)^q$. The unknown is $a:[0,T]\to L^2(\Omega)^q$ and $\mathcal{A}=\operatorname{diag}(a_1,...,a_q)$. Here $f_i(a) = a_i\sum_{j=1}^q p_{ij}(a_j - z_j)$.

Theorem 4.2 *Assume there exists* $\Sigma = \operatorname{diag}(\sigma_1,...,\sigma_q)$ *with* $\sigma_i > 0$ *such that*

$$\forall w\in\mathbf{R}^q,\ (\Sigma w)^t P w = \sum_{i,j=1}^q \sigma_i w_i p_{ij} w_j \le 0. \tag{24}$$

Then, the system (23) has a global weak solution on $[0,+\infty)$ *for any nonnegative data* $a_0\in L^2(\Omega)^q$.

Proof: We obtain an a priori $L^2(Q_T)$-estimate on a by applying Theorem 3.1 with

$$h_i(r) = \sigma_i(r-z_i) - \sigma_i z_i\log(r/z_i)\ for\ r\ge z_i\ and\ h_i(r)=0\ for\ r\in[0,z_i].$$

Indeed, for $r_i\ge z_i$ for all i, and by (24)

$$\sum_{i=1}^q h_i'(r_i)f_i(r) = \sum_{i=1}^q \sigma_i(1-z_i/r_i)r_i\sum_{j=1}^q p_{ij}(r_j-z_j)\le 0.$$

We use an approximation process as in the previous proof ($f_i^n = \psi_n f_i$ which preserves the structure). Since the system is quadratic, the $L^2(Q_T)$-estimate provides an $L^1(Q_T)$-bound on the nonlinear part of the approximate system. According to the results in [Pie], up to a subsequence, the approximate solution a_n converges a.e. and in $L^1(Q_T)$ for all T to some function a which is a supersolution of the problem. This means that there exist nonnegative measures $\mu_i, i=1,...,q$ on Q_T such that

$$\partial_t a_i - d_i\Delta_x a_i = f_i(a) + \mu_i. \tag{25}$$

Now, we use the fact that

$$\partial_t(\sum_i \sigma_i a_i^n) - \Delta_x(\sum_i \sigma_i d_i a_i^n) = \sum_i \sigma_i f_i^n(a^n), \tag{26}$$

where, by (24)

$$\sum_i \sigma_i f_i^n(a^n) \le \sum_{i,j}\sigma_i z_i p_{ij}(a_j^n - z_j).$$

We now pass to the limit in the sense of distributions in (26): we may use Fatou's Lemma for the nonlinear terms, thanks to the previous bound from above which is linear with respect to a^n. We obtain

$$\partial_t(\sum_i \sigma_i a_i) - \Delta_x(\sum_i \sigma_i d_i a_i) \leq \sum_i \sigma_i f_i(a).$$

But, together with (25), this proves that the measure $\sum_i \sigma_i \mu_i$ is equal to 0 and so is each μ_i so that the limit a is solution of the system in the sense of distributions.

To complete the proof, we also need to check that the initial data of a is indeed a_0 and that the boundary conditions are preserved. For the Dirichlet conditions, we may use the bound on a in $L^1(0,T;W_0^{1,1}(\Omega))$ coming from the L^1-bound on the right-hand side of the system (see e.g. [BP]). For the Neumann conditions, we repeat the above approach but with test functions in $C^\infty(\overline{Q_T})$ rather than only in $C_0^\infty(Q_T)$. Similarly, we control the initial data by using test-functions which do not vanish at $t = 0$. The details are left to the reader (see also [Pie]).

Remark: other choices of functions h_i. Theorem 3.1 may be used with other choices of convex functions h_i:

1. $h_i(x) = \sigma_i x$ with $\sigma_i \in (0, +\infty)$ is the simplest and corresponds to the fundamental case where $\sum_i \sigma_i f_i(r) \leq 0$. We then get an L^2-estimate on the solution itself and, if f is at most quadratic, global existence of weak solutions. Since we do not have in general uniform integrability of the nonlinearity, we act as in the previous proof.
2. $h_i(x) = x^2$. This corresponds to the "quadratic" Lyapunov structure $\sum_i r_i f_i(r) \leq 0$. Theorem 3.1 says that the solution of (1) is then bounded in $L^4(Q_T)$ for all T. We may conclude to global existence as in Theorem 4.1 if the growth of $f(r)$ at infinity is strictly lower than $|r|^4$. The limit case of growth $|r|^4$ may be addressed as in the previous proof.
3. Similarly, the same will hold with "$h-$ *subquadratic*" systems satisfying $\sum_i h_i'(r_i) f_i(r) \leq 0$ and such that the growth of $|f|$ at infinity is strictly less than $|h|^2$ (see the last Section).

5 Degenerate coefficients

Let us now consider the case of degenerate coefficients, for instance on the example given in the introduction, with variable C^1-coefficients, namely, for $i = 1, 2, 3, 4$

$$\partial_t a_i - \nabla_x \cdot (d_i \nabla_x a_i) = (-1)^i [a_1 a_3 - a_2 a_4], \tag{27}$$

$$\nabla_x a_i \cdot n = 0 \text{ on } \partial\Omega, \tag{28}$$

$$a_i(0) = a_{i0} \geq 0. \tag{29}$$

We approximate the problem by regularizing the diffusions with $d_i^n = d_i + n^{-1}$. Existence of a solution a^n to the approximate problem is a consequence of

Theorem 4.1. The main point is that, according to Theorem 3.1, we keep the uniform estimate

$$\int_{Q_T} (\sum_i z_i^n)(\sum_i d_i^n z_i^n) \leq M \ (independent\ of\ n).$$

Theorem 5.1 *Assume the d_i satisfy the assumptions of Section 2 and that $\exists\, d_0 \in (0, +\infty)$ such that*

$$d_1 + d_2 + d_3 + d_4 \geq d_0 > 0.$$

Assume $a_0 \in L^2(\Omega)^4$. Then, $g_n(a) = a_1^n a_3^n - a_2^n a_4^n$ is bounded in $L^1(Q_T)$ for all T independently of n.

Remark: Obviously, the condition on the d_i's allows that, for instance, three of them be identically equal to zero, the last one being bounded away from zero. Or, they may all degenerate, as long as they do not all vanish at the same place.

Proof: We drop the indexation by n. We denote by $M_1, M_2, \ldots$ positive constants independent of n. Since $\sum_i f_i \leq 0$, by Theorem 3.1, we have

$$\int_{Q_T} (\sum_i a_i)(\sum_i d_i a_i) \leq M_1.$$

This implies

$$\int_{Q_T} d_0 \min\{a_1 a_3, a_2 a_4\} \leq \int_{Q_T} (d_1 + d_3) a_1 a_3 + (d_2 + d_4) a_2 a_4 \leq M_1. \quad (30)$$

Now, integrating the second relation $\sum \log(a_i) f_i(a) \leq 0$, we obtain for all $t \in (0, T)$

$$\sum_i \int_\Omega |a_i \log(a_i) - a_i|(t) + \int_{Q_T} |\log\big(a_1 a_3 / a_2 a_4\big)||a_1 a_3 - a_2 a_4| \leq M_2. \quad (31)$$

This implies that, for the set $K := [a_1 a_3 \geq 2 a_2 a_4] \cup [a_2 a_4 \geq 2 a_1 a_3]$,

$$\int_K |a_1 a_3 - a_2 a_4| \leq \int_K \frac{1}{\log 2} \big|\log(a_1 a_3) - \log(a_2 a_4)\big| \big|a_1 a_3 - a_2 a_4\big| \leq M_2.$$

The complement of K is $\omega_1 \cup \omega_2$ where

$$\omega_1 = [a_2 a_4 \leq a_1 a_3 < 2 a_2 a_4], \ \omega_2 = [a_2 a_4 / 2 < a_1 a_3 \leq a_2 a_4].$$

But, using (30), we obtain

$$\int_{\omega_1} |a_1 a_3 - a_2 a_4| \leq \int_{\omega_1} a_2 a_4 = \int_{\omega_1} \min\{a_1 a_3, a_2 a_4\} \leq M_1 / d_0,$$

$$\int_{\omega_2} |a_1 a_3 - a_2 a_4| \leq \int_{\omega_2} a_1 a_3 = \int_{\omega_2} \min\{a_1 a_3, a_2 a_4\} \leq M_1 / d_0.$$

Since $Q_T = K \cup \omega_1 \cup \omega_2$, this proves the result.

Let us show on one situation how we may pass to the limit with the help of Theorem 5.1.

Corollary 5.1 *Assume hypotheses of Theorem 5.1 and*

$$\forall\, i = 1,2,3,4, \ \ d_i > 0 \ \ a.e..$$

Then, the approximate solution a^n converges to a weak solution of the system.

Remark: Here each d_i may vanish on a set of zero Lebesgue measure, but not all at the same time.

Proof: We take the same approximation as in Theorem 5.1. All the "formal" computations which follow are justified since the (weak) solution is obtained as the limit of regular solutions (see Theorem 4.1). We set $g_n = a_1^n a_3^n - a_2^n a_4^n$. We know by Theorem 5.1 that it is bounded in $L^1(Q_T)$. Let us use the truncation function T_k introduced in (6) and show that, for fixed k, $T_k(a_i^n)$ *converges almost everywhere* (up to a subsequence). Indeed, multiplying the equation in a_i^n by $T_k(a_i^n)$, we get

$$\int_{Q_T} d_i^n T_k'(a_i^n)|\nabla_x\, a_i^n|^2 \le k \int_{Q_T} |g_n| + \int_\Omega j_k(a_i^n(0)),$$

where $j_k'(r) = T_k(r)$ so that $\int_\Omega j_k(a_i^n(0))$ is bounded by a constant $M(k)$. It follows that if $\sigma_n := d_i^n T_k(a_i^n)$, then

$$\nabla_x \sigma_n = \nabla d_i^n T_k(a_i^n) + d_i^n T_k'(a_i^n)\nabla_x(a_i^n)$$

is bounded in $L^2(Q_T)$ for fixed k (we use $(T_k')^2 \le T_k'$ and the assumptions on the $d_i's$). Now,

$$\begin{aligned} \partial_t T_k(a_i^n) &= T_k'(a_i^n)\,\partial_t a_i^n = T_k'(a_i^n)(\nabla_x \cdot (d_i^n \nabla_x\, a_i^n) + (-1)^i g_n) \\ &= \nabla_x \cdot (T_k'(a_i^n) d_i^n \nabla_x\, a_i^n) - T_k^{''}(a_i^n) d_i^n |\nabla_x\, a_i^n|^2 + T_k'(a_i^n)(-1)^i g_n. \end{aligned} \tag{32}$$

We deduce that $\partial_t T_k(a_i^n) = \nabla_x u_n + v_n$ where u_n is bounded in $L^2(Q_T)$ and v_n bounded in $L^1(Q_T)$. It follows that

$$\partial_t \sigma_n = (\partial_t d_i^n) T_k(a_i^n) + \nabla_x(d_i^n u_n) - (\nabla_x d_i^n) u_n + d_i^n v_n = \nabla_x \hat{u}_n + \hat{v}_n,$$

where $\hat{u}_n$ is bounded in $L^2(Q_T)$ and $\hat{v}_n$ is bounded in $L^1(Q_T)$. It follows that $\sigma_n = d_i^n T_k(a_i^n)$ is compact in $L^1(Q_T)$ (see e.g. [Sim87]) so that we may assume that it converges almost everywhere. Since d_i^n converges a.e. to d_i which is > 0 *a.e.*, it follows that $T_k(a_i^n)$ converges itself a.e.. Since $T_k(a_i^n) = a_i^n$ on $a_i^n \le k-1$, up to a diagonal extraction, we may assume that a_i^n converges a.e. to some a_i. Moreover, this is true for any i, and $a_1 a_3 - a_2 a_4 \in L^1(Q_T)$.

Now, for all i, since $T_k^{''} \le 0$,

$$\partial_t T_k(a_i^n) - \nabla_x \cdot d_i^n \nabla_x\, T_k(a_i^n) \ge T_k'(a_i^n)(-1)^i (a_1^n a_3^n - a_2^n a_4^n). \tag{33}$$

Let us show that *the negative part $G_n := T_k'(a_i^n)[(-1)^i g_n]^-$ of the right-hand side is uniformly integrable.*

Let us choose $i = 1$ (the analysis is the same for the other values of i). Then, $[G_n > 0] \subset [a_1^n < k+1] \cap [a_2^n a_4^n < a_1^n a_3^n]$ and, for any $K \subset Q_T$ measurable,

$$\int_K G_n = \int_{K\cap[G_n>0]} G_n \le \int_{K\cap[a_1^n<k+1]} [a_1^n a_3^n - a_2^n a_4^n]^+ \le (k+1)\int_K a_3^n. \tag{34}$$

But, a_3^n is uniformly integrable on Q_T since $|a_3^n \log(a_3^n)|$ is bounded in $L^1(Q_T)$ (see (31)). Whence the uniform integrability of G_n.

Passing to the limit in (33): For fixed k, $T_k(a_i^n)$ converges in $L^1(Q_T)$ to $T_k(a_i)$ and $d_i^n \nabla_x T_k(a_i^n)$ converges at least weakly in $L^2(Q_T)$ to $d_i \nabla_x T_k(a_i)$. Thus, we may pass to the limit in the sense of distributions in the linear part.

Since G_n converges a.e. and is uniformly integrable, it converges in $L^1(Q_T)$. By a.e. convergence and Fatou's Lemma applied to the positive part of the right-hand side, we may deduce that, for all i

$$\partial_t T_k(a_i) - \nabla_x \cdot d_i \nabla_x T_k(a_i) \geq T_k'(a_i)(-1)^i(a_1 a_3 - a_2 a_4). \tag{35}$$

Then, letting k tend to $+\infty$, we obtain that a_i is a super-solution of the equation, that is

$$\partial_t a_i - \nabla_x \cdot d_i \nabla_x a_i = (-1)^i(a_1 a_3 - a_2 a_4) + \mu_i,$$

where μ_i is a positive measure on Q_T. But, on the other hand, we may pass directly to the limit in

$$\partial_t(a_1^n + a_2^n) - \nabla_x \cdot (d_1^n \nabla_x a_1^n + d_2^n \nabla_x a_2^n) = 0,$$

so that we obtain that $\mu_1 + \mu_2 = 0$ (and similarly $\mu_3 + \mu_4 = 0$). It follows that the limit is a solution in the sense of distributions.

To control also the initial data and the boundary conditions, we use test-functions in $C^\infty(\overline{Q_T})$ (see the end of the proof of Theorem 4.2).

6 Renormalized solutions for very degenerate cases

We still consider the following system for $i = 1, 2, 3, 4$:

$$\left.\begin{array}{rcl} \partial_t a_i - \nabla_x \cdot (d_i \nabla_x a_i) &=& (-1)^i \, [a_1 \, a_3 - a_2 \, a_4] \\ \nabla_x a_i \cdot n &=& 0 \text{ on } \partial\Omega, \\ a_i(0) &=& a_{i0} \geq 0. \end{array}\right\} \tag{36}$$

However, we only assume that $d_i > 0$ a.e. In particular *we do not assume* that $\sum d_i$ is uniformly bounded from below as in Theorem 5.1 or Corollary 5.1. As a consequence, we loose the L^1-estimate on the nonlinearity given in this theorem. Therefore, it is not possible to work with what we called "weak solutions" any more since the definition requires that the nonlinearity be at least integrable. However, a main point is that the functions $T_k'(a_i)[a_1 a_3 - a_2 a_4]$ are uniformly integrable for all $k > 0$ and we can reproduce the main steps in the approximating process of the previous paragraph to prove (see the definition in Section 2):

Theorem 6.1 *Under the above assumptions and $|a_0| \log |a_0| \in L^1(\Omega)$, the system (36) has a renormalized solution with defect measure.*

Proof: We introduce $d_i^n = d_i + \frac{1}{n}$ and a_i^n the (weak) solution of the system

$$\partial_t a_i^n - \nabla_x \cdot (d_i^n \, \nabla_x \, a_i^n) = (-1)^i \, (a_1^n \, a_3^n - a_2^n \, a_4^n), \tag{37}$$

with the homogeneous Neumann boundary condition, and $a_i^n(0) = a_i(0)$. Its existence is stated in Theorem 4.1. Since it is obtained as a limit of regular solutions to approximate systems, all subsequent "formal" computations are justified. For instance, the entropy estimate shows that

$$\frac{d}{dt}\int_\Omega \sum a_i^n \, \log a_i^n + \int_\Omega \sum d_i^n \, \frac{|\nabla_x \, a_i^n|^2}{a_i^n} + \int_\Omega (a_1^n \, a_3^n - a_2^n \, a_4^n) \, \log(\frac{a_1^n \, a_3^n}{a_2^n \, a_4^n}) \le 0, \quad (38)$$

so that for all $T > 0$,

$$\left. \begin{array}{l} \sup_{t\in[0,T]} \int_\Omega \sum a_i^n(t,x) \, \log a_i^n(t,x) + \int_{Q_T} \sum d_i^n \, \frac{|\nabla_x \, a_i^n|^2}{a_i^n} \\ + \int_{Q_T} (a_1^n \, a_3^n - a_2^n \, a_4^n) \, (\log(a_1^n \, a_3^n) - \log(a_2^n \, a_4^n)) \le C_T. \end{array} \right\} \quad (39)$$

We successively prove the following *for all $k > 0$ fixed* and all $i = 1, 2, 3, 4$, where $g_n = a_1^n a_3^n - a_2^n a_4^n$:

- **(i)** $d_i^n T_k'(a_i^n) |\nabla_x \, a_i^n|^2$ is bounded in $L^1(Q_T)$.
- **(ii)** $T_k'(a_i^n) g_n$ is uniformly integrable on Q_T.
- **(iii)** There exists $a_i \in L^1(Q_T)$ such that, up to a subsequence, $T_k(a_i^n)$ converges to $T_k(a_i)$ a.e..

Then, we may pass to the limit in

$$\partial_t T_k(a_i^n) - \nabla_x \cdot (d_i^n \nabla_x \, T_k(a_i^n)) = T_k'(a_i^n)(-1)^i (a_1^n a_3^n - a_2^n a_4^n) - T_k{}''(a_i^n) d_i^n |\nabla_x \, a_i^n|^2.$$

to obtain that, for all $k > 0$

$$\partial_t T_k(a_i) - \nabla_x \cdot (d_i \nabla_x \, T_k(a_i)) \ge T_k'(a_i)(-1)^i (a_1 a_3 - a_2 a_4) - T_k{}''(a_i) d_i |\nabla_x \, a_i|^2.$$

Indeed, by (iii) and dominated convergence, $T_k(a_i^n)$ converges in $L^1(Q_T)$ to $T_k(a_i)$; by (i), $d_i^n \nabla_x \, T_k(a_i^n)$ converges also weakly in $L^2(Q_T)$. Hence, we may pass to the limit in the sense of distributions in the left-hand side. For the right hand-side, we use (ii) and the weak-L^2-convergence of $\nabla_x \, a_i^n$ on the sets $[a_i^n \le k]$.

Proof of (i): It comes from the second term in (39).
Proof of (ii): Let us do it for $i = 1$ (the other cases are similar).
Let $p > 1$. Either $a_2^n a_4^n \le p \, a_1^n a_3^n$ or $a_2^n a_4^n \ge p \, a_1^n a_3^n$ and then

$$0 \le a_2^n a_4^n - a_1^n a_3^n \le \frac{1}{\log p} [a_2^n a_4^n - a_1^n a_3^n][\log a_2^n a_4^n - \log a_1^n a_3^n].$$

Using this together with (39), we obtain that, for $K \subset Q_T$ measurable

$$\int_{[a_1^n \le k] \cap K} |a_1^n a_3^n - a_2^n a_4^n| \le (1+p) k \int_K a_3^n + C(T) [\log p]^{-1},$$

which proves the uniform integrability of $T_k'(a_i^n) g_n$ since p is arbitrary and a_3^n is uniformly integrable.
Proof of (iii): We go back to the proof of Corollary 5.1 and check that the compactness of $\sqrt{d_i^n} T_k(a_i^n)$ requires only the bounds claimed in (i) and (ii) (see (32) and the paragraph which follows it).

7 Reaction terms of higher degree

To show how far our approaches may be carried out, we now consider systems with higher nonlinearities of the following form where $p_i \in [1, +\infty)$, d_i are positive constants and for $i = 1, 2, 3, 4$

$$\left.\begin{array}{rcl} \partial_t a_i - d_i\,\Delta_x a_i &=& (-1)^i\,(a_1^{p_1}\,a_3^{p_3} - a_2^{p_2}\,a_4^{p_4}), \\ \nabla_x\,a_i(t,x)\cdot n &=& 0 \qquad \text{for} \qquad x \in \partial\Omega, \\ a_i(0,x) &=& a_{i0}(x) \geq 0. \end{array}\right\} \tag{40}$$

The general philosophy is the following: if we can obtain a priori $L^1(Q_T)$-estimates on the nonlinearities $g = a_1^{p_1} a_3^{p_3} - a_2^{p_2} a_4^{p_4}$, then we obtain existence of a weak solution. If we can at least obtain uniform integrability on $T_k'(a_i)g$ for all $k > 0$, then we obtain renormalized solutions.

We are able to prove the following.

Proposition 7.1 *Assume $|a_0|\log|a_0| \in L^2(\Omega)$. If $p_i \leq 2$ for all i, then (40) has a renormalized solution (with defect measure) in any dimension. In dimension 1, it is also the case as soon as $p_i \leq 3$ for all i and it is then a weak solution if moreover $p_1 + p_3 \leq 3$ and $p_2 + p_4 \leq 3$.*

Remark: open problems. The situation is unclear if the values of the p_i are higher. According to the structure of the right-hand side, one has $L^2(Q_T)$- and uniform $L^1(\Omega)$-bounds on the a_i, but this is not sufficient to conclude to global existence, even of renormalized solutions.

Proof of Proposition 7.1: We only indicate the necessary a priori estimates. The analysis is then the same as in the previous sections (see the three points (i)-(iii) in the proof of Theorem 6.1).

Since $\sum_{i=1}^4 \log a_i^{p_i} f_i(a) \leq 0$, by Theorem 3.1 applied with $h_i(r) = p_i[r_i \log r_i - r_i]^+$, we obtain that $|a|\log|a|$ is bounded in $L^2(Q_T)$. Morover we have

$$\left.\begin{array}{l} \sup_{t\in[0,T]} \int_\Omega \sum a_i(t)\,\log a_i(t) + \int_{Q_T} \sum d_i\,p_i\,\frac{|\nabla_x\,a_i|^2}{a_i} \\ \qquad + \int_{Q_T} (a_1^{p_1}\,a_3^{p_3} - a_2^{p_2}\,a_4^{p_4})\log\left(\frac{a_1^{p_1}\,a_3^{p_3}}{a_2^{p_2}\,a_4^{p_4}}\right) \leq C. \end{array}\right\} \tag{41}$$

We then deduce that $T_k'(a_i)f_i(a)$ is uniformly integrable for all $k > 0$ (whence the existence of the global renormalized solution). Indeed, if $p > 1$, either $a_2^{p_2} a_4^{p_4} \leq p\,a_1^{p_1} a_3^{p_3}$ or $a_2^{p_2} a_4^{p_4} \geq p\,a_1^{p_1} a_3^{p_3}$ in which case

$$0 \leq a_2^{p_2} a_4^{p_4} - a_1^{p_1} a_3^{p_3} \leq \frac{1}{\log p}\,(a_2^{p_2}\,a_4^{p_4} - a_1^{p_1}\,a_3^{p_3})\log\left(\frac{a_2^{p_2}\,a_4^{p_4}}{a_1^{p_1}\,a_3^{p_3}}\right).$$

We deduce that for all $K \subset Q_T$ measurable,

$$\int_{[a_1^n \leq k]\cap K} |a_1^{p_1} a_3^{p_3} - a_2^{p_2} a_4^{p_4}| \leq (1+p)k^{p_1}\int_K a_3^{p_3} + C[\log p]^{-1},$$

whence the required uniform integrability since $p_3 \leq 2$ and $a_3 \log a_3$ is bounded in $L^2(Q_T)$.

Next we turn to the case of dimension 1. The above analysis shows that uniform integrability of $T'_k(a_i)f_i(a)$ may be obtained as soon as the $a_i^{p_i}$ are themselves uniformly integrable. This is true when $p_i \leq 3$ in dimension 1 since, as proved next :

$$\text{If } N = 1,\ |a|^3(\log|a|)^2 \text{ is bounded in } L^1(Q_T). \tag{42}$$

The last assertion of the theorem is also a consequence of (42) since, if $p_1+p_3 \leq 3, p_2+p_4 \leq 3$, then $a_1^{p_1}a_3^{p_3} - a_2^{p_2}a_4^{p_4}$ is itself uniformly integrable in $L^1(Q_T)$ and we can obtain a weak solution.

The proof will then be complete after proving the estimate (42). This may be obtained as follows (here $\log e = 1$ and C denotes any constant depending only on T and the initial data):

$$\int_{Q_T} (a_i)^3 \ \log(e+a_i)^2 \leq \int_0^T \left(\int_\Omega a_i \ \log(e+a_i)\right) \sup_{x\in\Omega} \left((a_i)^2 \ \log(e+a_i)\right)$$

We use

$$\sup_{x\in\Omega} \left[(a_i)^2 \ \log(e+a_i)\right] \leq C\Big[\int_\Omega |\partial_x \left((a_i)^2 \log(e+a_i)\right) + \int_\Omega (a_i)^2 \log(e+a_i)\Big],$$

$$\int_{Q_T} |\partial_x \left((a_i)^2 \ \log(e+a_i)\right)| \leq C \ \int_{Q_T} a_i^{3/2} \log(e+a_i) \left|\frac{\partial_x a_i}{\sqrt{a_i}}\right|$$

$$\leq C \ \left(\int_{Q_T} (a_i)^3 \ \log(e+a_i)^2\right)^{1/2},$$

$$\int_\Omega (a_i)^2 \log(e+a_i) \leq C \left(\int_\Omega (a_i)^3 \log(e+a_i)^2\right)^{2/3}.$$

This yields (42).

8 Appendix

The purpose of this Appendix is double: first, for the reader's convenience, we recall on a particular quadratic system the main steps (taken from [PSch], [Pie]) in proving $L^2(Q_T)$-estimates by duality as well as global existence of weak solutions. Then, we show how general embedding properties of independent interest may be used to obtain $L^2(Q_T)$-estimates for the system (1) in dimensions 1 and 2.

Theorem 8.1 *For the system (4), assume that the d_i are positive constants, the f_i are at most quadratic in r (that is the function $k(\cdot)$ of (3) is at most linear) and*

$$\forall r \in [0,+\infty)^q, \sum_i f_i(r) \leq 0.$$

Then, (4) has a global weak solution for initial data in $L^2(\Omega)$.

Steps of the proof: We truncate the nonlinearities f_i, keeping the same properties for the f_i^n, and we estimate the solution a^n of the approximate problem.

Estimate of a^n in $L^2(Q_T)$: From the above structure, we deduce

$$(\sum_i a_i^n)_t - \Delta_x(\sum_i d_i a_i^n) \le 0.$$

Set $z = \sum_i a_i^n, z_d = \sum_i d_i a_i^n, A = z_d/z$ (we suppose here that $z > 0$ a.e. for the sake of simplicity. Then $z_t - \Delta_x(Az) \le 0$. Let us consider the positive solution of the dual problem:

$$-(w_t + A\Delta_x w) = H \in \mathcal{C}_0^\infty(Q_T), H \ge 0,\ w(T) = 0,\ \nabla_x w{\cdot}n = 0 \text{ [or } w = 0] \text{ on } \partial\Omega.$$

We have $\int_{Q_T} z\,H \le \int_\Omega z(0)w(0)$. Let us estimate $w(0)$ in $L^2(\Omega)$. Multiplying the equation in w by $-\Delta_x w$ gives

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla_x w(t)|^2 + \int_\Omega A(\Delta_x w)^2 = -\int_\Omega H\Delta_x w \le \int_\Omega \frac{d_0}{2}(\Delta_x w(t))^2 + C(d_0)H^2,$$

where $0 < d_0 = \min_i\{d_i\} \le A$. It follows, after integration in time that $\int_{Q_T}(\Delta_x w)^2 \le C\int_{Q_T} H^2$. Going back to the equation in w, we deduce a bound for w_t in $L^2(Q_T)$ and therefore a bound $w(0)$ in $L^2(\Omega)$ in terms of $\int_{Q_T} H^2$. Therefore

$$\int_{Q_T} z\,H \le \int_\Omega z(0)w(0) \le \|z(0)\|_{L^2}\|w(0)\|_{L^2} \le C\|z(0)\|_{L^2}\|H\|_{L^2(Q_T)},$$

which, by duality, gives a bound of z in $L^2(Q_T)$ in terms of $\|z(0)\|_{L^2(\Omega)}$.

We deduce that *the nonlinearities are bounded in $L^1(Q_T)$*. This provides compactness of a^n in $L^1(Q_T)$ (and convergence of a subsequence a.e.) (see e.g. [BP]).

Now we may use the approach in [Pie] to prove that *the limit is a supersolution of the system.* The technique consists in considering the truncated equations as in (33). In general, we are not able to obtain any uniform integrability. The method consists -for instance for the first equation- in considering $w^n = T_k(a_1^n + \eta(a_2^n + ... + a_q^n))$ where $\eta > 0$ is small. The equation satisfied by w^n is in general not simple, due to the fact that the diffusion operators are different from each other: it looks like

$$\partial_t w^n - d_1\Delta_x w^n \ge T_k'(..)(f_1^n + \eta\sum_{2\le i\le q} f_i^n) + G(\eta, k, n). \tag{43}$$

It is easy to pass to the limit except in the extra term $G(\eta, k, n)$ which contains the difficulty. The main point in the proof of [Pie] is to prove the estimate

$$\forall\,\varphi \in \mathcal{C}_0^\infty(Q_T),\ |< G(\eta, k, n), \varphi >| \le C(k, \varphi)\eta^{1/2}.$$

Then, we may pass to the limit as $n \to +\infty$, as $\eta \to 0$, and as $k \to +\infty$.

To prove that the limit a is also a subsolution, we use again the structure $\sum_i f_i \le 0$ like in the last part of the proof of Theorem 4.2 above.

We now turn to the question of obtaining bounds for system (1) without using the duality method.

For the system (1), the entropy estimate leads naturally to the following bounds (for $i = 1..4$, and all $T > 0$) :

$$\sup_{t\in[0,T]} \|a_i(t,\cdot)\,\log a_i(t,\cdot)\|_{L^1(\Omega)} + \|\nabla_x\sqrt{a_i}\|\|_{L^2(Q_T)} \le C_T. \tag{44}$$

Proposition 8.1 *Suppose a_i is a function satisfying (44). Then*

$$\|a_i\,\log(e+a_i)^{2/3}\|_{L^3(Q_T)} \le C_T\,, \qquad \text{if} \quad N=1 \tag{45}$$

$$\|a_i\|^{1+2/N}_{L^{1+2/N}(Q_T)} \le C_T\,, \qquad \text{if} \quad N\ge 2 \tag{46}$$

Remark 8.1 *For $N=1$ and $N=2$, we have therefore $\|a_i\|^2_{L^2(Q_T)} \le C_T$.*

Proof: For $N=1$, the proof is given in Section 7. We also recall that this estimate is the key for further smoothness of the solutions of equation (1) in this case.

For $N\ge 3$, the results follows similarly to the 1D-case using the classical Sobolev estimates

$$\|u\|_{L^{\frac{2N}{N-2}}(\Omega)} \le C(N,\Omega)\left(\|u\|_{L^2(\Omega)} + \|\nabla_x u\|_{L^2(\Omega)}\right).$$

The case $N=2$ is a limit case and requires more work. It is based on Trudinger's inequality saying that there are two absolute strictly positive constants s_0 and C_0 such that, for all $u\in H^1(\Omega)$,

$$\int_\Omega \exp\left(\frac{s_0\,u(x)^2}{\|u\|^2_{H^1(\Omega)}}\right) \le C_0\,. \tag{47}$$

As a consequence, we can also find two strictly positive absolute constants s and C such that (for all functions $u\in H^1(\Omega)$),

$$\int_\Omega \frac{u(x)^2}{\|u\|^2_{H^1(\Omega)}}\,\exp\left(\frac{s\,u(x)^2}{\|u\|^2_{H^1(\Omega)}}\right) \le C\,. \tag{48}$$

Hence,

$$\int_{Q_T} a_i(t,x)\,\exp\left(\frac{s\,a_i(t,x)}{\|\sqrt{a_i}(t,\cdot)\|^2_{H^1(\Omega)}}\right) \le C\int_0^T \|\sqrt{a_i}(t,\cdot)\|^2_{H^1(\Omega)} \le C_T. \tag{49}$$

We note that thanks to Young's inequality (valid for $x,y,\gamma>0$)

$$x\,y \le e^{\gamma x} + \frac{y}{\gamma}(\log(\frac{y}{\gamma})-1),$$

applied to $\gamma = \frac{\log(a)}{a}+\frac{s}{q}$ and $x=y=a$, we have for all $a>e$ and $s,q>0$,

$$a^2 \le a\,e^{\frac{sa}{q}} + \frac{a}{\frac{\log(a)}{a}+\frac{s}{q}}\left(\log\left(\frac{a}{\frac{\log(a)}{a}+\frac{s}{q}}\right)-1\right) \le a\,e^{\frac{sa}{q}} + \frac{aq}{s}\log(a^2).$$

Using this last inequality with $q=\|\sqrt{a_i}(t,\cdot)\|^2_{H^1(\Omega)}$ and $a=\max(e,a_i(t,x))$, we conclude the lemma (thanks to estimate (49))

$$\begin{aligned}
\|a_i\|^2_{L^2(Q_T)} &\le \|\min\{a_i,e\}\|^2_{L^2(Q_T)} + \|\max\{a_i,e\}\|^2_{L^2(Q_T)}\\
&\le e^2\,|\Omega|T + \int_{Q_T} a_i(t,x)\,\exp\left(\frac{s\,a_i(t,x)}{\|\sqrt{a_i}(t,\cdot)\|^2_{H^1(\Omega)}}\right)\\
&\quad + \frac{2}{s}\int_0^T\left(\int_\Omega a_i(t,x)\,\log(a_i(t,x))\right)\|\sqrt{a_i}(t,\cdot)\|^2_{H^1(\Omega)}\\
&\le e^2\,|\Omega|T + C_T + \frac{2}{s}\,C_T^2.
\end{aligned}$$

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