Probabilistic Interpretation and Numerical Approximation of a Kac Equation without Cutoff

Laurent Desvillettes[1], Carl Graham[2] and Sylvie Méléard[3]

May 28, 1999

Abstract

A nonlinear pure-jump Markov process is associated with a singular Kac equation. This process is the unique solution in law for a non-classical stochastic differential equation. Its law is approximated by simulable stochastic particle systems, with rates of convergence. An effective numerical study is given at the end of the paper.

1 The set-up

1.1 The physical model

In the upper atmosphere, a gas is described by the nonnegative density $f(t,x,v)$ of particles which at time t and point x move with velocity v. Such a density satisfies a Boltzmann equation, see for example Cercignani *et al.* [3],

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f),$$

where Q is a quadratic collision kernel acting only on the variable v, preserving momentum and kinetic energy, of the form

$$Q(f,f)(t,x,v) = \int_{v_* \in \mathbb{R}^3} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \Big(f(t,x,v')f(t,x,v'_*) - f(t,x,v)f(t,x,v_*) \Big) \\ B(|v-v_*|,\theta) \sin\theta \, d\phi d\theta dv_*$$

with $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma$ and $v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$, the unit vector σ having colatitude θ and longitude ϕ in the spherical coordinates in which $v - v_*$ is the polar axis. The nonnegative function B is called the cross section.

If the molecules in the gas interact according to an inverse power law in $1/r^s$ with $s \geq 2$, then $B(x,\theta) = x^{\frac{s-5}{s-1}} d(\theta)$ where $d \in L^\infty_{loc}(]0,\pi])$ and $d(\theta)\sin\theta \sim K(s)\theta^{-\frac{s+1}{s-1}}$ when θ goes to zero, for some $K(s) > 0$. Physically, this explosion comes from the accumulation of

[1]CMLA, ENS Cachan, 61 av. du Pdt Wilson, 94235 Cachan, France, desville@cmla.ens-cachan.fr
[2]CMAP, École Polytechnique, F-91128 Palaiseau (UMR CNRS 7641), carl@cmapx.polytechnique.fr
[3]Université Paris 10, MODAL'X, 200 av. de la République, 92000 Nanterre et Labo. de Proba., Université Paris 6, 75252 Paris, France (URA CNRS 224), sylm@ccr.jussieu.fr

grazing collisions. This equation is said to be non-cutoff because it is classical to consider the simpler case when $d \in L^1(]0,\pi])$, which is in turn named the cutoff case.

The integral term in the nonlinear Boltzmann equation comes from the randomness in the geometric configuration of collisions, and it is natural to study its probabilistic interpretation. This interpretation will allow to define stochastic interacting particle systems which will be used to approximate, in a certain sense, the solution of this equation.

The two main difficulties for the probabilistic interpretation are that the interaction appearing in the collision term is localized in space (it is not mean-field) and the cross section B is non-cutoff.

Graham and Méléard [7] give a probabilistic interpretation of a mollified Boltzmann equation, in which the interaction is delocalized in space and the cross section B is cutoff. They prove that some stochastic interacting particle systems converge in law to a solution of this equation and give a precise rate of convergence. Méléard [12] considers the full Boltzmann equation (non mollified) and proves that under a cutoff assumption and for small initial conditions which ensure the existence and uniqueness of the solution of this equation, some interacting particle systems converge to this solution. These results give a theoretical justification of the Nanbu and Bird algorithms, see [3] and [2].

1.2 A simplified model: the non-cutoff Kac equation

We are interested in this work in omitting the cutoff assumption on the cross section B. The full non-cutoff Boltzmann equation is very difficult to study. There is a restricted existence result in Ukai [21]. The definition of renormalized solutions, used in the existence proof for the cutoff case by DiPerna and Lions [5], is difficult in the non-cutoff case, see [1] for work in this direction.

We restrict ourselves here to the study of non cutoff spatially homogeneous Boltzmann equations. The methods in this paper can be easily extended for such equations in any dimension, when the cross section B depends only on θ (Maxwellian molecules), see [9]. For the sake of simplicity we consider the non cutoff Kac equation

$$\frac{\partial f}{\partial t} = K_\beta(f,f), \tag{1.1}$$

where $f \equiv f(t,v)$, $t \geq 0$, $v \in \mathbb{R}$, and

$$K_\beta(f,f)(t,v) = \int_{v_* \in \mathbb{R}} \int_{\theta=-\pi}^{\pi} \Big(f(t,v')f(t,v'_*) - f(t,v)f(t,v_*) \Big) \beta(\theta) d\theta dv_* \tag{1.2}$$

with

$$v' = v\cos\theta - v_*\sin\theta\,, \quad v'_* = v\sin\theta + v_*\cos\theta\,. \tag{1.3}$$

This can be seen as a particular case of the Boltzmann equation, namely when 2D radial solutions are considered, see [4].

By analogy with the Boltzmann cross section B described earlier, the cross section $\beta : [-\pi,\pi]-\{0\} \mapsto \mathbb{R}_+$ will be an even function satisfying the L^2 assumption $\int_0^\pi \theta^2\beta(\theta)\,d\theta < +\infty$. If the weaker assumption $\int_0^\pi \beta(\theta)\,d\theta < +\infty$ holds, the equation and cross section are said to be "cutoff", which justifies the terminology of "non cutoff" Kac equation.

Note that there is a result of existence for the non cutoff equation (1.1), cf. [4]:

Theorem 1.1 *Let $f_0 \geq 0$ be such that*

$$\int_{v\in\mathbb{R}} f_0(v)(1+|v|^2+|\log f_0(v)|)dv < +\infty$$

and $\beta \geq 0$ be a cross section such that $\int_0^\pi \theta^2\beta(\theta)\,d\theta < +\infty$ (non cutoff case).

Then there exists a nonnegative solution $f^\beta(t,v) \in L^\infty([0,+\infty[_t; L^1(\mathbb{R}_v,(1+|v|^2)dv)) \cap C([0,+\infty[_t;\mathcal{D}'(\mathbb{R}_v))$ to (1.1) with initial datum f_0 in the following weak sense: for any $\phi \in W^{2,\infty}(\mathbb{R}_v)$,

$$\frac{d}{dt}\int_{v\in\mathbb{R}} f^\beta(t,v)\phi(v)dv = \int_{v\in\mathbb{R}}\int_{v_*\in\mathbb{R}} K^\phi_\beta(v,v_*)f^\beta(t,v)f^\beta(t,v_*)dv_*dv, \tag{1.4}$$

where

$$K^\phi_\beta(v,v_*) = \int_{-\pi}^{\pi}\Big(\phi(v\cos\theta - v_*\sin\theta)-\phi(v)-(v(\cos\theta-1)-v_*\sin\theta)\phi'(v)\Big)\beta(\theta)d\theta - bv\phi'(v) \tag{1.5}$$

and

$$b = \int_{-\pi}^{\pi}(1-\cos\theta)\beta(\theta)\,d\theta. \tag{1.6}$$

Remark: There is a compensated term in the operator (1.5). If we moreover assume that $\int_0^\pi \theta\beta(\theta)\,d\theta < +\infty$, then $\int_{-\pi}^{\pi}\sin\theta\beta(\theta)\,d\theta$ is well-defined and equal to 0 (since β is even), and eq. (1.5) can be rewritten as $K^\phi_\beta(v,v^*) = \int_{-\pi}^{\pi}\{\phi(v')-\phi(v)\}\beta(\theta)\,d\theta$. In the cutoff case $\beta \in L^1([0,\pi])$ there is existence and uniqueness of a solution in the sense of Theorem 1.1 even if $f_0|\log f_0|$ is not integrable, see [4], Appendix A.

We shall see in the sequel that it is in fact possible to obtain measure-valued solutions to eq. (1.1) as soon as f_0 is a nonnegative finite measure, even in the non-cutoff case. Pulvirenti and Toscani [14] also give an existence result in this context.

We will associate with the Kac equation a nonlinear martingale problem. Section 2 studies the cutoff case and Section 3 the non-cutoff case. For the latter case, following Tanaka [18], we will give a non classical nonlinear stochastic differential equation representation for such a solution. The collision kernel will be interpreted as a stochastic integral with respect to a fixed driving Poisson process.

We will prove the existence and uniqueness of a solution of this non classical equation using a sophisticated Picard iteration method. We use this to prove existence and uniqueness of a solution for the nonlinear martingale problem.

We use results of Graham and Méléard [7], and exhibit in Section 4 some simulable interacting particle systems, of which the laws converge to a solution of the Kac equation with a precise rate of convergence. The idea is the following. Since we cannot directly simulate eq. (1.1) when $\int_0^\pi \beta(\theta)\, d\theta = +\infty$, we introduce a cutoff equation by considering $\beta_\ell = \beta \wedge \ell$ and simulate it with a system of n particles (using Nanbu's or Bird's method for example). Then, when $\ell \to 0$ and $n \to +\infty$, a sufficient condition is given on the speed of convergence of those two quantities, so that the law of the particle system converges to the solution of our non cutoff equation. Estimates of convergence are also given.

Finally, in Section 5, an empirical study of the convergence when $\ell \to 0$ and $n \to +\infty$ of the particle systems is performed. A function $n \to \ell(n)$ is computed numerically, in such a way that a criterion on the error is optimized (basically, the error due to the cutoff and the error due to the discretization must be of the same order).

Tanaka [18], [17], studies a spatially homogenous Boltzmann equation with Maxwellian molecules, under the stronger L^1 assumption $\int \theta\beta(\theta)\, d\theta < +\infty$. He introduces the non classical nonlinear stochastic differential equation, and proves the existence and uniqueness in law of the solution using a complicated L^1 method, based on an Euler scheme, using the fact that the metric for probability measures with a second moment

$$\rho(p,q) = \inf\Big\{\Big(\int (x-y)^2 r(dx,dy)\Big)^{1/2} : r \text{ has marginals } p \text{ and } q\Big\}, \qquad p,q \in \mathcal{P}_2(\mathbb{R}^3),$$

is non-expansive along solutions of the equation, a result recently extended by Toscani and Villani [19] to all dimensions of space under the L^2 assumption $\int \theta^2\beta(\theta)\, d\theta < +\infty$.

This gives a uniqueness result for these equations, but is very specialized and does not give uniqueness for the corresponding Markov process. His method does not adapt easily to our L^2 setting. We do not use this non-expansive property, and develop a contraction method allowing very precise computations, which we use also for convergence estimates.

Sznitman in [16] studied a spatially homogeneous hard-sphere Boltzmann equation taking in account the large velocities. In that model, there is no angular dependence of the collision kernel. He obtains convergence results without estimates using a compactness-uniqueness method.

There exists a deterministic spectral method for simulating non-cutoff spatially homogeneous Boltzmann equations for Maxwellian molecules, see Pareschi, Toscani and Villani [13] and the references therein.

1.3 The probabilistic interpretation

The probabilistic interpretation of the Kac equation (1.1) comes from its weak form, but in a slightly more general setting than that of Theorem 1.1.

For a function f and a measure μ we denote $\int f(x)\mu(dx)$ by $\langle f, \mu\rangle$ or $\langle f(x), \mu(dx)\rangle$.

Definition 1.2 *Let β be a cross section such that $\int_0^\pi \theta^2\beta(\theta)\,d\theta < +\infty$ and P_0 in $\mathcal{P}_2(\mathbb{R})$ (the space of probability measures with a second moment).*

A probability measure flow $(P_t)_{t\geq 0}$ is said to solve eq. (1.1) if for any ϕ in $C_b^2(\mathbb{R})$,

$$\langle\phi, P_t\rangle = \langle\phi, P_0\rangle + \int_0^t \langle K_\beta^\phi, P_s\otimes P_s\rangle\,ds = \langle\phi, P_0\rangle + \int_0^t \langle K_\beta^\phi(v,v^*), P_s(dv)\,P_s(dv^*)\rangle\,ds, \quad (1.7)$$

where K_β^ϕ is defined in (1.5).

It is natural to interpret (1.7) as the evolution equation of the flow of marginals of a Markov process which corresponds to a nonlinear martingale problem. Let X denote the canonical process on the Skorohod space $\mathbb{D} = \mathbb{D}(\mathbb{R}_+, \mathbb{R})$.

Definition 1.3 *Let β be a cross section such that $\int_0^\pi \theta^2\beta(\theta)\,d\theta < +\infty$, and P_0 in $\mathcal{P}_2(\mathbb{R})$.*

We say that the probability measure $P \in \mathcal{P}(\mathbb{D}(\mathbb{R}_+, \mathbb{R}))$ solves the nonlinear martingale problem starting at P_0 if under P, the law of X_0 is P_0 and for any ϕ in $C_b^2(\mathbb{R})$,

$$\phi(X_t) - \phi(X_0) - \int_0^t \langle K_\beta^\phi(X_s, v^*), P_s(dv^*)\rangle\,ds \quad (1.8)$$

is a square-integrable martingale. Here, P_s denotes the marginal of P at time s, and K_β^ϕ is defined in (1.5).

Note that if P solves (1.8), then $(P_t)_{t\geq 0}$ solves (1.7).

2 Probabilistic interpretation and approximations for the Kac equation with cutoff

We consider first the simpler cutoff Kac equation for which $\beta \in L^1([0,\pi[)$. Existence and uniqueness of a solution P^β to (1.8) and $(P_t^\beta)_{t\geq 0}$ to (1.7) can be easily proved. Moreover, we are able to describe some simulable interacting particle systems whose laws converge to P^β when the size of the system tends to infinity.

2.1 The solution of the nonlinear martingale problem

Theorem 2.1 *Let β be a cross section such that $\int_0^\pi \beta(\theta)\,d\theta < +\infty$ and $P_0 \in \mathcal{P}(\mathbb{R})$.*
1) There is a unique solution P^β to the nonlinear martingale problem starting at P_0 in the sense of Definition 1.3. Its flow of time-marginals $(P_t^\beta)_{t\geq 0}$ is the unique solution of the Kac equation (1.1) in the sense of Definition 1.2.
2) If moreover P_0 has a density f_0, then P_t^β has a density for any $t \geq 0$ and can be written $P_t^\beta(dv) = f^\beta(t,v)\,dv$, where f^β is the unique weak solution to (1.1) in the sense of Theorem 1.1 (cf. [4], Appendix A).

Proof. 1) We follow Shiga and Tanaka [15], Lemma 2.3. Since β is in $L^1([0,\pi[)$, we have for any flow $(Q_t)_{t\geq 0}$ in $\mathcal{P}(\mathbb{R})$,

$$\langle K_\beta^\phi(v,v^*), Q_s(dv^*)\rangle = \langle \int_{-\pi}^{\pi} \{\phi(v\cos\theta - v^*\sin\theta) - \phi(v)\}\beta(\theta)\,d\theta, Q_s(dv^*)\rangle$$

and

$$\phi \in L^\infty(\mathbb{R}) \mapsto \langle K_\beta^\phi(\cdot,v^*), Q_s(dv^*)\rangle \in L^\infty(\mathbb{R}) \tag{2.1}$$

is a bounded pure-jump Markov operator generating a unique law P^Q in $\mathcal{P}(\mathbb{D})$ starting at P_0. Its flow of marginals solves a linearized version of (1.7): for all ϕ in $L^\infty(\mathbb{R})$,

$$\langle \phi, P_t^Q\rangle = \langle \phi, P_0\rangle + \int_0^t \langle K_\beta^\phi(v,v^*), Q_s(dv^*)\,P_s^Q(dv)\rangle\,ds\,. \tag{2.2}$$

Let $|\mu| = \sup\{\langle\phi,\mu\rangle : \|\phi\|_\infty \leq 1\}$ denote the variation norm, and $\|\beta\|_1 = \int_{-\pi}^{\pi}\beta(\theta)\,d\theta$. For $i = 1,2$, take $(Q_t^i)_{t\geq 0}$, and consider corresponding solutions $(P_t^i)_{t\geq 0}$ of (2.2). Then,

$$\langle \phi, P_t^1 - P_t^2\rangle = \int_0^t \langle K_\beta^\phi, Q_s^1 \otimes (P_s^1 - P_s^2) + (Q_s^1 - Q_s^2)\otimes P_s^2\rangle\,ds$$

and hence

$$|P_t^1 - P_t^2| \leq 2\|\beta\|_1 \int_0^t |P_s^1 - P_s^2| + |Q_s^1 - Q_s^2|\,ds.$$

Then, by iteration,

$$|P_t^1 - P_t^2| \leq 2\|\beta\|_1 e^{2\|\beta\|_1 t}\int_0^t |Q_s^1 - Q_s^2|\,ds\,. \tag{2.3}$$

Taking $Q_t^1 = Q_t^2 = Q_t$ we see that there is a unique probability measure flow solving the linearized equation (2.2) associated with any $(Q_t)_{t\geq 0}$, which must then be equal to the flow of marginals of P^Q generated by (2.1).

We now consider the nonlinear equation (1.7). Uniqueness easily follows from (2.3). Let $P_t^0 = P_0$ and for $k \geq 0$, $(P_t^{k+1})_{t\geq 0}$ be the solution associated with $(P_t^k)_{t\geq 0}$ by (2.2):

$$\langle \phi, P_t^{k+1}\rangle = \langle \phi, P_0\rangle + \int_0^t \langle K_\beta^\phi(v,v^*), P_s^k(dv^*)\,P_s^{k+1}(dv)\rangle\,ds$$

for $\phi \in L^\infty(\mathbb{R})$. Iteration of the estimate (2.3) yields

$$|P_t^{k+1} - P_t^k| \leq (2\|\beta\|_1 e^{2\|\beta\|_1 t})^k \frac{t^k}{k!}\sup_{0\leq s\leq t}|P_s^1 - P_s^0|.$$

Then $(P_t^k)_{t\geq 0}$ converges uniformly on compact sets to $(\tilde{P}_t)_{t\geq 0}$ solving (1.7).

We now turn to the problem of existence and uniqueness for the nonlinear martingale problem.

Let P be the law generated by (2.1) for $(Q_t)_{t\geq 0}$ equal to $(\tilde{P}_t)_{t\geq 0}$. Then the flow $(P_t)_{t\geq 0}$ satisfies (2.2) for $(Q_t)_{t\geq 0}$ equal to $(\tilde{P}_t)_{t\geq 0}$, as does $(\tilde{P}_t)_{t\geq 0}$, and by uniqueness for (2.2) we obtain $(P_t)_{t\geq 0} = (\tilde{P}_t)_{t\geq 0}$. Thus P solves the nonlinear martingale problem (1.8).

Finally, if we assume that there exist two solutions P^1 and P^2 for the nonlinear martingale problem (1.8), $(P_t^1)_{t\geq 0}$ and $(P_t^2)_{t\geq 0}$ will be solutions to (1.7) and hence will be equal to $(\tilde{P}_t)_{t\geq 0}$. Then P^1 and P^2 solve a linearized martingale problem with this fixed $(\tilde{P}_t)_{t\geq 0}$ in (1.8), and it is well-known that they are then both generated by (2.1) with $(Q_t)_{t\geq 0}$ equal to $(\tilde{P}_t)_{t\geq 0}$, and hence $P^1 = P^2$.

2) Assume now that $P_0(dv) = f_0(v)\,dv$. We are going to show that if $Q_t(dv) = g(t,v)\,dv$ for $t \geq 0$, then the marginal P_t^Q of the law P^Q of the Markov process with generator (2.1) has a density. We use its explicit probabilistic evolution. Let $(X_t)_{t\geq 0}$ be the canonical process, $T_0 = 0$, and $(T_n)_{n\geq 1}$ its jump times (possibly $+\infty$). Since the jump rate is bounded, any sample path jumps a finite number of times in $[0,T]$. Since the jump measure $\beta(\theta)d\theta ds$ is absolutely continuous with respect to time, it is easy to see, following for example [11] p.136, that the law of the first jump-time T_1 conditionally to $X_0 = v$ has a density with respect to the Lebesgue measure. Since the law of X_0 has the density f_0, then (X_0, T_1) has a density with respect to the Lebesgue measure. Moreover, conditionally to (X_0, T_1), the law of the jump ΔX_{T_1} has clearly a density and thus the law of (X_{T_1}, T_1) has a density. By the Markov property, we then deduce that for every T_n, the law of (X_{T_n}, T_n) has a density, and so P_t^Q has a density. Applying this result to the Picard iteration sequence, if P^k has a density and if $P_0(dv) = f_0(v)\,dv$ then P_t^{k+1} has a density. Since $P_t^0(dv) = f_0(v)\,dv$ then $P_t^k(dv) = f^k(t,v)\,dv$ for all $k \geq 0$ and $t \geq 0$. The variation norm on measures with a density is the same as the L^1 norm on their densities, hence $(f^k)_{k\geq 0}$ is a Cauchy sequence and then converges in L^1 norm (in v) uniformly on compact sets (in t) to a function $f(t,v)$ which is the density of the unique solution to (1.7). The function f is then the unique weak solution of (1.1) in the sense of Theorem 1.1. $\square$

2.2 Stochastic approximations

Under the cutoff assumption $\int_0^\pi \beta(\theta)\,d\theta < +\infty$, we define two different mean-field interacting particle systems which approximate the solution of the nonlinear martingale problem (1.8). Let $\mathbf{v}^n = (v_1, v_2, ..., v_n)$ be the generic point in $\mathbb{R}^n$, and $\mathbf{e_i} : h \in \mathbb{R} \mapsto \mathbf{e_i}.h = (0, ..., 0, h, 0, ..., 0) \in \mathbb{R}^n$ with h at the i-th place. We consider $\phi \in C_b(\mathbb{R}^n)$.

The simple mean-field system is a Markov process in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^n)$ with generator

$$\frac{1}{n-1} \sum_{1\leq i\neq j\leq n} \int_{-\pi}^{+\pi} \Big(\phi(\mathbf{v}^n + \mathbf{e_i}.(v_i(\cos\theta - 1) - v_j \sin\theta)) - \phi(\mathbf{v}^n)\Big)\beta(\theta)d\theta. \tag{2.4}$$

The binary mean-field system is a Markov process on the same space with generator

$$\frac{1}{n-1}\sum_{1\le i\neq j\le n}^{n}\int_{-\pi}^{+\pi}\frac{1}{2}\Big(\phi(\mathbf{v}^n+\mathbf{e_i}.(v_i(\cos\theta-1)-v_j\sin\theta)\,+\,\mathbf{e_j}.(v_j(\cos\theta-1)-v_i\sin\theta))$$
$$-\ \phi(\mathbf{v}^n)\Big)\beta(\theta)d\theta. \qquad (2.5)$$

We denote in both cases the Markov process by $V^{\beta,n}=(V^{\beta,1n},...,V^{\beta,nn})$, and by $|.|_T$ the variation norm in the space of signed measures on $\mathbb{D}([0,T],\mathbb{R})$.

We use Graham and Méléard, [6] Theorem 6.1 or [7] Theorem 3.1, for the following.

Theorem 2.2 *1) Let $(V_0^{\beta,in})_{1\le i\le n}$ be i.i.d. with law P_0. Then we have propagation of chaos in strong sense: for given $T>0$ and $k\in\mathbb{N}^*$,*

$$|\mathcal{L}(V^{\beta,1n},...,V^{\beta,kn})-(P^\beta)^{\otimes k}|_T\le Kk^2\frac{\exp(\|\beta\|_1T)}{n}, \qquad (2.6)$$

where P^β is the unique solution of the nonlinear martingale problem with initial law P_0 in the sense of Definition 1.3. Here, K denotes a constant independent of k,T,β,n.
2) The empirical measure defined by

$$\mu^{\beta,n}=\frac{1}{n}\sum_{i=1}^{n}\delta_{V^{\beta,in}}$$

converges in probability to P^β in $\mathcal{P}(\mathbb{D}([0,T],\mathbb{R}))$ for the weak convergence for the Skorohod metric on $\mathbb{D}([0,T],\mathbb{R})$ with an estimate of convergence in $\sqrt{K\exp(\|\beta\|_1T)}/\sqrt{n}$.

3 Representation using Poisson point processes for the Kac equation without cutoff

We now concentrate on the non cutoff case and only assume $\int_0^\pi\theta^2\beta(\theta)\,d\theta<+\infty$.

We define a specific nonlinear stochastic differential equation corresponding to (1.8). This construction uses an appropriate Picard iteration method involving an auxiliary space. We give statements on $[0,T]$ for an arbitrary $T\in\mathbb{R}_+$.

In the sequel, $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},P)$ shall be a Polish filtered probability space satisfying the usual conditions. Such a space is Borel isomorphic to the Lebesgue space $([0,1],\mathcal{B}([0,1]),d\alpha)$ with generic point α, which we use as an auxiliary space. For clarity of exposition we reserve the notation E for the expectation and $\mathcal{L}$ for the law of a random variable on $(\Omega,\mathcal{F},P)$, and use E_α and $\mathcal{L}_\alpha$ for $([0,1],\mathcal{B}([0,1]),d\alpha)$ or specifically denote the α-dependence. Finally, the processes on $([0,1],\mathcal{B}([0,1]),d\alpha)$ are called α-processes.

Let us precise a few more notations.

A process V is $\mathbf{L}^2_T$ if it is adapted, has sample paths in $\mathbb{D}([0,T],\mathbb{R})=\mathbb{D}_T$, and $E(\int_0^T V_s^2\,ds)<\infty$. We consider the L^∞ norm $\sup_{0\le t\le T}|x_t|$ on $\mathbb{D}_T$, and the L^2 convergence of processes for this norm. Let $\mathcal{P}_2(\mathbb{D}_T)$ denote the space of probability measures on $\mathbb{D}_T$ such that the canonical process is L^2:

$$\mathcal{L}(V) \in \mathcal{P}_2(\mathbb{D}_T) \Leftrightarrow E\Big(\sup_{0\le t\le T} V_t^2\Big) < +\infty\,.$$

We similarly define $\mathcal{P}_p(\mathbb{D}_T)$ for $p \ge 1$. For P and Q in $\mathcal{P}_2(\mathbb{D}_T)$,

$$\rho_T(P,Q) = \inf\Big\{ \Big(\int_{\mathbb{D}_T\times\mathbb{D}_T} \sup_{0\le t\le T} (x_t - y_t)^2 \, R(dx,dy) \Big)^{1/2} : R \text{ has marginals } P \text{ and } Q \Big\}$$

defines a metric for weak convergence with test functions which are continuous for the uniform norm on $\mathbb{D}_T$, measurable for the product σ-field, and have growth dominated by the square of the uniform norm.

We use a special representation in order to have a fixed Poisson driving term. Let $N(d\theta d\alpha dt)$ be an adapted Poisson point process on $H = [-\pi,\pi] \times [0,1]$ with intensity measure $\beta(\theta) d\theta d\alpha dt$, and $\tilde N(d\theta d\alpha dt)$ be its compensated Poisson point process, see for instance Ikeda and Watanabe [10].

Definition 3.1 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ be a Polish filtered probability space satisfying the usual conditions, β be a cross section such that $\int_0^\pi \theta^2 \beta(\theta)\, d\theta < +\infty$, $N(d\theta d\alpha dt)$ be an adapted Poisson point process on $H = [-\pi,\pi]\times[0,1]$ with intensity measure $\beta(\theta) d\theta d\alpha dt$, and V_0 be an independent square-integrable initial condition.*

We say that an $\mathbf{L}^2_T$ process V solves the nonlinear stochastic differential equation if there exists an α-process W on $([0,1], \mathcal{B}([0,1]), d\alpha)$ such that for all t in $[0,T]$,

$$\begin{cases} V_t(\omega) = V_0(\omega) + \displaystyle\int_0^t \int_H \{(\cos\theta - 1) V_{s-}(\omega) - (\sin\theta) W_{s-}(\alpha)\} \tilde N(\omega, d\theta d\alpha ds) - b \int_0^t V_s(\omega)\, ds, \\ \mathcal{L}(V) = \mathcal{L}_\alpha(W)\,, \end{cases} \tag{3.1}$$

with b given by (1.6).

Remark: If Ω, β, N and V_0 are as in Definition 3.1, and if Z is a given $\mathbf{L}^2_T$ α-process, then one can consider the classical SDE

$$V_t = V_0 + \int_0^t \int_H \{(\cos\theta - 1) V_{s-} - (\sin\theta) Z_{s-}(\alpha)\} \tilde N(d\theta d\alpha ds) - b \int_0^t V_s\, ds. \tag{3.2}$$

Setting $Q_t = \mathcal{L}_\alpha(Z_t)$, an application of the Itô formula yields that for any $\phi \in C_b^2(\mathbb{R})$,

$$\phi(V_t) - \phi(V_0) - \int_0^t \int_{[0,1]} K_\beta^\phi(V_s, Z_s(\alpha))\, d\alpha ds = \phi(V_t) - \phi(V_0) - \int_0^t \langle K_\beta^\phi(V_s, z), Q_s(dz)\rangle\, ds$$

is a martingale. Thus the law $\mathcal{L}(V)$ on $\mathbb{D}$ of any solution of (3.1) is a solution of the nonlinear martingale problem in the sense of Definition 1.3 with initial datum $\mathcal{L}(V_0)$.

We are now interested in proving existence and uniqueness results for our nonlinear SDE (3.1). This is done in several steps.

Let us first give the following definition, which necessitates $\int_0^\pi \theta^2 \beta(\theta)\, d\theta < +\infty$.

Definition 3.2 *If Ω, β, N and V_0 are as in Definition 3.1, and if Y is an $\mathbf{L}^2_T$ process, Z an $\mathbf{L}^2_T$ α-process, the equation*

$$V_t = V_0 + \int_0^t \int_H \{(\cos\theta - 1)Y_{s-} - (\sin\theta)Z_{s-}(\alpha)\}\tilde{N}(d\theta d\alpha ds) - b\int_0^t Y_s\, ds \tag{3.3}$$

defines a mapping $Y, Z, V_0, N \mapsto V = \Phi(Y, Z, V_0, N)$. We also have $\mathcal{L}(V) \in \mathcal{P}_2(\mathbb{D}_T)$.

We now prove a key contraction estimate.

Proposition 3.3 *Let Ω, β, N and V_0 be as in Definition 3.1, and take $i = 1, 2$. Consider $\mathbf{L}^2_T$ processes Y^i and $\mathbf{L}^2_T$ α-processes Z^i, and set $V^i = \Phi(Y^i, Z^i, V_0, N)$.*

Then $V^i \in \mathcal{P}_2(\mathbb{D}_T)$ and

$$E\Big(\sup_{0\le s\le t}(V^1_s - V^2_s)^2\Big) \le (b' + 2b^2 t)\int_0^t E((Y^1_s - Y^2_s)^2)\, ds + b''\int_0^t E_\alpha((Z^1_s - Z^2_s)^2)\, ds \tag{3.4}$$

where $b' = 8\int_{-\pi}^{\pi}(\cos\theta - 1)^2\beta(\theta)\, d\theta$, $b'' = 8\int_{-\pi}^{\pi}\sin^2\theta\beta(\theta)\, d\theta$.

Note that b' and b'' are well defined under our assumption on β.

Proof. We have

$$\begin{aligned} E\Big(\sup_{0\le s\le t}(V^1_s - V^2_s)^2\Big) \le 2E\Big(\sup_{0\le s\le t}\Big(\int_0^s\int_H \{(\cos\theta - 1)(Y^1_{u-} - Y^2_{u-}) \\ - (\sin\theta)(Z^1_{u-} - Z^2_{u-})(\alpha)\}\tilde{N}(dud\theta d\alpha)\Big)^2\Big) \\ + 2b^2 E\Big(\sup_{0\le s\le t}\Big(\int_0^s (Y^1_u - Y^2_u)\, du\Big)^2\Big), \end{aligned}$$

and using the Doob and Jensen inequalities and the compensator of N,

$$\begin{aligned} E\Big(\sup_{0\le s\le t}(V^1_s - V^2_s)^2\Big) &\le 8E\Big(\int_0^t\int_H \{(\cos\theta - 1)(Y^1_s - Y^2_s) - (\sin\theta)(Z^1_s - Z^2_s)(\alpha)\}^2\beta(\theta)d\theta d\alpha ds\Big) \\ &\quad + 2b^2 t E\Big(\int_0^t (Y^1_s - Y^2_s)^2\, ds\Big) \\ &\le (b' + 2b^2 t)\int_0^t E((Y^1_s - Y^2_s)^2)\, ds + b''\int_0^t E_\alpha((Z^1_s - Z^2_s)^2)\, ds, \end{aligned} \tag{3.5}$$

since $\int_{-\pi}^{\pi}(\cos\theta - 1)\sin\theta\beta(\theta)\, d\theta = 0$ (β is even and $(\cos\theta - 1)\sin\theta$ is odd and $O(\theta^2)$). □

The classical SDE (3.2) corresponds to finding a fixed point $V = \Phi(V, Z, V_0, N)$. We now obtain an existence and uniqueness result for this classical SDE.

Theorem 3.4 *Let Ω, β, N and V_0 be as in Definition 3.1, and Z be an $\mathbf{L}^2_T$ α-process.*

Then there exists a unique strong solution V of the SDE (3.2), i.e. an $\mathbf{L}^2_T$ process V such that $V = \Phi(V, Z, V_0, N)$ in the sense of Definition 3.2. We denote it by $V = F(Z, V_0, N)$. Its law $\mathcal{L}(V)$ is in $\mathcal{P}_2(\mathbb{D}_T)$ and depends on $\mathcal{L}_\alpha(Z)$ only through the flow of marginals $(\mathcal{L}_\alpha(Z_t))_{t\ge 0}$.

Proof. Iteration of the contraction estimate (3.4) yields uniqueness and convergence of the Picard iteration scheme $Y^0 = V_0$, $Y^{k+1} = \Phi(Y^k, Z, V_0, N)$, which defines F (details will be given later in a more complex case).

We denote by $p = (p^\theta, p^\alpha)$ the point process on H corresponding to N, and introduce the inhomogeneous Poisson point process $p^*_t = (p^\theta_t, Z_t(p^\alpha_t))$ on $[-\pi,\pi]\times\mathbb{R}$ and its counting measure N^*. Then N^* has the intensity measure $\beta(\theta)d\theta\,\mathcal{L}_\alpha(Z_t)(dz)\,dt$,

$$V_t = V_0 + \int_0^t \int_{[-\pi,\pi]\times\mathbb{R}} \{(\cos\theta - 1)V_{s-} - (\sin\theta)z\}\tilde{N}^*(d\theta dz ds) - b\int_0^t V_s\,ds$$

and the same kind of contraction estimates and Picard iteration show that V is a well-defined function of V_0 and N^* and hence $\mathcal{L}(V)$ is a well-defined function of $\mathcal{L}(V_0)$ and $\mathcal{L}(N^*)$, the latter being completely specified by its intensity measure $\beta(\theta)d\theta\,\mathcal{L}_\alpha(Z_t)(dz)\,dt$ and hence by $(\mathcal{L}_\alpha(Z_t))_{t\geq 0}$. □

Let us now consider the nonlinear SDE (3.1). A new idea is to devise an appropriate generalization of the Picard iteration method. The corresponding sequences of processes are defined in the following way.

Definition 3.5 *Let Ω, β, N and V_0 be as in Definition 3.1.*

Let V^0 be the process with constant value V_0.

For $k \geq 0$, once $V^0, \dots, V^k$ and $Z^0, \dots, Z^{k-1}$ are defined, we choose an α-process Z^k such that

$$\mathcal{L}_\alpha(Z^k | Z^{k-1}, \dots, Z^0) = \mathcal{L}(V^k | V^{k-1}, \dots, V^0)$$

and set

$$V^{k+1} = \Phi(V^k, Z^k, V_0, N)\,.$$

Remark: Tanaka [18] introduces for his existence proof a similar sequence of processes V^k, but involving only the pairs $\mathcal{L}_\alpha(Z^k, Z^{k-1}) = \mathcal{L}(V^k, V^{k-1})$, which does not suffice to obtain a satisfying uniqueness result.

We now state a theorem of existence for the nonlinear SDE.

Theorem 3.6 *1) Let Ω, β, N and V_0 be as in Definition 3.1. The Picard sequences $(V^k)_{k\geq 0}$ and $(Z^k)_{k\geq 0}$ introduced in Definition 3.5 converge a.s. and in L^2 to $\hat{V}$ and $\hat{W}$ solving (3.1): $\hat{V} = \Phi(\hat{V}, \hat{W}, V_0, N)$ and $\mathcal{L}(\hat{V}) = \mathcal{L}_\alpha(\hat{W})$. The law P^β of $\hat{V}$ belongs to $\mathcal{P}_2(\mathbb{D}_T)$ and solves the nonlinear martingale problem (1.8) with initial datum $\mathcal{L}(V_0)$.*
2) The law P^β does not depend on the specific choice of Ω, N, and V_0, but only on $P_0 = \mathcal{L}(V_0)$.

Proof. 1) Since $\mathcal{L}_\alpha(Z^k - Z^{k-1}) = \mathcal{L}(V^k - V^{k-1})$ and $b' + b'' = 16b$, estimate (3.4) gives

$$E\Big(\sup_{0\le s\le t}(V_s^{k+1} - V_s^k)^2\Big) \le (16b + 2b^2 t)\int_0^t E((V_s^k - V_s^{k-1})^2)\,ds$$
$$\le (16b + 2b^2 t)^k \frac{t^k}{k!}\sup_{0\le s\le t} E((V_s^1 - V_s^0)^2).$$

Then, $(V^k)_{k\ge0}$ and $(Z^k)_{k\ge0}$ converge for the L^2 norm and a.s. (using the Borel-Cantelli lemma) to a process $\hat V$ and an α-process $\hat W$. This L^2 convergence implies that $\hat V = \Phi(\hat V, \hat W, V_0, N)$. The sequences $(V^k)_{k\ge0}$ and $(Z^k)_{k\ge0}$ have same law, hence $\mathcal{L}(\hat V) = \mathcal{L}_\alpha(\hat W)$ The Itô formula shows that P^β is a solution to (1.8).
2) Since $\mathcal{L}((V^k)_{k\ge0})$ does not depend on the particular choice of Ω, V_0, N, and Z^k, $k\ge 0$, then $\mathcal{L}(\hat V)$ depends only on $\mathcal{L}(V_0)$. □

We now prove that the law of any solution of (3.1) is equal to P^β.

Theorem 3.7 *1) Let Ω, β, N, V_0, and $\hat V$ be as in Theorem 3.6, and let $U = \Phi(U, Y, V_0, N)$, $\mathcal{L}(U) = \mathcal{L}_\alpha(Y)$ be another solution of (3.1). Then $\mathcal{L}(U) = \mathcal{L}(\hat V) = P^\beta$.*
2) There is uniqueness in law for (3.1).

Proof. 1) We can suppose that $U = \Phi(U, Y, V_0, N)$, $\mathcal{L}(U) = \mathcal{L}_\alpha(Y) = Q$, and $\hat V = \Phi(\hat V, \hat W, V_0, N)$, $\mathcal{L}(\hat V) = \mathcal{L}_\alpha(\hat W) = P^\beta$.

We cannot directly compare $\hat V$ and U because we have no information on $\hat W$ and Y. Theorem 3.4 implies that $P^\beta, Q \in \mathcal{P}_2(\mathbb{D}_T)$. Then, for any $\tau\in[0,T]$

$$\rho_\tau(P^\beta, Q) = \inf\Big\{E_\alpha\Big(\sup_{0\le t\le\tau}(W'_t - Y'_t)^2\Big)^{1/2} : \mathcal{L}_\alpha(W') = P^\beta,\ \mathcal{L}_\alpha(Y') = Q\Big\},$$

and for any $\varepsilon > 0$ there exists W^ε and Y^ε such that $\mathcal{L}_\alpha(W^\varepsilon) = P^\beta$, $\mathcal{L}_\alpha(Y^\varepsilon) = Q$, and

$$\rho_\tau(P^\beta, Q)^2 \le E_\alpha\Big(\sup_{0\le t\le\tau}(W_t^\varepsilon - Y_t^\varepsilon)^2\Big) < \rho_\tau(P^\beta, Q)^2 + \varepsilon\,. \tag{3.6}$$

Theorem 3.4 defines F in such a way that $\hat V = F(\hat W, V_0, N)$ and $U = F(Y, V_0, N)$. We set $V^\varepsilon = F(W^\varepsilon, V_0, N)$ and $U^\varepsilon = F(Y^\varepsilon, V_0, N)$, and since $\mathcal{L}_\alpha(W^\varepsilon) = \mathcal{L}_\alpha(\hat W)$ and $\mathcal{L}_\alpha(Y^\varepsilon) = \mathcal{L}_\alpha(Y)$ we have $\mathcal{L}(V^\varepsilon) = \mathcal{L}(\hat V) = P^\beta$ and $\mathcal{L}(U^\varepsilon) = \mathcal{L}(U) = Q$. Since $V^\varepsilon = \Phi(V^\varepsilon, W^\varepsilon, V_0, N)$ and $U^\varepsilon = \Phi(U^\varepsilon, Y^\varepsilon, V_0, N)$ we use (3.4) and (3.6) to obtain

$$E\Big(\sup_{0\le s\le\tau}(V_s^\varepsilon - U_s^\varepsilon)^2\Big) \le (b' + 2b^2\tau)\int_0^\tau E((V_s^\varepsilon - U_s^\varepsilon)^2)\,ds + b''\tau(\rho_\tau(P^\beta, Q)^2 + \varepsilon)$$
$$\le b''\tau\exp(b'\tau + 2b^2\tau^2)(\rho_\tau(P^\beta, Q)^2 + \varepsilon).$$

Fixing $\tau > 0$ in such a way that $K = b''\tau\exp(b'\tau + 2b^2\tau^2) < 1$, we have

$$\rho_\tau(P^\beta, Q)^2 \le E\Big(\sup_{0\le t\le\tau}(V_t^\varepsilon - U_t^\varepsilon)^2\Big) < K(\rho_\tau(P^\beta, Q)^2 + \varepsilon)$$

and $\rho_\tau(P^\beta, Q) = 0$ since $\varepsilon > 0$ is arbitrary. Hence we have uniqueness in law on $[0, \tau]$.

For $n \geq 0$ we set $T_n = n\tau$ and $V^n = (\hat{V}_{T_n+t})_{t\geq 0}$ and similarly define U^n, etc. Assume we have uniqueness in law on $[0, T_n]$. Then in particular $\mathcal{L}(\hat{V}_{T_n}) = \mathcal{L}(U_{T_n})$, thus $\bar{U}^n = F(Y^n, \hat{V}_{T_n}, N^n - N_{T_n})$ has same law as $U^n = F(Y^n, U_{T_n}, N^n - N_{T_n})$ and thus

$$\bar{U}^n = \Phi(\bar{U}^n, Y^n, \hat{V}_{T_n}, N^n - N_{T_n}), \quad \mathcal{L}(\bar{U}^n) = \mathcal{L}(U^n) = \mathcal{L}_\alpha(Y^n),$$

and we obtain that $\mathcal{L}(\bar{U}^n) = \mathcal{L}(V^n)$ on $[0, \tau]$ and hence $\mathcal{L}(U^n) = \mathcal{L}(V^n)$ on $[0, \tau]$. Hence the flow of marginals $(\mathcal{L}_\alpha(Y_t))_{0\leq t\leq T_{n+1}}$ and $(P_t^\beta)_{0\leq t\leq T_{n+1}}$ are equal. Using Theorem 3.4 we conclude that $\mathcal{L}(U) = P^\beta$ on $[0, T_{n+1}]$. Hence recursively $\mathcal{L}(U) = P^\beta$ on $[0, T]$.
2) The result comes immediately from Theorem 3.6, 2). □

Now at last we can give an existence and uniqueness statement for the nonlinear martingale problem of Definition 1.3.

Theorem 3.8 *Let β be a cross section such that $\int_0^\pi \theta^2 \beta(\theta)\, d\theta < +\infty$, and suppose that $P_0 \in \mathcal{P}_2(\mathbb{R})$. Then, there exists a unique solution P^β to the nonlinear martingale problem with initial datum P_0 in the sense of Definition 1.3.*

Moreover, P^β is in $\mathcal{P}_2(\mathbb{D}_T)$, and the flow $(P_t^\beta)_{t\geq 0}$ is a measure solution to eq. (1.1) in the weak sense of Definition 1.2. This flow satisfies the following properties of momentum and energy: for any $t \in \mathbb{R}_+$, $\langle v, P_t^\beta(dv)\rangle = \exp(-bt)\langle v, P_0(dv)\rangle$ and $\langle v^2, P_t^\beta(dv)\rangle = \langle v^2, P_0(dv)\rangle$. Finally, if $\langle |v|^p, P_0(dv)\rangle < +\infty$ for $p \geq 2$, then P^β is in $\mathcal{P}_p(\mathbb{D}_T)$.

Proof. The existence result is given in Theorem 3.6, and the result on the flow of marginals follows by taking the expectation of (1.8). The moment result follows classically.

Let us now prove the result of uniqueness.

Let $Q \in \mathcal{P}_2(\mathbb{D}_T)$ be a solution to (1.8). It follows from the martingale problem that for Borel positive ϕ on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ such that $\phi(\cdot, \cdot, z) \leq Kz^2$, the compensated sum

$$\sum_{0\leq s\leq t} \phi(s, X_{s-}, \Delta X_s) - \int_0^t \int_{-\pi}^{\pi} \langle \phi(s, X_s, (\cos\theta - 1)X_s - (\sin\theta)v^*), Q_s(dv^*)\rangle \beta(\theta)\, d\theta ds$$

is a L^2 martingale under Q which can be written using an α-process X^* of law Q as

$$\sum_{0\leq s\leq t} \phi(s, X_{s-}, \Delta X_s) - \int_0^t \int_H \phi(s, X_s, (\cos\theta - 1)X_s - (\sin\theta)X_s^*(\alpha))\beta(\theta)\, d\theta d\alpha ds.$$

Moreover $X_t = X_0 + M_t - b\int_0^t X_s\, ds$, where M is the martingale compensated sum of jumps of X, which is an L^2 martingale with Doob-Meyer Bracket

$$\int_0^t \int_H \{(\cos\theta - 1)X_s - (\sin\theta)X_s^*(\alpha)\}^2 \beta(\theta)\, d\theta d\alpha ds.$$

This characterizes the compensator of the point process ΔX.

Following Tanaka [17] Section 4, we can build on an enlarged probability space Ω a Poisson point process $\tilde{N}$ on $H = [-\pi, \pi] \times [0, 1]$ with intensity measure $\beta(\theta)\, d\theta d\alpha$, independent of X_0, such that

$$M_t = \int_0^t \int_H \{(\cos\theta - 1)X_{s-} - (\sin\theta)X^*_{s-}(\alpha)\}\tilde{N}(d\theta d\alpha ds).$$

Then $X = \Phi(X, X^*, X_0, N)$ and $\mathcal{L}(X) = Q = \mathcal{L}_\alpha(X^*)$ and Theorem 3.7 implies that Q must be the probability P starting at P_0 defined in Theorem 3.6. $\square$

4 Stochastic approximations for the non cutoff Kac equation

We consider the non cutoff Kac equation, when only $\int_0^\pi \theta^2\beta(\theta)\, d\theta$ is known to be finite.

We want to approximate the solution of the nonlinear martingale problem (1.8) in this case by using a simulable interacting particle system. As an intermediate step, we introduce cutoff approximations of this nonlinear martingale problem.

4.1 Convergence of cutoff approximations

We consider cross sections $(\beta_\ell)_{\ell\geq 0}$ and β and corresponding b_ℓ and b (defined in (1.6)), and set

$$\delta_\ell = \int_{-\pi}^{\pi} (1-\cos\theta)|\beta - \beta_\ell|(\theta)\, d\theta, \quad c_\ell = \int_{-\pi}^{\pi}(1-\cos\theta)(\beta\wedge\beta_\ell)(\theta)\, d\theta \leq b_\ell \wedge b. \tag{4.7}$$

We endow $\mathcal{P}_2(\mathbb{R})$ with the metric

$$\rho(p,q) = \inf\Big\{\Big(\int_{\mathbb{R}\times\mathbb{R}} (x-y)^2\, r(dx,dy)\Big)^{1/2} : r \text{ has marginals } p \text{ and } q\Big\}$$

corresponding to weak convergence plus convergence of the second moment.

Theorem 4.1 *Let $P_0 \in \mathcal{P}_2(\mathbb{R})$ be given, and let P^β and P^{β_ℓ} be the solutions given in Theorem 3.8 to the martingale problems (1.8) with cross sections β and β_ℓ respectively.*

Then

$$\sup_{0\leq t\leq T} \rho(P_t^{\beta_\ell}, P_t^\beta)^2 \leq \rho_T(P^{\beta_\ell}, P^\beta)^2 \leq (16\delta_\ell T + 2\delta_\ell^2 T^2)\exp(16 c_\ell T + 2c_\ell^2 T^2)\langle v^2, P_0(dv)\rangle.$$

Hence if $\lim_{\ell\to\infty}\delta_\ell = 0$, then $\lim_{\ell\to\infty}\sup_{0\leq t\leq T}\rho(P_t^{\beta_\ell}, P_t^\beta) = \lim_{\ell\to\infty}\rho_T(P_t^{\beta_\ell}, P_t^\beta) = 0$. This is the case when the β_ℓ are cutoff versions of β, such as $\beta\wedge\ell$ or $\beta(\theta)\mathbf{1}_{|\theta|\geq 1/\ell}$.

Proof. We use coupling techniques, and adopt the notations of the previous section. Let $\ell \geq 0$ be fixed, and let there be Ω with independent Poisson random measures $N^\wedge(d\theta d\alpha ds)$ with characteristic measure $(\beta\wedge\beta_\ell)(\theta)\, d\theta d\alpha$, $N^+(d\theta d\alpha ds)$ with characteristic measure

$(\beta-\beta_\ell)^+(\theta)\,d\theta d\alpha$, $N^-(d\theta d\alpha ds)$ with characteristic measure $(\beta-\beta_\ell)^-(\theta)\,d\theta d\alpha$. Then $N = N^\wedge + N^+$ and $N_\ell = N^\wedge + N^-$ are Poisson random measures with characteristic measures $\beta(\theta)\,d\theta d\alpha$ and $\beta_\ell(\theta)\,d\theta d\alpha$. We perform a Picard iteration scheme. We take V_0 of law P_0 and define $V^{\ell,0} = V^0 = V_0$, and for $k \geq 0$ we choose α-processes Z^k and $Z^{\ell,k}$ such that

$$\mathcal{L}_\alpha(Z^k, Z^{\ell,k}|Z^{k-1},\ldots,Z^0,Z^{\ell,k-1},\ldots,Z^{\ell,0}) = \mathcal{L}(V^k,V^{\ell,k}|V^{k-1},\ldots,V^0,V^{\ell,k-1},\ldots,V^{\ell,0})$$

and set (cf. (3.3), using naturally $b_\ell = \int_{-\pi}^{\pi}(1-\cos\theta)\beta_\ell(\theta)\,d\theta$ instead of b for $V^{\ell,k+1}$)

$$V^{k+1} = \Phi(V^k, Z^k, V_0, N)\,, \quad V^{\ell,k+1} = \Phi(V^{\ell,k}, Z^{\ell,k}, V_0, N_\ell)\,.$$

Then, following Theorem 3.6 there are a.s. and L^2 limits V and V^ℓ to the sequences $(V^k)_{k\geq 0}$ and $(V^{\ell,k})_{k\geq 0}$, and Z and Z^ℓ to the sequences $(Z^k)_{k\geq 0}$ and $(Z^{\ell,k})_{k\geq 0}$, and necessarily $\mathcal{L}_\alpha(Z,Z^\ell) = \mathcal{L}(V,V^\ell)$.

We easily adapt the proofs of Proposition 3.3 and Theorem 3.6 to this situation in which the Poisson point processes are not quite the same. Using $E((V_t^\ell)^2) = E(V_t^2) = E(V_0^2)$,

$$\rho_T(P^{\beta_\ell}, P^\beta) \leq E\bigg(\sup_{0\leq s\leq T}(V_s^\ell - V_s)^2\bigg)$$

$$\leq (16c_\ell + 2c_\ell^2 T)E\bigg(\int_0^T (V_s^\ell - V_s)^2\,ds\bigg) + (16\delta_\ell + 2\delta_\ell^2 T)TE(V_0^2)$$

and an iteration gives the bound in the theorem. □

Corollary 4.2 *Assume $P_0 \in \mathcal{P}_2(\mathbb{R})$ has a density f_0, and $\int f_0|\log f_0| < \infty$. Then the solution P^β to the nonlinear martingale problem (1.8) is such that for any $t \geq 0$, $P_t^\beta(dv) = f^\beta(t,v)\,dv$ where $f^\beta(t,v) \in L^\infty([0,\infty[_t; L^2(\mathbb{R}_v))$ is the weak-sense solution of the Kac equation (1.1) obtained in Theorem 1.1.*

Proof. We consider the solutions $P_t^{\beta_\ell}$ to the nonlinear martingale problem with cutoff cross sections $\beta_\ell = \beta\wedge\ell$. Theorem 2.1 implies that $P_t^{\beta_\ell} = f^{\beta_\ell}(t,v)\,dv$, and it is shown in the proof of Theorem 2.1 of Desvillettes [4] that there is a subsequence of $(f^{\beta_\ell})_{\ell\geq 0}$ converging to a function f^β in $L^\infty([0,\infty[_t, L^1(\mathbb{R}_v))$ weak *. Since $\lim_{\ell\to\infty}\sup_{0\leq t\leq T}\rho(P_t^{\beta_\ell}, P_t^\beta) = 0$ by Theorem 4.1, necessarily $P_t^\beta(dv) = f^\beta(t,v)\,dv$. □

Remark. In a forthcoming paper [8], we use the Malliavin calculus to obtain the existence of a density $f^\beta(t,\cdot)$ for P_t^β for any $t > 0$, assuming only that the initial datum is a nonnegative finite measure with a second moment.

4.2 Convergence estimates for particle systems

We consider here a cross section β satisfying $\beta(x) \leq C_1|x|^{-\alpha}$ for some $C_1 > 0$ and $\alpha \in]1,3[$, and its cutoff approximation $\beta_\ell(\theta) = \beta(\theta)\mathbf{1}_{\frac{1}{\ell}\leq|\theta|}$. Then $\beta_\ell \in L^1([0,\pi[)$ and

$$\|\beta_\ell\|_1 = \int_{-\pi}^{+\pi} \beta_\ell(\theta)d\theta \leq \frac{2C_1}{\alpha-1}(\ell^{\alpha-1} - \pi^{1-\alpha}) .$$

To every function β^ℓ, we can associate a particle system $(V^{\beta_\ell,n})$ as defined in Section 2. Since the metric ρ is not directly comparable to the variation metric, we introduce the weaker metric

$$\tilde\rho(p,q) = \inf\Big\{\Big(\int_{\mathbb{R}\times\mathbb{R}}((x-y)^2\wedge 1)\, r(dx,dy)\Big)^{1/2} : r \text{ has marginals } p \text{ and } q\Big\}$$

on $\mathcal{P}_2(\mathbb{R})$, and a similar metric $\tilde\rho_T$ on $\mathcal{P}_2(\mathbb{D}_T)$.

Theorem 4.3 *Let β be a cross section such that $\beta(x) \leq C_1|x|^{-\alpha}$ for some $C_1 > 0$ and $\alpha \in]1,3[$, and $\ell(n)$ be a sequence of integers going to $+\infty$ in such a way that $\exp(\frac{2C_1}{\alpha-1}\ell(n)^{\alpha-1}T) = o(n)$. Let $(V_0^{\beta_{\ell(n)},in})_{1\leq i\leq n}$ be i.i.d. with a second order law P_0.*
1) For every $k \in \mathbb{N}^*$, the sequence $\mathcal{L}(V^{\beta_{\ell(n)},1n},\dots,V^{\beta_{\ell(n)},kn})$ converges to $(P^\beta)^{\otimes k}$, where P^β is the unique solution of the nonlinear martingale problem with initial datum P_0 obtained in Theorem 3.8.

Moreover we have the convergence estimate

$$\sup_{0\leq t\leq T}\tilde\rho(\mathcal{L}(V_t^{\beta_{\ell(n)},kn}),P_t^\beta) \leq \tilde\rho_T(\mathcal{L}(V^{\beta_{\ell(n)},kn}),P^\beta)$$
$$\leq K\bigg(\frac{\exp(\frac{2C_1}{\alpha-1}\ell(n)^{\alpha-1}T)}{n} + (16\delta_{\ell(n)}T + 2\delta_{\ell(n)}^2T^2)\exp(16bT+2b^2T^2)\langle v^2, P_0(dv)\rangle\bigg),$$

where $\delta_\ell \leq 2C_1\int_0^{1/\ell}(1-\cos\theta)\theta^{-\alpha}d\theta$ tends to zero when ℓ tends to infinity since $\alpha \in]1,3[$.
2) The empirical measures $\mu^{\beta_{\ell(n)}}$ defined in Theorem 2.2 converge in probability to P^β in $\mathcal{P}(\mathbb{D}_T)$.

Proof. We simply associate Theorems 2.2 and 4.1. □

4.3 The simulation algorithms

We deduce from the above study two algorithms associated respectively with the simple mean-field interacting particle system and the binary mean-field interacting particle system. The description of the algorithms is the same in both cases, since the theoretical justification is unified for the two systems.

As seen previously, the empirical measures $\mu^{\beta_{l(n)},n}$ approximate the law of the Kac process whose marginal at time t is equal to the solution $f(t,.)$ of the Kac equation.

We simulate the particle system of size n. The total jump rate is $n\|\beta_{\ell(n)}\|_1$ for (2.4) and $n\|\beta_{\ell(n)}\|_1/2$ for (2.5). A Poisson process of same rate gives the sequence of collision times, at each of which we choose uniformly among the $n(n-1)/2$ possibilities the pair of particles which collide. We then choose the impact parameter θ according to $\beta_{\ell(n)}(\theta)\,d\theta/\|\beta_{\ell(n)}\|_1$, and in the simple mean-field particle system we only update the velocity of one of the colliding particles, while in the binary one we update both. This simulation is exact if we simulate exactly the exponential variables related to the Poisson process, instead of discretizing time. See Graham and Méléard [7] for more details.

5 Numerical results

In Subsection 4.2, a criterion on the function $n \to \ell(n)$ was established, in order to ensure the convergence of the algorithms described in Subsection 4.3 when $n \to +\infty$ towards the solution of the non cutoff Kac equation. In this last part, we study how to choose, in practice, the dependence of ℓ with respect to n, in order to optimize the computations.

We select a typical solution of the non cutoff Kac equation (1.1). We choose

$$\beta(\theta) = |\sin\theta|^{-2}\,\mathbf{1}_{\{\theta\in[-\pi/2,\pi/2]\}}\,(2\,\pi)^{-1}$$

as a typical non cutoff cross section. Note that it is not integrable and does not have a first moment.

We also choose the initial datum

$$f_0(v) = \mathbf{1}_{\{v\in[-1/2,1/2]\}},$$

because its particle discretization is extremely simple.

The corresponding solution of Kac equation is denoted by $f(t,v)$.

We also introduce for $\ell > 1$ (as in Subsection 4.2) the cutoff cross section $\beta_\ell(\theta) = \beta(\theta)\,\mathbf{1}_{\{|\theta|\geq 1/\ell\}}$, and the corresponding solution $f^\ell(t,v)$ of the cutoff Kac equation (with the same initial datum).

The mass and energy of f as well as f^ℓ are independent of t and given by

$$a_0^f(t) = a_0^{f^\ell}(t) = \int_{\mathbb{R}} f(t,v)\,dv = 1\,, \qquad \frac{a_2^f(t)}{2} = \frac{a_2^{f^\ell}(t)}{2} = \int_{\mathbb{R}} f(t,v)\,\frac{|v|^2}{2}\,dv = \frac{1}{24}\,.$$

Therefore, f and f^ℓ have the same (Gaussian) limit when t tends to infinity, given by

$$\lim_{t\to+\infty} f(t,v) = \lim_{t\to+\infty} f^\ell(t,v) = \sqrt{\frac{6}{\pi}}\,e^{-6\,|v|^2}.$$

The fact that f and f^ℓ are identical at times 0 and $+\infty$ makes it difficult to choose a time t_0 where it is interesting to compare $f(t_0,\cdot)$ and $f^\ell(t_0,\cdot)$, that is, a time t_0 such that $||f(t_0,\cdot) - f^\ell(t_0,\cdot)||$ is of the same order of magnitude as $\sup_{t\in\mathbb{R}} ||f(t,\cdot) - f^\ell(t,\cdot)||$, for some reasonable norm $||\ ||$. In our case, after an empirical study, we choose $t_0 = 1.8$.

For the initial datum chosen here, the only known explicitly computable quantities (depending on f or f^ℓ) for an arbitrary time t are the moments of order $2N$, where $N \in \mathbb{N}$ (cf. [20]), that is

$$a^f_{2N}(t) = \int_{\mathbb{R}} f(t,v)\,|v|^{2N}\,dv, \qquad a^{f^\ell}_{2N}(t,v) = \int_{\mathbb{R}} f^\ell(t,v)\,|v|^{2N}\,dv.$$

But a^f_0 and a^f_2 (as well as $a^{f^\ell}_0$ and $a^{f^\ell}_2$) are independent of t, so that the first moment which is explicitly computable and really depending on time is $a^f_4(t)$ (and $a^{f^\ell}_4(t)$). The formulas are the following

$$a^f_4(t) = \frac{1}{48}\,(1 - e^{-t/2}) + \frac{1}{80}\,e^{-t/2}, \qquad a^{f^\ell}_4(t) = \frac{1}{48}\,(1 - e^{-R_\ell\,t/2}) + \frac{1}{80}\,e^{-R_\ell\,t/2}, \tag{5.8}$$

where

$$R_\ell = 1 - \frac{1}{2\,\pi\,\ell} - \frac{1}{4\,\pi}\,\sin(\frac{2}{\ell}). \tag{5.9}$$

We shall compare in the sequel the theoretical values of $a^f_4(t_0), a^{f^\ell}_4(t_0)$ (given by eq. (5.8), (5.9)) with the values obtained by the Nanbu (that is, simple mean-field) algorithm described in subsections 2.2 and 4.3.

The initial datum is discretized under the form

$$f_0(v) := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\frac{i}{n}-\frac{1}{2}}(v),$$

and the Poisson process corresponding to the Nanbu algorithm is implemented in the way described in Subsection 4.3: at each iteration, two particles are selected randomly (with a uniform law), an exponential time is added to a time counter, and the velocity of only one particle is changed (except if the time counter becomes bigger than t_0), according to the usual rule of collisions (i.-e. eq. (1.3)). The angle θ used in this collision is taken randomly according to the cutoff cross section β_ℓ.

We then get a discretized version of f^ℓ, denoted by

$$\tilde{f}^{\ell,n}(t_0,v) = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{v_i(t_0)}(v),$$

and the corresponding fourth moment is computed by the formula

$$a^{\tilde{f}^{\ell,n}}_4(t_0) = \frac{1}{n}\sum_{i=0}^{n-1} v_i(t_0)^4.$$

We are now interested in the behavior of the quantity

$$|a^{\tilde{f}^{\ell,n}}_4(t_0) - a^f_4(t_0)|$$

when ℓ and n vary.

More precisely, we choose to estimate how ℓ and n have to be related in order to give an error of discretization and an error due to the cutoff which are the same.

It means that we try to find the quantity $\ell(n)$ (when n varies) such that

$$| < a_4^{\tilde{f}^{\ell(n),n}}(t_0) > - a_4^{f^{\ell(n)}}(t_0)| = |a_4^{f^{\ell(n)}}(t_0) - a_4^{f}(t_0)|. \tag{5.10}$$

In this equality, the right-hand side quantity is explicitly computable thanks to eq. (5.8), (5.9), and the notation $< \cdots >$ means the mean value"over all possible experiments".

Of course, in order to estimate the quantity $< a_4^{\tilde{f}^{\ell,n}}(t_0) >$ (for a given ℓ, n), we can carry out only a finite number of numerical experiments.

Therefore, for each n, we choose a number $m(n)$ of simulations, made each time with a different set of random numbers. The corresponding mean value is denoted by $< a_4^{\tilde{f}^{\ell,n}}(t_0) >_{m(n)}$, and replaces $< a_4^{\tilde{f}^{\ell,n}}(t_0) >$ when we try to estimate $\ell(n)$ in such a way that (5.10) holds. The number $m(n)$ is chosen as large as possible. It is limited by the speed of the computer.

In order to find $\ell(n)$, we use a fixed point method, (this is easy because the dependence with respect to ℓ of the values of $| < a_4^{\tilde{f}^{\ell,n}}(t_0) >_{m(n)} - a_4^{f^{\ell}}(t_0)|$ is almost undetectable as soon as ℓ is confined in a "reasonable" interval).

In this process, we can also compute a confidence interval $[\ell^+(n), \ell^-(n)]$, in which $\ell(n)$ lies with a "large" probability.

We now present the numerical results. For each n belonging to a geometric progression, we give $m(n)$, and the computed quantities $\ell^+(n)$, $\ell(n)$, and $\ell^-(n)$.

n	m(n)	$\ell_1^+(n)$	$\ell_1(n)$	$\ell_1^-(n)$
125	5E5	1.6085	1.6135	1.6185
250	5E5	1.954	1.958	1.962
500	2E5	2.420	2.426	2.431
1E3	1E5	3.0530	3.0595	3.0645
2E3	5E4	3.921	3.931	3.940
4E3	5E4	5.125	5.135	5.150
8E3	5E4	6.79	6.81	6.83
16E3	2E4	9.11	9.15	9.19
32E3	2E4	12.380	12.485	12.550
64E3	1E4	17.10	17.15	17.20
128E3	1E3	23.12	23.50	24.00
256E3	600	32.6	33.6	34.4
512E3	400	45.0	45.5	46.0
1024E3	300	64.0	65.0	66.0
2048E3	100	90.0	92.5	95.0

Table 1

We now display curves made with Table 1. In Figure 1, $\ell^+(n)$, $\ell(n)$, and $\ell^-(n)$ are represented as functions of n. In Figure 2, they are represented in a log/log scale. The dashed lines correspond to $\ell^+(n)$ and $\ell^-(n)$, while the continuous lines are related to $\ell(n)$.

Fig. 1 clearly shows a concave curve, which is in accord with the guess that $\ell(n)$ should increase less rapidly than n. Remember that in Subsection 4.2, a sufficient condition of convergence of the method was that (up to different constants) $\exp(\ell(n)) = o(n)$ ($\alpha = 2$ in our example).

However, we can see on Fig. 2 that the curve giving $\ell(n)$ with respect to n is convex when represented on a log/log scale (and in fact almost a straight line). Therefore, a good approximation for $\ell(n)$ seems to be some power n^k, for $k \in]0,1[$. This means of course that the condition $\exp(\ell(n)) = o(n)$ is not at all fulfilled, and suggests that Theorem 4.2 is far from optimal.

Of course our numerical study is limited and one should not draw hasty conclusions from it. We think however that in practice, a choice of $\ell(n)$ as a power of n might not be so bad.

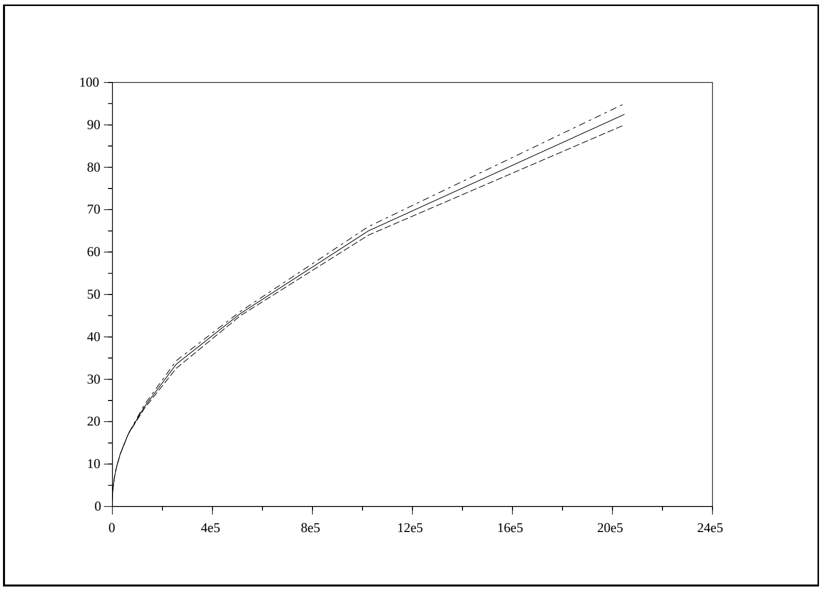

Figure 1: $\ell(n)$ as function of n

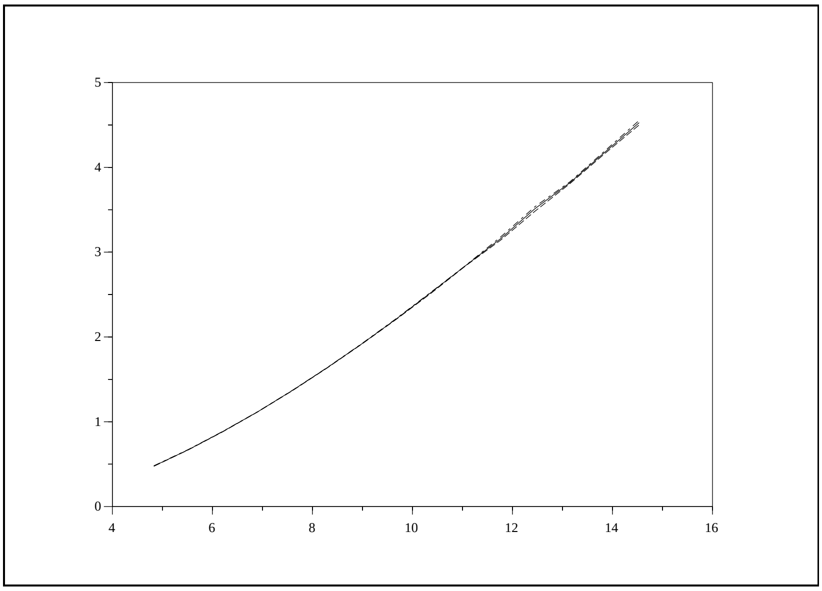

Figure 2: $\ell(n)$ as function of n in log/log scale

References

[1] ALEXANDRE, R.: Une définition des solutions rénormalisées pour l'équation de Boltzmann sans troncature, Comptes Rendus Acad. Sciences (1999).

[2] BABOVSKY, H.; ILLNER, R.: A convergence proof for Nanbu's simulation method for the full Boltzmann equation, SIAM J. Num. Anal. 26, 46–65 (1989).

[3] CERCIGNANI, C.; ILLNER, R.; PULVIRENTI, M.: *The mathematical theory of dilute gases*, Applied Math. Sciences, Springer-Verlag, Berlin (1994).

[4] DESVILLETTES, L.: About the regularizing properties of the non cutoff Kac equation. Comm. Math. Physics 168, 416–440 (1995).

[5] DIPERNA, R.J.; LIONS, P-L.: On the Cauchy problem for Boltzmann equations, global existence and weak stability, Ann. Math. 130, 321–366 (1989).

[6] GRAHAM, C.; MÉLÉARD, S.: Chaos hypothesis for a system interacting through shared resources, Prob. Th. Rel. Fields 100, 157–173 (1993).

[7] GRAHAM, C.; MÉLÉARD, S.: Stochastic particle approximations for generalized Boltzmann models and convergence estimates, Ann. Prob. 25, 115–132 (1997).

[8] GRAHAM, C.; MÉLÉARD, S.: Existence and regularity of a solution to a Kac equation without cutoff using Malliavin calculus, (preprint 1998).

[9] FOURNIER, N.: Existence and regularity study for a 2-dimensional Kac equation without cutoff by a probabilistic approach, Prépublication Laboratoire Probabilités Paris VI, 468 (1998)

[10] IKEDA, N.; WATANABE, S.: *Stochastic differential equations and diffusion processes*, North-Holland, Amsterdam, (1981).

[11] JACOD, J.; SHIRYAEV, A.N.: *Limit theorems for stochastic processes*, Springer-Verlag, Berlin, (1987).

[12] MÉLÉARD, S.: Stochastic approximations of the solution of a full Boltzmann equation for small initial data, to appear in ESAIM P&S (1998).

[13] PARESCHI, L.; TOSCANI, G.; VILLANI, C.: Spectral methods for the non cut-off Boltzmann equation and numerical grazing collision limit, Preprint (1998).

[14] PULVIRENTI, A.; TOSCANI, G.: The theory of the nonlinear Boltzmann equation for Maxwell molecules in Fourier representation, Ann. Math. Pura ed Appl. 171, 181–204 (1996).

[15] SHIGA, T.; TANAKA, H.: Central limit theorem for a system of Markovian particules with mean-field interactions, Z. Wahrsch. Verw. Geb. 69, 439–459 (1985).

[16] SZNITMAN, A.S.: Équations de type de Boltzmann, spatialement homogènes, Z. Wahrsch. Verw. Geb. 66, 559–592 (1984).

[17] TANAKA, H.: On the uniqueness of Markov process associated with the Boltzmann equation of Maxwellian molecules, Proc. Intern. Symp. SDE, Kyoto, 409–425 (1976).

[18] TANAKA, H.: Probabilistic treatment of the Boltzmann equation of Maxwellian molecules, Z. Wahrsch. Verw. Geb. 46, 67–105 (1978).

[19] TOSCANI, G.; VILLANI, C.: Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas, to appear in Journal Stat. Phys. (1998).

[20] TRUESDELL, C.: On the pressure and the flux of energy in a gas according to Maxwell's kinetic theory II, J. Rat. Mech. Anal. 5, 55 (1980).

[21] UKAI, S.: Local solutions in Gevrey classes to the nonlinear Boltzmann equation without cutoff, Japan J. Appl. Math., 1, 141–156, (1984).