NON MARKOVIANITY OF THE BOLTZMANN-GRAD LIMIT OF A SYSTEM OF RANDOM OBSTACLES IN A GIVEN FORCE FIELD

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Abstract. In this paper we consider a particle moving in a random distribution of obstacles. Each obstacle is absorbing and a fixed force field is imposed. We show rigorously that certain (very smooth) fields prevent the process obtained by the Boltzmann-Grad limit from being Markovian. Then, we propose a slightly different setting which allows this difficulty to be removed.

Abstract. On considère dans ce travail une particule qui se déplace à travers une distribution aléatoire d'obstacles. Chaque obstacle est absorbant, et un champ de forces fixe est imposé. On montre rigoureusement que certains champs (très réguliers) empêchent le processus obtenu par la limite de Boltzmann-Grad d'être Markovien. Ensuite, on décrit une situation légèrement différente dans laquelle la difficulté précédente ne peut apparaître.

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1. Introduction

In this paper, we investigate the rigorous derivation of linear kinetic transport equations starting from the basic particle dynamics in a random context.

The first result in this direction was obtained many years ago by G. Gallavotti, who showed how to derive the linear Boltzmann equation (with hard–sphere cross section) starting from the dynamics of a single particle in a random distribution of fixed hard scatterers in the so–called Boltzmann–Grad limit. This paper (Cf.[G]), published in [G1] and unfortunately not widely known, is technically simple but has a deep content. In particular it is proven there for the first time that it is perfectly consistent to obtain an irreversible stochastic behavior as a limit of a sequence of deterministic Hamiltonian systems (in a random medium). Later on this result was improved (see [S1], [S2] and

[BBuS]). More recently, the Boltzmann–Grad limit in the case when the distribution of scatterers is periodic (and not random) has also been considered in [BoGoW] (see also the references therein). Note that in this case, the result is totally different.

It is sometimes assumed that a given force field does not change anything in the derivation of the (linear) Boltzmann equation. However, it was noticed (at the formal level) by Bobylev, Hansen, Piasecki and Hauge (Cf. [Bob]), and verified (at the numerical level) by Kuzmany and Spohn (Cf. [Spo]) that charged particles in a constant magnetic field give rise to a non Markovian behavior.

We wish here to analyse rigorously such a behavior (though for a given force coming out of a smooth potential rather than for a constant magnetic field) and to prove the convergence of the system (taking into account only absorbing obstacles) towards the solution of an equation which is not the standard linear Vlasov-Boltzmann equation, but an equation with coefficients depending on time (this equation is close to that obtained in the setting of [Bob], that is when the force field is a magnetic field, and when the obstacles are not absorbing but instead give rise to a rebound of the particle).

Then, we propose a setting in which the difficulty disappears, so that the usual Boltzmann-Grad limit holds. Namely, we consider obstacles which are not fixed, but which move along straight lines with a random velocity.

In the first part of our article, we assume that the scatterers are distributed according to a Poisson law with parameter $\mu_\varepsilon = \mu\,\varepsilon^{-1}$ on $\mathbb{R}^2$ (the case of $\mathbb{R}^3$ can be treated similarly), and are comprised of balls of radius ε. More precisely, a given scatterer localized in $c(\in \mathbb{R}^d)$ is assumed to be absorbing (that is, our test particle disappears when it enters the obstacle).

The probability distribution of finding exactly N obstacles in a bounded measurable set $\Lambda \subset \mathbb{R}^2$ is given by:

$$P(d\mathbf{c}_N) = e^{-\mu_\varepsilon|\Lambda|}\frac{\mu_\varepsilon^N}{N!}dc_1 \dots dc_N, \tag{1}$$

where $c_1 \dots c_N = \mathbf{c}_N$ are the positions of the scatterers and $|\Lambda|$ denotes the Lebesgue measure of Λ.

The expectation with respect to the Poisson repartition of parameter μ_ε will be denoted by $\mathbb{E}^\varepsilon$.

We consider a fixed force $F(t,x)$ acting on the test particle, so that the equation of motion of this particle (having initial position x and initial velocity v) is given by

$$\frac{d}{dt}(T_1^t(x,v)) = T_2^t(x,v), \quad \frac{d}{dt}(T_2^t(x,v)) = F(t, T_1^t(x,v)), \tag{2}$$

up to the first time $\tau_{\mathbf{c}}(x,v)$ when the particle enters an obstacle.

For a given initial datum $f_{in} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, we can define the quantity

$$f_\varepsilon(t,x,v) = \mathbb{E}^\varepsilon[f_{in}(T^{-t}(x,v))\, 1_{\{t \le \tau_{\mathbf{c}}(x,v)\}}]. \tag{3}$$

Then, our first theorem is the following :

Theorem 1: Let $\mathbf{c}$ be given by a Poisson's repartition of parameter $\mu_\varepsilon = \mu\, \varepsilon^{-1}$ (on $\mathbb{R}^2$) and $F \equiv F(t,x)$ be a given force in $C(\mathbb{R}; W^{1,+\infty}(\mathbb{R}^2))$ (that is, globally Lipschitz in x, locally uniformly in t). We denote by T^t the flow defined (for $t \in \mathbb{R}$) by (2). We suppose moreover that F is such that for a.e. initial data $(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2$, the velocity never reaches 0 (in other words, $T_2^s(x,v) \neq 0$ for $s \in \mathbb{R}$). Then (for a given $f_{in} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$), the quantity f_ε defined by (3) converges (when $\varepsilon \to 0$) in $L^1([0,T] \times \mathbb{R}^2 \times \mathbb{R}^2)$ for all $T > 0$ towards the (unique) solution f in $L^1([0,T] \times \mathbb{R}^2 \times \mathbb{R}^2)$ of the equation

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = -\, 2\, \mu\, |v|\, f\, 1_{\{x \neq T_1^{-s}(x,v), s \in]0,t[\}}. \tag{4}$$

together with the initial condition

$$f(0,x,v) = f_{in}(x,v).$$

Remarks:

1. Equation (4) is at variance with the expected equation

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = -2\, \mu\, |v|\, f. \tag{5}$$

as soon as the trajectories (in the space of x only) of the ODE (2) cross themselves (for a set of times of strictly positive measure) for a non zero measure set of initial data. This happens for very smooth forces (which do not even depend on t), for example for the harmonic oscillator $F(t,x) = -x$, when $t \ge \pi/2$. This phenomenon also appears for forces depending on the velocity of the particles, such as the Lorentz force : this is exactly the case studied in [Bob].

2.. The assumption that F is globally Lipschitz is used only to ensure that the flow T^t is well-defined for all t (it could be replaced by any locally Lipschitz force provided that one studies the solution for times t such that T^t is well-defined). The assumption that for a.e. v, $T_2^s(x,v) \neq 0$ for $s \in \mathbb{R}$ is generic (and is satisfied by the harmonic oscillator for example). It can be relaxed somehow (for example, one could allow a finite number of points where $T_2^s(x,v) = 0$ if, at those points, the derivative of the velocity is not 0). It seems however very difficult to completely remove these kinds of assumptions (one could imagine very singular trajectories, with many points where $T_2^s(x,v)$ and many (or all of) its derivatives are 0).

We now turn to a way of recovering the "right" equation, that is an equation describing a Markovian process, at the end of the Boltzmann-Grad asymptotics. We introduce a new configuration of obstacles, which are no longer at rest. Their initial position $\mathbf{c}$ is still given by the Poisson law with parameter $\mu_\varepsilon = \mu\,\varepsilon^{-1}$, but they also move with a (fixed) velocity $\mathbf{w} = (w_1, .., w_N)$ which is distributed according to a centered Gaussian law with variance 1. The velocities of the obstacles are independent from each other and independent of $\mathbf{c}$.

The expectation with respect to the measure we just described will be denoted by $\mathbb{E}^{\varepsilon'}$.

We still consider the force $F(t,x)$, the test particle obeying eq. (2), and the condition of absorption (together with the definition of τ, which now also depends on w) to be maintained.

For a given initial datum $g_{in} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, we define the quantity

$$g_\varepsilon(t,x,v) = \mathbb{E}^{\varepsilon'}[g_{in}(T^{-t}(x,v))\, 1_{\{t \le \tau_{\mathbf{c},\mathbf{w}}(x,v)\}}]. \tag{6}$$

We now state our second theorem :

Theorem 2: Let $\mathbf{c}, \mathbf{w}$ be given by a repartition as described above (that is, Poisson with parameter $\mu_\varepsilon = \mu\,\varepsilon^{-1}$ for $\mathbf{c}$, and centered Gaussian with variance 1 for $\mathbf{w}$, with independence of $\mathbf{c}$ and $\mathbf{w}$), and $F \equiv F(t,x)$ be a given force in $C(\mathbb{R}; W^{1,+\infty}(\mathbb{R}^2))$ (that is, globally Lipschitz in x, locally uniformly in t). Then (for a given $g_{in} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$), the quantity g_ε defined by (6) converges (when $\varepsilon \to 0$) in $L^1([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$ for all $T > 0$ towards the (unique) solution g in $L^1([0,T] \times \mathbb{R}^2 \times \mathbb{R}^2)$ of the equation

$$\partial_t g + v \cdot \nabla_x g + F \cdot \nabla_v g = -\,2\,\mu\, g \int_{w\in\mathbb{R}^2} |v-w|\, \frac{e^{-\frac{|w|^2}{2}}}{2\pi}\, dw \tag{7}$$

together with the initial condition

$$g(0,x,v) = g_{in}(x,v).$$

Remarks :

1. This theorem gives a way of finding the "right" equation as a Boltzmann-Grad limit. There are certainly many other ways of doing so (for example considering another reasonable distribution of velocities for the scatterers, or letting the scatterers vibrate around an equilibrium position). The idea consists in adding some extra randomness to the system.

2. Though we treat here only the simplest case (absorption by the obstacles), we believe that a similar behavior arises when a more general interaction between the test particle and the obstacles is considered. That is, the nonmarkovian behavior which results in the Boltzmann-Grad limit in the presence of self crossings of trajectories (which of course still appears in this case), can be cured by the same addition of randomness.

3. Note that in this theorem, no assumption on F (or on the flow T^t) is made, apart from the smoothness assumption (F Lipschitz) which allows the flow to be defined. This point is significant since in more complicated contexts, one might only have very little information about F.

The remainder of this paper is organized as follows : we first prove theorem 1 in section 2, and then theorem 2 in section 3.

2. Proof of theorem 1

We first write down the series giving the explicit value of f_ε. For this purpose, we first observe that thanks to the assumption that F is globally Lipschitz, the trajectory $T_1^{-t}(x,v)$ (for $t \in [0,T]$) of the test particle is included in some ball $B(0,R(T))$ (depending on x,v). Then we can write the explicit formula (for $t \in [0,T]$) :

$$f_\varepsilon(t,x,v) = \sum_{N\geq 0} e^{-\mu_\varepsilon\,|B(0,R(T))|}\frac{\mu_\varepsilon^N}{N!}\int_{c_1\in B(0,R(T))}..\int_{c_N\in B(0,R(T))} f_{in}(T^{-t}(x,v))$$

$$\times\, 1_{\{T_1^{-s}(x,v)\notin B(c_i,\varepsilon),s\in[0,t],i=1..N\}}\,d\mathbf{c}. \tag{8}$$

Then, denoting by

$$\theta_\varepsilon(t,x,v) = \{y\in\mathbb{R}^2, \exists s\in[0,t], |y-T_1^{-s}(x,v)|\leq\varepsilon\} \tag{9}$$

the tube of width ε around the trajectory (in the space of x), and noticing that this does not depend on the configuration of obstacles, we see that

$$f_\varepsilon(t,x,v) = e^{-\mu_\varepsilon\,|\theta_\varepsilon(t,x,v)|}\, f_{in}(T^{-t}(x,v)). \tag{10}$$

Therefore, in order to get theorem 1, and thanks to Lebesgue's dominated convergence theorem, it is sufficient to prove the following lemma :

Lemma 1: Under the assumptions of theorem 1, for all $t \in [0,T]$ and a.e. x, v, the volume of the tube $\theta_\varepsilon(t,x,v)$ satisfies the following asymptotic property :
(11)

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} |\theta_\varepsilon(t,x,v)| = 2 \int_0^t |T_2^{-s}(x,v)| \, 1_{\{T_1^{-s}(x,v) \notin \cup_{\sigma\in[0,s[} \{T_1^{-\sigma}(x,v)\}\}} ds.$$

Proof of lemma 1 : We consider only those x and v such that the velocity $T_2^{-s}(x,v)$ does not go to 0 between times 0 and t. Note that by assumption, the (x,v) which do not satisfy this condition belong to a set of measure 0. For trajectories with such initial data, it is possible to define by $\nu(-u)$ and $R(-u)$ resp. the normal vector to the trajectory and its (signed) radius of curvature at the point $T_1^{-u}(x,v)$.

Thanks to our assumptions on F, for $u \in [0,t]$, the modulus of the velocity $|T_2^{-u}(x,v)|$ is bounded between v_{min} and v_{max}. Since (still thanks to our assumptions on F) an upper bound is also available for the derivative of the velocity, we can find a strictly positive lower bound (called R_{min}) for the (absolute value of the) radius of curvature $|R(-u)|$.

We only consider in the sequel ε such that $0 < \varepsilon < R_{min}/2$. We define the following change of variable (remember that t, x, v is given)

$$\begin{aligned} \zeta : \quad [0,t] \times [-\varepsilon, \varepsilon] \quad &\longrightarrow \quad \mathbb{R}^2 \\ (s,z) \quad &\mapsto \quad \zeta(s,z) = \int_0^s T_2^{-h}(x,v)dh + \nu(-s)\, z. \end{aligned} \tag{12}$$

Though ζ is not necessarily globally one-to-one (because of the self-crossings of the trajectory in the space of x), we know at least that for any given s_0, it is indeed one-to-one for s such that $|s - s_0| < 2\pi\,(R_{min} - \varepsilon)/v_{max}$. Its jacobian determinant is

$$J(s,z) = |T_2^{-s}(x,v)|(1 - \frac{z}{R(-s)}).$$

We consider the set of times for which a self-crossing occurs and denote it by

$$B = \left\{ s \in [0,t] : T_1^{-s}(x,v) \in \cup_{\sigma\in[0,s[} \{T_1^{-\sigma}(x,v)\} \right\}.$$

We then bound the $\mathbb{R}^2$-measure of the flow tube from above.

Using the change of variables ζ, we see that :

$$|\theta_\varepsilon(t,x,v)| \le \int_{s\in B^c}\int_{z=-\varepsilon}^{\varepsilon} |T_2^{-s}(x,v)|(1-\frac{z}{R(-s)})\, dsdz + \pi\,\varepsilon^2$$
$$\le 2\varepsilon\,(1+\varepsilon/R_{min})\int_{s\in B^c} |T_2^{-s}(x,v)|\, ds + \pi\,\varepsilon^2,$$

so that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1}\,|\theta_\varepsilon(t,x,v)| \le 2\int_0^t |T_2^{-s}(x,v)|\, 1_{\{s\in B^c\}}\, ds.$$

Note that the extremities of the trajectory need some special attention, since the corresponding part of the tube is not in the image of ζ. This explains where the term $\pi\,\varepsilon^2$ comes from in the above computation.

Let us now turn to the proof of a lower bound. This is slightly more intricate since we have to take into account the points where our change of variable is in fact not one-to-one (typically, for ε small enough, those are points close to some self-crossing of the trajectory in the x space).

We first define (for any $\delta > 0$) the constant

$$K_\delta = \inf_{0\le s_1<s_2\le t;|s_1-s_2|\ge\pi\, R_{min}/v_{max};d(s_2,B)\ge\delta} |T_1^{-s_1}(x,v) - T_1^{-s_2}(x,v)|.$$

Note that $K_\delta > 0$ because of the definition of B. Taking now $\varepsilon < K_\delta$ (and still $\varepsilon < R_{min}/2$), we can use the change of variable ζ and write the lower bound

$$|\theta_\varepsilon(t,x,v)| \ge$$
$$\int_{\{s\in[0,t]:\ d(s,B)\ge\delta\}}\int_{z=-\varepsilon}^{\varepsilon} |T_2^{-s}(x,v)|(1-\frac{z}{R(-s)}) dsdz$$
$$\ge 2\,\varepsilon\,(1-\varepsilon/R_{min})\int_{\{s\in[0,t]:\ d(s,B)\ge\delta\}} |T_2^{-s}(x,v)| ds,$$

so that

$$\liminf_{\varepsilon\to 0} \varepsilon^{-1}\,|\theta_\varepsilon(t,x,v)| \ge 2\int_{\{s\in[0,t]:\ d(s,B)\ge\delta\}} |T_2^{-s}(x,v)|\, ds.$$

We conclude by letting δ go to 0, thanks to Lebesgue's dominated convergence theorem. Since for all $s\in[0,t]$, $1_{\{s\in[0,t]:\ d(s,B)\ge\delta\}}$ converges to $1_{\bar{B}^c}$, it is sufficient to prove that B is a closed set of $[0,t]$. Indeed, this is a consequence of the fact that the (absolute value of the) radius of curvature is bounded below, which prevents self-crossings at points corresponding to times which are close.

This ends the proof of the lemma.

3. Proof of theorem 2

Once again, we write down the series giving the explicit value of g_ε. Note however that since there is no bound on the velocity of the obstacles, we can't estimate a priori the set (in the x space) of the positions (at time 0) of the scatterers met later (before time T) by the test particle.

Then, we use the (less explicit) formula :

$$g_\varepsilon(t,x,v) = \lim_{R\to+\infty} \sum_{N\geq 0} e^{-\mu_\varepsilon\,|B(0,R)|} \frac{\mu_\varepsilon^N}{N!}$$

$$\times \int_{c_1\in B(0,R)} .. \int_{c_N\in B(0,R)} \int_{w_1\in\mathbb{R}^2} .. \int_{w_N\in\mathbb{R}^2} g_{in}(T^{-t}(x,v))$$

$$\times\, 1_{\{T_1^{-s}(x,v)\notin B(c_i,\varepsilon), s\in[0,t], i=1..N\}}\, e^{-\frac{|\mathbf{w}|^2}{2}} \frac{d\mathbf{w}}{(2\pi)^N} d\mathbf{c}. \tag{13}$$

We now need to slightly modify our definition of the tube θ. We define for each $w \in \mathbb{R}^2$:

$$\theta'_\varepsilon(t,x,v,w) = \{y\in\mathbb{R}^2, \exists s\in[0,t], |y - T_1^{-s}(x,v) + w\,s| \leq \varepsilon\}. \tag{14}$$

Then,

$$g_\varepsilon(t,x,v) = \lim_{R\to+\infty} e^{-\mu_\varepsilon \int_{w\in\mathbb{R}^2} |\theta'_\varepsilon(t,x,v,w)|\, e^{-\frac{|w|^2}{2}} \frac{dw}{2\pi}}\, g_{in}(T^{-t}(x,v)). \tag{15}$$

Therefore, in order to get theorem 2, it is sufficient to prove the following lemma :

Lemma 2: The volume of the tube $\theta'_\varepsilon(t,x,v,w)$ satisfies the following asymptotic property : for all $(t,x,v)\in[0,T]\times\mathbb{R}^2\times\mathbb{R}^2$,

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} \int_{w\in\mathbb{R}^2} |\theta'_\varepsilon(t,x,v,w)|\, e^{-\frac{|w|^2}{2}} \frac{dw}{2\pi} \tag{16}$$

$$= 2\int_0^t \int_{w\in\mathbb{R}^2} |T_2^{-s}(x,v) - w|\, e^{-\frac{|w|^2}{2}} \frac{dw}{2\pi}\, ds.$$

Proof of lemma 2 : We consider a given $(t,x,v)\in[0,T]\times\mathbb{R}^2\times\mathbb{R}^2$. We prove that $\varepsilon^{-1}\,|\theta'_\varepsilon(t,x,v,w)|$ converges to $\int_0^t |T_2^{-s}(x,v) - w|\, ds$ for a.e. w. Then the convergence of the integral will be a consequence of Lebesgue's dominated convergence theorem.

We first notice that for a.e. $w\in\mathbb{R}^2$, the (translated) velocity $T_2^{-s}(x,v) - w$ is different from 0 for all s. This is due to the fact that $\{T_2^{-s}(x,v), s\in[0,t]\}$ is a Lipschitz curve of $\mathbb{R}^2$.

Then, we can apply the same technique as in lemma 1 and get the convergence of $\varepsilon^{-1}\,|\theta'_\varepsilon(t,x,v,w)|$ towards $2\int_{B_w^c}|T_2^{-s}(x,v)-w|\,ds$, where $B_w=\{s\in[0,t]:\exists\sigma<s, T_1^{-\sigma}(x,v)-w\,\sigma=T_1^{-s}(x,v)-w\,s\}$. As a consequence, it is sufficient to prove that for a.e. w, the set B_w is negligible.

In order to do so, we first note that the set

$$U=\left\{\left(s,\frac{T_1^{-s}(x,v)-T_1^{-\sigma}(x,v)}{s-\sigma}\right),0\le\sigma<s\le t\right\}$$

is a Lipschitz surface of a 3-dimensional space, so that its (3-dimensional) Lebesgue measure is 0. Thanks to Fubini's theorem, we know then that for a.e. $w\in\mathbb{R}^2$,

$$B_w=\left\{s\in[0,t]:\exists\sigma<s, w=\frac{T_1^{-s}(x,v)-T_1^{-\sigma}(x,v)}{s-\sigma}\right\}$$

is negligible (as a 1-dimensional space).

This concludes the proof of Lemma 2.

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