Entropy dissipation estimates for the Landau equation: General cross sections

Laurent Desvillettes

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1 Introduction and main result

1.1 Description of the Landau operator and equation

We are concerned here with the Landau operator appearing in plasma theory (cf. [11, 21]), defined by

$$Q_\psi(f,f)(v) = \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} a^\psi(v-w) \left(f(w)\,\nabla f(v) - f(v)\,\nabla f(w) \right) dw \right\}. \quad (1)$$

Here, $a^\psi := a^\psi(z) := (a^\psi_{ij}(z))_{ij}$ ($z \in \mathbb{R}^3$) is a (nonnegative symmetric) matrix-valued function with only one degenerate direction, namely that of z. More precisely,

$$a^\psi_{ij}(z) = \Pi_{ij}(z)\,\psi(|z|), \quad (2)$$

where ψ is a (scalar valued) nonnegative function, and

$$\Pi_{ij}(z) = \delta_{ij} - \frac{z_i z_j}{|z|^2} \quad (3)$$

is the i,j-component of the orthogonal projection Π onto $z^\perp := \{y \,/\, y \cdot z = 0\}$.

Laurent Desvillettes
Univ. Paris Diderot, Sorbonne Paris Cité, Institut de Mathématiques de Jussieu - Paris Rive Gauche, UMR 7586, CNRS, Sorbonne Universités, UPMC Univ. Paris 06, F-75013, Paris, France, e-mail: desvillettes@math.univ-paris-diderot.fr

We observe that at the formal level (that is, when both f and φ are smooth functions having a reasonable behavior at infinity), the (symmetric) weak version of the Landau operator can be defined by the following formula:

$$\int_{\mathbb{R}^3} Q_\psi(f,f)(v)\,\varphi(v)\,dv \tag{4}$$

$$= -\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(v)\,f(w)\,a^\psi(v-w)\left(\frac{\nabla f}{f}(v)-\frac{\nabla f}{f}(w)\right)$$

$$\Big(\nabla\varphi(v)-\nabla\varphi(w)\Big)\,dvdw,$$

where the symmetric matrix a^ψ acts as a bilinear form on two vectors.

Using the test functions $\varphi(v)=1$, v_i (for $i=1,..,3$), $\frac{|v|^2}{2}$, we see that (still at the formal level), the Landau operator conserves mass, momentum and kinetic energy, that is:

$$\int_{\mathbb{R}^3} Q(f,f)(v)\begin{pmatrix}1\\ v_i\\ |v|^2/2\end{pmatrix}dv = 0. \tag{5}$$

We also get (once again at the formal level) the formula for the entropy dissipation $D_\psi := D_\psi(f)$ (defined on functions f from $\mathbb{R}^3$ to $\mathbb{R}_+$) by considering $\varphi(v)=\ln f(v)$:

$$D_\psi(f) := -\int_{\mathbb{R}^3} Q_\psi(f,f)(v)\ln f(v)\,dv \tag{6}$$

$$= \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(v)\,f(w)\,\psi(|v-w|)\,\Pi(v-w)\left(\frac{\nabla f}{f}(v)-\frac{\nabla f}{f}(w)\right)$$

$$\left(\frac{\nabla f}{f}(v)-\frac{\nabla f}{f}(w)\right)dvdw \geq 0.$$

The most physically relevant function ψ appearing in operator (1), (2) is $\psi(z) = |z|^{-1}$. It corresponds to the case when f is the density of charged particles (moving according to Coulomb interaction) in a plasma, cf. [21]. It also naturally appears in the so-called weak coupling asymptotics of Boltzmann equation (cf. [6] and the older reference [7]).

It is however also interesting, at least from the mathematical viewpoint, to consider more general functions ψ. We refer for example to [13] to see how the Landau kernel with arbitrary ψ can be obtained from the Boltzmann kernel (with arbitrary cross section) through a scaling involving the concept of grazing collisions.

We shall use in this paper the terminology of [14], which is very close to that of [26]. When the dimension of the space is $N=3$ (we shall always make that assumption in the sequel), if ψ is given by a power law, we say that

$$\psi(|z|) = |z|^{\gamma+2} \tag{7}$$

is coming out of hard potentials when $\gamma \in]0,1]$, Maxwell molecules when $\gamma = 0$, moderately soft potentials when $\gamma \in [-2,0]$, and very soft potentials when $\gamma \in]-4,-2[$. We also shall call general soft potentials the case $\gamma < 0$ (that is, γ can be smaller than -4), and general hard potentials the case $\gamma \in]0,2[$ (that is, γ can be larger than 1). Note that the Coulomb case falls within the category of very soft potentials.

We now introduce the spatially homogeneous Landau equation

$$\partial_t f(t,v) = Q_\psi(f(t,\cdot), f(t,\cdot)). \tag{8}$$

with initial data

$$f(0,v) = f_{in}(v). \tag{9}$$

As a consequence of formula (5), the solutions of the Landau equation (8), (9) satisfy (at least formally) the conservation of mass, momentum and energy, that is

$$\int_{\mathbb{R}^3} f(t,v) \begin{pmatrix} 1 \\ v_i \\ |v|^2/2 \end{pmatrix} dv = \int_{\mathbb{R}^3} f_{in}(v) \begin{pmatrix} 1 \\ v_i \\ |v|^2/2 \end{pmatrix} dv. \tag{10}$$

They also satisfy (at the formal level) the entropy identity (first part of Boltzmann's H-theorem)

$$\frac{d}{dt} H(f(t,\cdot)) = -D_\psi(f(t,\cdot)) \leq 0, \tag{11}$$

where $H := H(f)$ is the entropy functional (defined on functions from $\mathbb{R}^3$ to $\mathbb{R}_+$):

$$H(f) := \int_{\mathbb{R}^3} f(v) \ln f(v)\, dv, \tag{12}$$

and D_ψ is the entropy dissipation functional defined in (6).

As stated in detail in [14], identities (10) and (11) naturally furnish an *a priori* estimate (when the initial data have a finite mass, energy and entropy): indeed

$$\sup_{t\in[0,T]} \int_{v\in\mathbb{R}^3} f(t,v) \left(1 + \frac{|v|^2}{2} + |\ln f(t,v)|\right) dv \tag{13}$$
$$+ \int_0^T D_\psi(f(t,\cdot))\, dt \leq C(T, \mathscr{M}_{in}),$$

where the constant $C(T,\mathscr{M}_{in})$ only depends on T and

$$\mathscr{M}_{in} = \int_{v\in\mathbb{R}^3} f_{in}(v) \left(1 + \frac{|v|^2}{2} + |\ln(f_{in}(v))|\right) dv.$$

As a consequence (remembering that $D_\psi(f)$ is a nonnegative quantity), any (nonnegative) lower bound of $D_\psi(f)$ will naturally yield an *a priori* estimate for the solutions of the Landau equation (when the initial data have a finite mass, energy and entropy).

One of the first such lower bounds appeared in [17] in the case when ψ is a so-called “overMaxwellian” cross section, that is $\psi(z) \geq c_0\,|z|^2$, for some c_0. The context there was the study of the large time behavior of the Landau equation, and the lower bound was the relative Fisher information of f.

This result was substantially improved in [14], in order to include soft potentials including the Coulomb potential, but with the (non relative) Fisher information as a lower bound. An estimate of the same type but involving the relative Fisher information will be provided in a paper in preparation, cf. [10], and is related to Cercignani’s conjecture (cf. [15] and the references therein).

In the sequel, we denote by $L^1_p(\mathbb{R}^3)$ the set of functions which have a moment of order p, that is $f\,(1+|v|^p)$ in $L^1(\mathbb{R}^3)$, and by $L\ln L$ the set of functions such that $f\,\ln f$ is in $L^1(\mathbb{R}^3)$.

The main theorem of [14] writes (in dimension 3 here, cf. [14] for the same result in higher dimension)

Theorem 1 *Let $f := f(v) \geq 0$, belonging to $L^1_2 \cap L\,\ln L(\mathbb{R}^3)$, be such that $\int f\,|\ln f|\,dv \leq \bar{H}$, for some $\bar{H} > 0$. Let ψ satisfy*

$$\forall z \in \mathbb{R}^3, \qquad \psi(z) \geq c_0\,\inf(1, |z|^{\gamma_1+2}),$$

for some $c_0 > 0$ and $\gamma_1 \leq 0$.

Then, there exists a constant $C := C(\int f\,dv, \int f\,v\,dv, \int f\,|v|^2/2\,dv, \bar{H}, \gamma_1, c_0) > 0$ which (explicitly) depends only on the mass, momentum, energy, (an upper bound of the) entropy and the parameters of the lower bound on ψ (that is, γ_1 and c_0), such that

$$\int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2\,(1+|v|^2)^{\inf(\gamma_1/2,-1)}\,dv \leq C\,(1+D_\psi(f)),$$

where $D_\psi(f)$ is defined in (6).

The inequality in this theorem is an entropy dissipation estimate which enables to control a weighted $H^1(\mathbb{R}^3)$ norm of $\sqrt{f}$ (that is, a weighted Fisher information) by the Landau entropy dissipation D_ψ of the Landau operator.

It is related to some other results linking smoothness to the dissipation of a Lyapounov (entropy) functional. In the case of the Boltzmann equation without cutoff, such estimates were proven in [2], [22], [1], [25], [18], [19], [20], [3], [4] (cf. also the older attempts, more related to the large time behavior of the equation than to the issue of smoothness) in [12], [8], [9], and used for example in [5] (formula (70) p. 30, Lemma 13, p. 35, and Remark p. 36).

This theorem can be seen as a mixture of the estimates proven in [2], where the Boltzmann equation with general cross sections is considered, but where the large velocities are not part of the estimate, and the estimates proven in [17], in which the large velocities are treated, but only the Landau operator with overMaxwellian molecules is considered. It is also related to the Remark p. 36 in [5].

An important feature of Theorem 1 is the fact that the constant C appearing in the estimate only depends on quantities which are known to be controlled in the evolution of the spatially homogeneous Landau equation (8), (9), provided that they are initially finite (namely, the mass, momentum and (an upper bound of the) entropy, cf. the *a priori* estimate (13)). This feature ensures that applying Theorem 1 to the solution at time t of this equation (and with such initial data), we end up with a new *a priori* estimate for its solutions. In this way, it was possible to improve in [14] the existence theory for the Landau equation with very soft potentials (including the Coulomb case) as well as to recover recent results obtained on moderately soft potentials by K.-C. Wu (cf. [26]).

Our goal in this work is to establish an extension of Theorem 1 to the case of more general cross sections ψ (that is which are not moderately soft potentials or very soft potentials like in [14], and not of Maxwell molecules type like in [17]). We indeed would like to be able to treat general soft potentials, that is ψ which decay at infinity like a negative power law (more precisely, cross sections ψ which are such that $\psi(z) \sim |z|^{\gamma+2}$ when $z \to 0$, for $\gamma < 0$), or even ψ which decay very quickly at infinity (like a negative exponential of a power law, or even a negative exponential of an exponential of a power law). We would also like to be able to treat cross sections ψ which are such that $\psi(z) \sim |z|^{\gamma+2}$ when $z \to 0$, for $\gamma \in]0,2[$, that is, general hard potentials.

Though those extensions have no direct applications to physics, they enable to understand the proof of Theorem 1 more deeply than in [14] (especially the treatment of the determinant appearing in the denominator of Cramer's formula, see below), and provide the occasion of computing explicitly bounds for the constants appearing in the estimates of $D_\psi(f)$ (something that was done in [14] only for radially symmetric functions f).

We propose first an abstract (functional) result, which holds for any cross section $\psi \geq 0$, and will then be used in order to provide estimates for specific cases of ψ.

Theorem 2 *Let $f := f(v) \geq 0$, $M := M(v) \geq 0$, and $\phi := \phi(|v|^2/2) \geq 0$ be functions such that the right-hand side of inequality (14) below is finite.*

Then

$$\int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 M(v)\, dv \leq 3\, \Delta_\phi(f)^{-2} \left(\int_{\mathbb{R}^3} \phi(|w|^2/2)\, f(w) < w >^2 dw \right)^4 \tag{14}$$
$$\times \Bigg\{ 12 \left[\int_{\mathbb{R}^3} f(v) < v >^2 M(v)\, dv \right] \Bigg[3 \left(\int_{\mathbb{R}^3} f(w) < w > \phi(|w|^2/2)\, dw \right)^2$$
$$+ 8 \left(\int_{\mathbb{R}^3} f(w) < w >^2 |\phi'(|w|^2/2)|\, dw \right)^2 \Bigg]$$

$$+24 D_\psi(f) \sup_{v\in\mathbb{R}^3} M(v) \left(\int_{\mathbb{R}^3} f(w)\,\phi^2(|w|^2/2)\,\frac{|v-w|^2}{\psi(|v-w|)}\, <w>^2\, dw \right) \Bigg\},$$

where (here and in all the rest of the paper) $<v>= (1+|v|^2)^{1/2}$*, and*

$$\Delta_\phi(f) := Det\left(\int_{\mathbb{R}^3} f(w)\,\phi(|w|^2/2) \begin{bmatrix} 1 & w_i & w_j \\ w_i & w_i^2 & w_i w_j \\ w_j & w_i w_j & w_j^2 \end{bmatrix} dw \right).$$

This functional estimate leads to the following corollary, which still holds for any cross section $\psi \geq 0$, but can in practice be used only when $z \mapsto \psi(z)/|z|^2$ is bounded below by a strictly positive constant on each bounded set of $\mathbb{R}^3$ (typically, for general soft potentials, but not (general or not) hard potentials).

Corollary 2.1 *Let* $f := f(v) \geq 0$*, belonging to* $L^1_2 \cap L\ln L(\mathbb{R}^3)$ *and such that* $D_\psi(f)$ *is finite. We also suppose that* $M := M(v) \geq 0$ *is bounded, and that* $\phi \geq 0$ *is* C^1*, bounded, with* ϕ' *also bounded.*

Then

$$\int_{\mathbb{R}^3} |\nabla\sqrt{f(v)}|^2\, M(v)\, dv \leq 72\,\Delta_\phi(f)^{-2}\, ||\phi||_\infty^4\, \mathscr{E}_f^5 \tag{15}$$

$$\times \left[||M||_\infty \left(\frac{3}{2}\,||\phi||_\infty^2 + 4\,||\phi'||_\infty^2 \right) \mathscr{E}_f^2 + \beta\, D_\psi(f) \right],$$

where (here and in all the rest of the paper)

$$\mathscr{E}_f := \int_{\mathbb{R}^3} f(v)\,(1+|v|^2)\,dv,$$

and $\beta > 0$ *is any number such that*

$$\forall v, w \in \mathbb{R}^3, \qquad M(v)\,\phi^2(|w|^2/2) \leq \beta\, \frac{\psi(|v-w|)}{|v-w|^2}\,. \tag{16}$$

Note also that in this result and its corollaries below (Corollaries 2.2, 2.3 and 2.4), up to the quantity $\Delta_\phi(f)$ which will be discussed later, the only dependence of the constants w.r.t. f is that of $\mathscr{E}_f$, that is, a dependence through quantities which are constant in the evolution of the spatially homogeneous Landau equation (8), (9).

This estimate leads in turn to a family of corollaries (Corollaries 2.2, 2.3 and 2.4), which hold for functions ψ satisfying various lower bounds.

We start with the case when ψ satisfies a lower bound corresponding to a non-positive power law including general soft potentials.

Corollary 2.2 *Let* $f := f(v) \geq 0$*, belonging to* $L^1_2 \cap L\ln L(\mathbb{R}^3)$*, and* ψ *satisfying the lower bound* $\psi(z) \geq c_0\,|z|^{2+\gamma}$ *for some* $c_0 > 0$*,* $\gamma \leq 0$*. We assume that* $D_\psi(f)$ *is finite.*

Then

$$\int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 < v >^{\gamma} \, dv \le 72 \, \Delta_{\phi}(f)^{-2} \, \mathscr{E}_f^5 \tag{17}$$

$$\times \left[(\frac{3}{2} + |\gamma|^2) \, \mathscr{E}_f^2 + c_0^{-1} \, 2^{\sup(0,|\gamma|-1)+\sup(2-|\gamma|,0)} \, D_{\psi}(f) \right],$$

and

$$\phi(z) = (1+2z)^{\gamma/4}.$$

Note that Corollary 2.2 (together with Proposition 4 below) gives a completely explicit estimate in Theorem 1 of [14]. Our feeling is that the exponent γ in the weight appearing in estimate (17) is optimal. This result (like those of Corollaries 2.3 and 2.4 below), together with the bound appearing in Proposition 4 on $\Delta_{\phi}(f)$, enables the building of an existence theory of standard weak solutions (that is, the concept of H-solutions appearing in [24] is not needed here) for the related spatially homogeneous Landau equations, provided that ψ has no too strong singularities (for example singularities at point 0 strictly weaker than $\psi(z) \sim |z|^{-2}$ can be handled). We refer to [14] for that kind of applications to the spatially homogeneous Landau equation

Next we turn to the case when ψ can decay much more rapidly at infinity, namely like an exponential of a power.

Corollary 2.3 *Let $f := f(v) \ge 0$, belonging to $L^1_2 \cap L \ln L(\mathbb{R}^3)$, and ψ satisfying the lower bound: $\frac{\psi(z)}{|z|^2} \ge c_0 \, e^{-c_1 \, |v|^{\delta}}$, for some $c_0, c_1, \delta > 0$. We assume that $D_{\psi}(f)$ is finite.*

Then

$$\int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 \, e^{-\tilde{c_1} \, |v|^{\delta}} \, dv \le 72 \, \Delta_{\phi}(f)^{-2} \, e^{-2\tilde{c_1}} \tag{18}$$

$$\times \mathscr{E}_f^5 \left(\left[\frac{3}{2} \, e^{-\tilde{c_1}} + \sup\left(4, 4 \, (\tilde{c_1} \, \delta)^{2-4/\delta} \, (2\,\delta - 2)^{4/\delta} \, e^{-2+2/\delta}\right) \right] \mathscr{E}_f^2 + c_0^{-1} \, D_{\psi}(f) \right),$$

and

$$\phi(z) = e^{-\frac{\tilde{c_1}}{2} \, (1+2z)^{\delta}}, \qquad \tilde{c_1} = c_1 \, 2^{\sup(0,\delta-1)}.$$

In estimate (18), it is not clear whether the weight $e^{-c_1 \, 2^{\sup(\delta-1,0)} \, |v|^{\delta}}$ is optimal. We believe however that the optimal weight, if it exists, should be of the same general shape (that is, an exponential of a power δ), or close to such a shape.

Finally we turn to the case when ψ can decay even more rapidly at infinity, namely like an exponential of an exponential of a power. For this extreme situation, rather than giving a result concerning all possible decays, we focus on a special case, namely when $\psi(z) \ge e^{-e^{|z|}}$, for which it is possible to write a quite simple estimate.

Corollary 2.4 *Let $f := f(v) \ge 0$, belonging to $L^1_2 \cap L \ln L(\mathbb{R}^3)$, and ψ satisfying the lower bound: $\frac{\psi(z)}{|z|^2} \ge \exp(-e^{|z|})$. We assume that $D_{\psi}(f)$ is finite.*

Then

$$\int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 \, e^{-3 e^{3|v|}} \, dv \le 72 \, \Delta_\phi(f)^{-2} \, \mathscr{E}_f^5 \, e^{-9} \left(e^{-3} \left(\frac{3}{2} e^{-3} + 81 \right) \mathscr{E}_f^2 + 3 D_\psi(f) \right), \tag{19}$$

and

$$\phi(z) = \exp(-\frac{3}{2} e^{3\sqrt{2z}}).$$

As in the previous corollary, one can observe that the weight appearing in the Fisher information is different from the the cross section $z \mapsto \frac{\Psi(|z|)}{|z|^2}$, it is also not optimal.

We then consider the case of cross sections which are not strictly positive at point 0, so that Corollary 2.1 cannot be used, and we have to come back to Theorem 2. We propose first the following result, which enables to treat as a special case hard potentials.

Corollary 2.5 *Let $f := f(v) \ge 0$, belonging to $L^1_2 \cap L \ln L(\mathbb{R}^3)$, and ψ satisfying the following lower bound: $\psi(z) \ge c_0 \inf(|z|^2, |z|^{\gamma+2})$, with $c_0 > 0$ and $\gamma \in]0,3[$. We assume that $D_\psi(f)$ is finite.*

Then

$$\int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 \, dv \le 3 \, \Delta_\phi(f)^{-2} \, \mathscr{E}_f^4 \Bigg\{ 132 \, \mathscr{E}_f^3 \tag{20}$$

$$+ 24 \, c_0^{-1} D_\psi(f) \left(\mathscr{E}_f + \left(\frac{4\pi (p-1)}{3(p-1) - \gamma p} \right)^{1-1/p} ||f||_{L^p} \right) \Bigg\},$$

for all $p > \frac{3}{3-\gamma}$, and

$$\phi(z) = (1+2z)^{-1/2}.$$

In Corollary 2.5, we made no effort to obtain a better weight (than 1) in the r.h.s. of estimate (20). We will discuss this issue in Corollary 2.7. Note also that an L^p (with $p > 1$) norm of f appears in the constants of the estimate. Such a quantity is not constant in the evolution of the Landau equation, but is sometimes propagated (or even created), cf. for example [16].

When γ is not too large (that is, for general hard potentials, i.-e. $\gamma < 2$), it is possible to use a Sobolev estimate and an interpolation inequality, in order to get rid of the L^p norm in the result above. The price to pay is the appearance of an exponent larger than 1 for the entropy dissipation $D(f)$ appearing in the estimate.

Corollary 2.6 *Let $f := f(v) \ge 0$, belong to $L^1_2 \cap L \ln L(\mathbb{R}^3)$, and ψ satisfying the following lower bound: $\psi(z) \ge c_0 \inf(|z|^2, |z|^{\gamma+2})$, with $c_0 > 0$ and $\gamma \in]0,2[$. We assume that $D_\psi(f)$ is finite.*

Then for any $p \in]\frac{3}{3-\gamma}, 3[$,

$$\int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2\, dv \le 792\, \Delta_\phi(f)^{-2}\, \mathscr{E}_f^7 + 144\, \Delta_\phi(f)^{-2}\, \mathscr{E}_f^5\, c_0^{-1}\, D_\psi(f) \qquad (21)$$

$$+144^{1/(1-\theta)}\,(1-\theta)\,\theta^{\theta/(1-\theta)} \left(\frac{4\pi\,(p-1)}{3\,(p-1)-\gamma p} \right)^{\frac{1-1/p}{1-\theta}} c_0^{-1/(1-\theta)}\, C_s^{\theta/(1-\theta)}\, \mathscr{E}_f^{1+4/(1-\theta)}$$
$$\times \Delta_\phi(f)^{-2/(1-\theta)}\, D_\psi(f)^{1/(1-\theta)},$$

where $\theta := \frac{3(p-1)}{2p}$, and C_s is the constant appearing in a Sobolev estimate (cf. proof of Corollary 2.6).

This result, together with the bound appearing in Proposition 4 on $\Delta_\phi(f)$, enables to obtain a new *a priori* estimate for the solutions $f(t,v)$ of the Landau equation with (general) hard potentials, when the initial data have a finite mass, energy and entropy. It gives indeed a bound for $\left(\int |\nabla_v \sqrt{f(t,v)}|^2 dv \right)^{1-\theta}$ in $L^1([0,T])$ for all $T>0$. This result, related to the regularization effect of the Landau equation, is to be compared with the results of the same kind obtained in [16]. There, much more informations on the smoothness of the solution are provided, but only under extra assumptions on the initial data (and on the cross section ψ).

The interpolation procedure used here is reminiscent of those used in [17], or, in the context of the Boltzmann equation, in [23].

If we now suppose that ψ is growing at infinity at least as fast as $|\cdot|^{\gamma+2}$ (like in general hard potentials), then we can get a slightly better estimate than (20), in which the weight $<\cdot>^\gamma$ appears. Namely we obtain the

Corollary 2.7 *Let $f := f(v) \ge 0$, belong to $L^1_2 \cap L \ln L(\mathbb{R}^3)$, and ψ satisfying the following lower bound: $\psi(z) \ge c_0\,|z|^{\gamma+2}$, with $c_0 > 0$ and $\gamma \in]0,3[$. We assume that $D_\psi(f)$ is finite.*

Then

$$\int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 <v>^\gamma dv \le 36\,\Delta_\phi(f)^{-2}\,\mathscr{E}_f^4 \left\{ \left(3 + 8\left(\frac{\gamma}{2}+1\right)^2 \right) \mathscr{E}_f^2 \int_{\mathbb{R}^3} f(w) <w>^{2+\gamma} dw \right. \qquad (22)$$
$$\left. + 24\, c_0^{-1}\, D_\psi(f) \left([2^{\frac{3}{2}\gamma} + 2^\gamma]\,\mathscr{E}_f + 3^{\gamma/2} \left(\frac{4\pi\,(p-1)}{(3-\gamma)\,p-3} \right)^{1-1/p} ||f||_{L^p} \right) \right\},$$

where $p > \frac{3}{3-\gamma}$, and

$$\phi(z) = (1+2z)^{-\frac{\gamma}{4}-\frac{1}{2}}.$$

Note that in this result, the moment $\int f(w) <w>^{2+\gamma} dw$ appears in the estimate (as well as $||f||_{L^p}$, which already appeared in estimate (20)). This moment is not constant in the evolution of the spatially homogeneous Landau equation (with general hard potentials). It is however sometimes propagated, or even created (cf. [16]).

We now complete the estimates appearing in Corollaries 2.1 to 2.7 by a lower bound on $\Delta_\phi(f)$. We start with a proposition showing that $\Delta_\phi(f)$ is somehow bounded below by a quantity which can be equal to 0 only when f is concentrated on a hyperplane (provided that $\phi > 0$ a.e.).

Proposition 3 *Let $f := f(v) \geq 0$ belong to $L^1_2(\mathbb{R}^3)$.*

Then, for all $i, j \in \{1,2,3\}$ such that $i \neq j$, and $\varepsilon > 0$, $R > 0$,

$$\Delta_\phi(f) := \, Det \, \left(\int_{\mathbb{R}^3} \phi(|v|^2/2)\, f(v) \begin{bmatrix} 1 & v_i & v_j \\ v_i & v_i^2 & v_i v_j \\ v_j & v_i v_j & v_j^2 \end{bmatrix} dv \right)$$

$$\geq \varepsilon^6 \inf_{B(0,R)} \phi\left(\frac{|\cdot|^2}{2}\right)^3 \left(\int_{B(0,R)} f(v)\, dv \right. \tag{23}$$

$$\left. - \sup_{\lambda^2+\mu^2+\nu^2=1} \int_{B(0,R)} f(v)\, 1_{\{|\lambda+\mu\, v_i + \nu\, v_j| \leq \varepsilon\}}\, dv \right)^3.$$

With this result in mind, any estimate on f which prevents concentration on zero-measure sets or at infinity can now be used to bound $\Delta_\phi(f)$ from below. Concentration on large velocities will be prevented by using the energy of f (remember that this quantity is constant during the evolution of Landau's equation). Concentration on zero-measure sets can be achieved (with efficiency from the point of view of numerical constants) if one uses some L^p estimate for f (cf. [17]). Though L^p regularity is known to be propagated (or even created) in some cases for the spatially homogeneous Landau equation, we however prefer to use $L\ln L$ regularity, which is much less efficient (from the point of view of numerical constants), but which can be obtained for all solutions of the spatially homogeneous Landau equation, as soon as the initial mass, energy and entropy are finite.

We provide therefore the following estimate:

Proposition 4 *Let $f := f(v) \geq 0$ belong to $L^1_2(\mathbb{R}^3)$, and assume that $H(f) \leq \bar{H}$.*

Then, for all $i, j \in \{1,2,3\}$ such that $i \neq j$, we have the estimate

$$\Delta_\phi(f) \geq \left(\frac{1}{4} \int f(v)\, dv \right)^3 \inf_{B\left(0, \sup\left(1, 2\, (\int f(v)\, v^2\, dv / \int f(v)\, dv)^{1/2}\right)\right)} \phi\left(\frac{|\cdot|^2}{2}\right)^3 \tag{24}$$

$$\times \inf \left[2^{-6}, 2^{-42} \left(\int f(v)\, dv \right)^6 e^{-24\bar{H}\, (\int f(v)\, dv)^{-1}} \sup\left(1, 2^{-18} \left(\frac{\int f(v)\, dv}{\int f(v)\, v^2\, dv} \right)^9 \right) \right].$$

As can be seen, this estimate can be used (together with Corollaries 2.1 to 2.6) in order to yield *a priori* estimates for the solutions of the spatially homogeneous Landau equation, since it involves (when ϕ is strictly positive a.e.) only the mass, energy and (an upper bound of the) entropy of f, all quantities which are known to be controlled for those solutions.

We now briefly explain what are the possible extensions of the results presented in this section.

We first observe that for cross sections ψ which are bounded below (by a strictly positive constant) on all bounded subsets, it is most probably possible to extend the results stated in Corollaries 2.2, 2.3 and 2.4 to some ψ which are decaying at infinity even more rapidly than an exponential of exponential. Our feeling is that the more ψ rapidly decays at infinity, the less optimal the final weight appearing in the estimate of the entropy dissipation will be, if one uses Corollary 2.1. It becomes indeed more and more difficult to bound from below a function of $v-w$ by a tensor product (that is, a function of v multiplied by a function of w) when this function tends quickly towards 0 at infinity.

One can also deal with functions ψ which have more than one point of cancellation, at least if those points constitute a finite set, and if the cancellation at each point is not stronger than $|z-z_0|^q$, with $q<3$. As in Corollary 2.5, some L^p norm of f will then appear in the estimate of the entropy dissipation, which can be dealt with as in Corollary 2.6 if $q<2$.

Finally, one can in principle deal with functions ψ which both cancel at a finite number of points, and which have a specific behavior at infinity. When ψ is growing at infinity more than $z\mapsto |z|^2$, we can get results analogous to Corollary 2.7, while if $z\mapsto \psi(z)/|z|^2$ is decaying at infinity, one can get an estimate in which some (decaying) weight appears, and where some L^p norm of f also appears (that is, some mixture of Corollaries 2.2, 2.3 and 2.4 with Corollary 2.5).

All the results presented in section 1 are proven in section 2.

2 Proofs of the theorems

We begin with the

Proof of Theorem 2:

We start as in the proof of the corresponding theorem in [14]. We first observe that (for all $x,y\in\mathbb{R}^3$)

$$y^T\,(|x|^2\,Id - x\otimes x)\,y = \frac{1}{2}\sum_{i,j\in\{1,2,3\}} |x_i\,y_j - x_j\,y_i|^2,$$

so that

$$D_\psi(f) = \frac{1}{4}\sum_{i,j\in\{1,2,3\}}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)\,f(w)\,\frac{\psi(|v-w|)}{|v-w|^2}\,\left|q_{ij}^f(v,w)\right|^2 dvdw,$$

where (for $i,j\in\{1,2,3\}$, $i\neq j$),

$$q^f_{ij}(v,w) := (v_i-w_i)\left(\frac{\partial_j f(v)}{f(v)}-\frac{\partial_j f(w)}{f(w)}\right)-(v_j-w_j)\left(\frac{\partial_i f(v)}{f(v)}-\frac{\partial_i f(w)}{f(w)}\right)$$
$$=\left[v_i\,\frac{\partial_j f(v)}{f(v)}-v_j\,\frac{\partial_i f(v)}{f(v)}\right]+w_j\,\frac{\partial_i f(v)}{f(v)}-w_i\,\frac{\partial_j f(v)}{f(v)}$$
$$-v_i\,\frac{\partial_j f(w)}{f(w)}+v_j\,\frac{\partial_i f(w)}{f(w)}+\left[w_i\,\frac{\partial_j f(w)}{f(w)}-w_j\,\frac{\partial_i f(w)}{f(w)}\right].$$

Then, instead of using $w\mapsto\chi(w)\,e^{-\lambda w^2}\,f(w)$, where χ is a polynomial of degree 1 as in [14], we use the functions $w\mapsto\chi(w)\,\phi(|w|^2/2)$, where ϕ is a generic radially symmetric function. Picking $i,j\in\{1,2,3\}$, $i\neq j$, we see that for $\chi(w)=1$,

$$\int_{\mathbb{R}^3} q^f_{ij}(v,w)\,\phi(|w|^2/2)\,f(w)\,dw=\left[v_i\,\frac{\partial_j f(v)}{f(v)}-v_j\,\frac{\partial_i f(v)}{f(v)}\right]\left(\int_{\mathbb{R}^3}\phi(|w|^2/2)\,f(w)\,dw\right)$$
$$+\left(\int_{\mathbb{R}^3} w_j\,\phi(|w|^2/2)\,f(w)\,dw\right)\frac{\partial_i f(v)}{f(v)}-\left(\int_{\mathbb{R}^3} w_i\,\phi(|w|^2/2)\,f(w)\,dw\right)\frac{\partial_j f(v)}{f(v)}$$
$$+v_i\left(\int_{\mathbb{R}^3} w_j\,\phi'(|w|^2/2)\,f(w)\,dw\right)-v_j\left(\int_{\mathbb{R}^3} w_i\,\phi'(|w|^2/2)\,f(w)\,dw\right).$$

Then, for $\chi(w)=w_i$,

$$\int_{\mathbb{R}^3} q^f_{ij}(v,w)\,w_i\,\phi(|w|^2/2)\,f(w)\,dw=\left[v_i\,\frac{\partial_j f(v)}{f(v)}-v_j\,\frac{\partial_i f(v)}{f(v)}\right]\left(\int_{\mathbb{R}^3} w_i\,\phi(|w|^2/2)\,f(w)\,dw\right)$$
$$+\left(\int_{\mathbb{R}^3} w_j\,w_i\,\phi(|w|^2/2)\,f(w)\,dw\right)\frac{\partial_i f(v)}{f(v)}-\left(\int_{\mathbb{R}^3} w_i^2\,\phi(|w|^2/2)\,f(w)\,dw\right)\frac{\partial_j f(v)}{f(v)}$$
$$+v_i\left(\int_{\mathbb{R}^3} w_i\,w_j\,\phi'(|w|^2/2)\,f(w)\,dw\right)-v_j\left(\int_{\mathbb{R}^3}(\phi(|w|^2/2)+w_i^2\,\phi'(|w|^2/2))\,f(w)\,dw\right)$$
$$+\int_{\mathbb{R}^3} w_j\,\phi(|w|^2/2)\,f(w)\,dw.$$

Exchanging i and j (or, equivalently, taking $\chi(w)=w_j$), we get the identity

$$\int_{\mathbb{R}^3} q^f_{ij}(v,w)\,w_j\,\phi(|w|^2/2)\,f(w)\,dw=\left[v_i\,\frac{\partial_j f(v)}{f(v)}-v_j\,\frac{\partial_i f(v)}{f(v)}\right]\left(\int_{\mathbb{R}^3} w_j\,\phi(|w|^2/2)\,f(w)\,dw\right)$$
$$-\left(\int_{\mathbb{R}^3} w_j\,w_i\,\phi(|w|^2/2)\,f(w)\,dw\right)\frac{\partial_j f(v)}{f(v)}+\left(\int_{\mathbb{R}^3} w_j^2\,\phi(|w|^2/2)\,f(w)\,dw\right)\frac{\partial_i f(v)}{f(v)}$$
$$-v_j\left(\int_{\mathbb{R}^3} w_i\,w_j\,\phi'(|w|^2/2)\,f(w)\,dw\right)+v_i\left(\int_{\mathbb{R}^3}(\phi(|w|^2/2)+w_j^2\,\phi'(|w|^2/2))\,f(w)\,dw\right)$$

$$- \int_{\mathbb{R}^3} w_i\, \phi(|w|^2/2)\, f(w)\, dw.$$

Considering the above identities as a 3×3 system for the unknowns $v_i\,\frac{\partial_j f(v)}{f(v)} - v_j\,\frac{\partial_i f(v)}{f(v)}$, $\frac{\partial_i f(v)}{f(v)}$ and $\frac{\partial_j f(v)}{f(v)}$, and using Cramer's formulas, we end up with the following formula for $\frac{\partial_i f(v)}{f(v)}$:

$$\frac{\partial_i f(v)}{f(v)} = \Delta_\phi(f)^{-1}$$

$$\times\, Det \left(\int_{\mathbb{R}^3} \phi(|w|^2/2)\, f(w) \begin{bmatrix} 1 & w_i & q^f_{ij}(v,w) + P_1(f)(v,w) \\ w_i & w_i^2 & q^f_{ij}(v,w)\, w_i + P_2(f)(v,w) \\ w_j & w_i\, w_j & q^f_{ij}(v,w)\, w_j + P_3(f)(v,w) \end{bmatrix} dw \right),$$

where

$$P_1(f)(v,w) = v_j\, \frac{w_i\, \phi'(|w|^2/2)}{\phi(|w|^2/2)} - v_i\, \frac{w_j\, \phi'(|w|^2/2)}{\phi(|w|^2/2)},$$

$$P_2(f)(v,w) = v_j\, \frac{[\phi(|w|^2/2) + w_i^2\, \phi'(|w|^2/2)]}{\phi(|w|^2/2)} - v_i\, \frac{w_i\, w_j\, \phi'(|w|^2/2)}{\phi(|w|^2/2)} - w_j,$$

$$P_3(f)(v,w) = v_j\, \frac{w_i\, w_j\, \phi'(|w|^2/2)}{\phi(|w|^2/2)} - v_i\, \frac{[\phi(|w|^2/2) + w_j^2\, \phi'(|w|^2/2)]}{\phi(|w|^2/2)} + w_i.$$

Then,

$$\left| \frac{\partial_i f(v)}{f(v)} \right| \le 2\, \Delta_\phi(f)^{-1} \left(\int_{\mathbb{R}^3} \phi(|w|^2/2)\, f(w)\, (1+|w|^2)\, dw \right)^2$$

$$\times \left(\int_{\mathbb{R}^3} \phi(|w|^2/2)\, f(w) \left[\sum_{k=1}^{3} |P_k(f)(v,w)|\, dw + |q^f_{ij}(v,w)|\, (1+|w_i|+|w_j|) \right] dw \right)$$

$$\le 2\, \Delta_\phi(f)^{-1} \left(\int_{\mathbb{R}^3} \phi(|w|^2/2)\, f(w)\, (1+|w|^2)\, dw \right)^2$$

$$\times \Bigg((1+|v_i|+|v_j|) \int_{\mathbb{R}^3} f(w) \left[\phi(|w|^2/2)\, (1+|w_i|+|w_j|) + |\phi'(|w|^2/2)|\, (|w_i|+|w_j|+2\,|w|^2) \right] dw$$

$$+ \int_{\mathbb{R}^3} f(w)\, \phi(|w|^2/2)\, |q^f_{ij}(v,w)|\, (1+|w_i|+|w_j|)\, dw \Bigg)$$

$$\le 2\, \Delta_\phi(f)^{-1} \left(\int_{\mathbb{R}^3} \phi(|w|^2/2)\, f(w) < w >^2 dw \right)^2$$

$$\times \Bigg(\sqrt{3} < v > \int_{\mathbb{R}^3} f(w) \left[\sqrt{3} < w > \phi(|w|^2/2) + 2\sqrt{2} < w >^2\ |\phi'|(|w|^2/2) \right] dw$$

$$+ \sqrt{3} \int_{\mathbb{R}^3} f(w)\, \phi(|w|^2/2)\, |q^f_{ij}(v,w)| < w > dw \Bigg).$$

Then

$$
\begin{aligned}
&\int_{\mathbb{R}^3} f(v) \left| \frac{\partial_i f(v)}{f(v)} \right|^2 M(v)\, dv \\
&\le 4\, \Delta_\phi(f)^{-2} \left(\int_{\mathbb{R}^3} \phi(|w|^2/2)\, f(w) < w >^2 dw \right)^4 \\
&\times \left(6 \int_{\mathbb{R}^3} f(v) < v >^2 M(v)\, dv \right. \\
&\times \left| \int_{\mathbb{R}^3} f(w) \left[\sqrt{3} < w > \phi(|w|^2/2) + 2\sqrt{2} < w >^2 |\phi'|(|w|^2/2) \right] dw \right|^2 \\
&\left. + 6 \int_{\mathbb{R}^3} f(v) M(v) \left| \int_{\mathbb{R}^3} f(w)\, \phi(|w|^2/2)\, |q^f_{ij}(v,w)| < w > dw \right|^2 dv \right) \\
&\le 4\, \Delta_\phi(f)^{-2} \left(\int_{\mathbb{R}^3} \phi(|w|^2/2)\, f(w) < w >^2 dw \right)^4 \\
&\times \left\{ 12 \int_{\mathbb{R}^3} f(v) < v >^2 M(v)\, dv \left[3 \left(\int_{\mathbb{R}^3} f(w) < w > \phi(|w|^2/2)\, dw \right)^2 \right. \right. \\
&\left. + 8 \left(\int_{\mathbb{R}^3} f(w) < w >^2 |\phi'|(|w|^2/2)\, dw \right)^2 \right] \\
&+ 6 \int_{\mathbb{R}^3} f(v) M(v) \left(\int_{\mathbb{R}^3} f(w)\, |q^f_{ij}(v,w)|^2 \frac{\psi(|v-w|)}{|v-w|^2} dw \right) \\
&\left. \times \left(\int_{\mathbb{R}^3} f(w)\, \phi^2(|w|^2/2) < w >^2 \frac{|v-w|^2}{\psi(|v-w|)} dw \right) \right\} \\
&\le 4\, \Delta_\phi(f)^{-2} \left(\int_{\mathbb{R}^3} \phi(|w|^2/2)\, f(w) < w >^2 dw \right)^4 \\
&\times \left\{ 12 \int_{\mathbb{R}^3} f(v) < v >^2 M(v)\, dv \left[3 \left(\int_{\mathbb{R}^3} f(w) < w > \phi(|w|^2/2)\, dw \right)^2 \right. \right. \\
&\left. + 8 \left(\int_{\mathbb{R}^3} f(w) < w >^2 |\phi'|(|w|^2/2)\, dw \right)^2 \right] \\
&\left. + 24\, D_\psi(f) \sup_{v \in \mathbb{R}^3} M(v) \left(\int_{\mathbb{R}^3} f(w)\, \phi^2(|w|^2/2) \frac{|v-w|^2}{\psi(|v-w|)} < w >^2 dw \right) \right\}.
\end{aligned}
$$

We conclude the proof of Theorem 2 by noticing that

$$
\int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 M(v)\, dv = \frac{1}{4} \sum_{i=1}^{3} \int_{\mathbb{R}^3} f(v) \left| \frac{\partial_i f(v)}{f(v)} \right|^2 M(v)\, dv.
$$

□

We now turn to the proofs of the corollaries of this theorem.

Proof of Corollary 2.1: It is a direct consequence of Theorem 2 and the bounds assumed on M, ϕ and ψ. $\Box$

Proof of Corollary 2.2: We recall that $\psi(z) \geq c_0\,|z|^{\gamma+2}$. Using the elementary inequalities

$$\forall x,y,p \in \mathbb{R}_+, \qquad (x+y)^p \leq 2^{\sup(p-1,0)}\,(x^p+y^p),$$

$$\forall x,y,p \in \mathbb{R}_+, \qquad x^p+y^p \leq 2^{\sup(1-p,0)}\,(x+y)^p,$$

we see that (for any $\gamma < 0$, $v,w \in \mathbb{R}^3$)

$$|v-w|^{|\gamma|} \leq 2^{\sup(|\gamma|-1,0)}\,(|v|^{|\gamma|}+|w|^{|\gamma|})$$

$$\leq 2^{\sup(|\gamma|-1,0)+\sup(2-|\gamma|,0)} < v >^{|\gamma|} < w >^{|\gamma|}.$$

Then taking

$$M(v) = < v >^{\gamma}, \qquad \phi(z) = (1+2z)^{\gamma/4},$$

we see that assumption (16) holds provided that $\beta = c_0^{-1}\,2^{\sup(|\gamma|-1,0)+\sup(2-|\gamma|,0)}$. Noticing that

$$||M||_\infty = 1, \qquad ||\phi||_\infty = 1, \qquad ||\phi'||_\infty = \gamma/2.$$

and using Corollary 2.1, we get Corollary 2.2.
$\Box$

Proof of Corollary 2.3: We recall that $\psi(z) \geq c_0\,e^{-c_1\,|z|^\delta}$. Still using the elementary inequalities used in the proof of Corollary 2.2, we see that (for all $v,w \in \mathbb{R}^3$)

$$e^{c_1\,|v-w|^\delta} \leq e^{\tilde{c_1}\,|v|^\delta}\,e^{\tilde{c_1}\,<w>^\delta},$$

with $\tilde{c_1} = c_1\,2^{\sup(0,\delta-1)}$, so that taking

$$M(v) = e^{-\tilde{c_1}\,|v|^\delta}, \qquad \phi(z) = e^{-\frac{\tilde{c_1}}{2}\,(1+2z)^\delta},$$

we see that assumption (16) holds provided that $\beta = c_0^{-1}$. We then observe that

$$||M||_\infty = 1, \qquad ||\phi||_\infty = e^{-\frac{\tilde{c_1}}{2}}, \qquad ||\phi'||_\infty \leq \sup\left(1, \tilde{c_1}\,\delta\left(\frac{2(\delta-1)}{\tilde{c_1}\,\delta}\right)^{2/\delta} e^{-1+1/\delta}\right).$$

Using Corollary 2.1, we get Corollary 2.3.
$\Box$

Proof of Corollary 2.4: We recall that $\psi(z) \geq \exp(-e^{|z|})$. Then

$$\exp(e^{|v-w|}) \leq \exp(e^{|v|}\,e^{|w|})$$

$$\leq 1+e^{|v|}\,e^{|w|}+\sum_{k=2}^{\infty}\frac{1}{k\,(k-1)}\,\frac{e^{k|v|}}{[(k-2)!]^{1/2}}\,\frac{e^{k|w|}}{[(k-2)!]^{1/2}}.$$

If we introduce

$$u_k := \frac{e^{k|v|}}{[(k-2)!]^{1/2}},$$

we see that

$$u_{k+1}\leq u_k \qquad \Longleftrightarrow \qquad k\geq 1+e^{2\,|v|}.$$

Then, the sequence u_k reaches its maximum when $k=[2+e^{2\,|v|}]$, so that (for all $k\geq 2$)

$$u_k \leq \frac{\exp(|v|\,(2+e^{2\,|v|}))}{([e^{2\,|v|}]!)^{1/2}},$$

and finally

$$\begin{aligned}\exp(e^{|v-w|}) &\leq 1+e^{|v|}\,e^{|w|}+\left(\sum_{k=2}^{\infty}\frac{1}{k\,(k-1)}\right)\frac{\exp(|v|\,(2+e^{2\,|v|}))}{([e^{2\,|v|}]!)^{1/2}}\,\frac{\exp(|w|\,(2+e^{2\,|w|}))}{([e^{2\,|w|}]!)^{1/2}}\\ &\leq 1+e^{|v|}\,e^{|w|}+\exp(|v|\,(2+e^{2\,|v|}))\,\exp(|w|\,(2+e^{2\,|w|}))\\ &\leq 3\,\exp(3\,e^{3\,|v|})\,\exp(3\,e^{3<w>}).\end{aligned} \tag{25}$$

We then introduce

$$M(v) := \exp(-3\,e^{3\,|v|}), \qquad \phi(z)=\exp\left(-\frac{3}{2}\,e^{3\sqrt{1+2z}}\right),$$

so that

$$\phi'(z)=-\exp\left(-\frac{3}{2}\,e^{3\sqrt{1+2z}}\right)\frac{9}{2}\sqrt{\frac{1}{1+2z}}\,e^{3\sqrt{1+2z}}.$$

We see that

$$||M||_\infty = e^{-3}, \qquad ||\phi||_\infty = e^{-\frac{3}{2}}, \qquad ||\phi'||_\infty \leq \frac{9}{2}.$$

Using estimate (25), we obtain the estimate

$$\begin{aligned}M(v)\,\phi(|w|^2/2) &\leq 3\,\exp(-e^{|v-w|})\\ &\leq \beta\,\frac{\Psi(|v-w|^2)}{|v-w|^2},\end{aligned}$$

with $\beta=3$. Using Corollary 2.1, we end up with the statement of Corollary 2.4. □

Proof of Corollary 2.5: We introduce

$$M(v)=1, \qquad \phi(z)=(1+2z)^{-1/2},$$

so that

$$\phi'(z)=-(1+2z)^{-3/2}.$$

Then, using Theorem 2, we see that

$$\int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 \, dv \le 3\, \Delta_\phi(f)^{-2}\, \mathscr{E}_f^4 \Big\{ 132\, \mathscr{E}_f^3$$

$$+\, 24\, c_0^{-1}\, D_\psi(f) \sup_{v\in\mathbb{R}^3} \int f(w)\, \sup(1, |v-w|^{-\gamma})\, dw \Big\}.$$

We now observe that

$$\sup_{v\in\mathbb{R}^3} \int f(w)\, \sup(1, |v-w|^{-\gamma})\, dw \le \mathscr{E}_f + ||f * |\cdot|^{-\gamma} 1_{\{|\cdot|\le 1\}}||_{L^\infty}.$$

Then, Young's inequality for convolutions ensures that, for any $p > \frac{3}{3-\gamma}$,

$$||f * |\cdot|^{-\gamma} 1_{\{|\cdot|\le 1\}}||_{L^\infty} \le ||f||_{L^p} \left(\frac{4\pi(p-1)}{(3-\gamma)\, p - 3} \right)^{1-1/p}.$$

This concludes the proof of Corollary 2.5. □

Proof of Corollary 2.6: We first write on $\sqrt{f}$ the Sobolev inequality corresponding to the Sobolev embedding $H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, that is

$$||f||_{L^3} \le C_s\, ||\nabla \sqrt{f}||_{L^2}^2, \tag{26}$$

where $C_s > 0$ is the (best) constant appearing in the Sobolev inequality.

Denoting

$$a := 396\, \Delta_\phi(f)^{-2}\, \mathscr{E}_f^7,$$

$$b := 72\, \Delta_\phi(f)^{-2}\, \mathscr{E}_f^5\, c_0^{-1},$$

$$c := 72\, \Delta_\phi(f)^{-2}\, \mathscr{E}_f^4\, c_0^{-1} \left(\frac{4\pi\, (p-1)}{3\,(p-1) - \gamma p)} \right)^{1-1/p},$$

we see thanks to Corollary 2.5, Hölder's inequality, and the Sobolev inequality (26) that

$$\int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 \, dv \le a + D_\psi(f)\, (b + c\, ||f||_{L^p})$$

$$\le a + D_\psi(f)\, (b + c\, ||f||_{L^3}^{\theta}\, \mathscr{E}_f^{1-\theta})$$

$$\le a + D_\psi(f) \left[b + c\, C_s^{\theta}\, \mathscr{E}_f^{1-\theta} \left(\int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 \, dv \right)^{\theta} \right],$$

where $\theta = \frac{3\,(p-1)}{2p} \in]0,1[$ for some $p \in]\frac{3}{3-\gamma}, 3[$ small enough (remember that $\gamma \in]0,2[$, so that such a choice is possible).

Then, denoting $q = \int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 \, dv$, we end up with the inequality

$$q \le a + D_\psi(f)\left[b + c\, C_s^\theta\, \mathscr{E}_f^{1-\theta}\, q^\theta\right].$$

Thanks to Young's inequality (applied with conjugate numbers $1/\theta$ and $1/(1-\theta)$), we get for any $d>0$

$$q^\theta \le \theta\, d\, q + (1-\theta)\, d^{-\theta/(1-\theta)}.$$

As a consequence,

$$q \le a + D_\psi(f)\, b + c\, \mathscr{E}_f^{1-\theta}\, C_s^\theta\, D_\psi(f)\, \theta\, d\, q + c\, \mathscr{E}_f^{1-\theta}\, C_s^\theta\, D_\psi(f)\, (1-\theta)\, d^{-\theta/(1-\theta)}.$$

Selecting $d = \frac{1}{2}\, c^{-1}\, \mathscr{E}_f^{\theta-1}\, C_s^{-\theta}\, D_\psi(f)^{-1}\, \theta^{-1}$, in such a way that

$$c\, \mathscr{E}_f^{1-\theta}\, C_s^\theta\, D_\psi(f)\, \theta\, d\, q = q/2,$$

we end up with the estimate

$$q \le 2a + 2 D_\psi(f)\, b + c^{1/(1-\theta)}\, \mathscr{E}_f\, C_s^{\theta/(1-\theta)}\, (1-\theta)\, \theta^{\theta/(1-\theta)}\, 2^{1/(1-\theta)}\, D_\psi(f)^{1/(1-\theta)}.$$

Recalling the definition of a, b, c, q, θ, we obtain Corollary 2.6. $\Box$

Proof of Corollary 2.7: We introduce

$$M(v) = <v>^\gamma, \qquad \phi(z) = (1+2z)^{-\frac{\gamma}{4}-\frac{1}{2}},$$

so that

$$||\phi||_\infty \le 1, \qquad ||\phi'||_\infty \le \frac{\gamma}{2} + 1.$$

Then, using Theorem 2, we see that

$$\int |\nabla \sqrt{f(v)}|^2 <v>^\gamma\, dv \le 36\, \Delta_\phi(f)^{-2}\, \mathscr{E}_f^4 \left\{ \left(3 + 8\left(\frac{\gamma}{2}+1\right)^2\right) \mathscr{E}_f^2 \int f(w) <w>^{2+\gamma}\, dw \right.$$

$$\left. + 24\, c_0^{-1}\, D_\psi(f) \sup_{v\in\mathbb{R}^3} <v>^\gamma \int f(w)\, <w>^{-\gamma}\, |v-w|^{-\gamma} dw \right\}.$$

We now estimate

$$\sup_{v\in\mathbb{R}^3} <v>^\gamma \int f(w)\, <w>^{-\gamma}\, |v-w|^{-\gamma} dw$$

$$\le 3^{\gamma/2} \sup_{v\in\mathbb{R}^3} \int f(w)\, <w>^\gamma <w>^{-\gamma}\, |v-w|^{-\gamma}\, 1_{\{|v-w|\le 1\}}\, dw$$

$$+ \sup_{v\in\mathbb{R}^3} <v>^\gamma \int f(w)\, <w>^{-\gamma}\, |v-w|^{-\gamma}\, 1_{\{|v-w|\ge 1\}}\, 1_{\{|w|\le |v|/2\}}\, dw$$

$$+ \sup_{v\in\mathbb{R}^3} <v>^\gamma \int f(w)\, <w>^{-\gamma}\, |v-w|^{-\gamma}\, 1_{\{|v-w|\ge 1\}}\, 1_{\{|w|\ge |v|/2\}}\, dw$$

$$\leq 3^{\gamma/2}\,||f*|\cdot|^{-\gamma}\,1_{\{|\cdot|\leq 1\}}||_{L^\infty} + \sup_{v\in\mathbb{R}^3} \frac{2^\gamma < v >^\gamma}{\sup(1,|v|^\gamma)}\,||f||_{L^1}$$

$$+ \sup_{v\in\mathbb{R}^3} < v >^\gamma < v/2 >^{-\gamma}\,||f||_{L^1}$$

$$\leq 3^{\gamma/2}\,||f||_{L^p} \left(\frac{4\pi(p-1)}{(3-\gamma)\,p-3}\right)^{1-1/p} + (2^{3\gamma/2}+2^\gamma)\,||f||_{L^1}.$$

Using this estimate and the previous one, we get the statement of Corollary 2.7

Proof of Proposition 3 : Observing that $\Delta_\phi(f)$ is a Grad determinant, we see that (for all $\varepsilon > 0$, $R > 0$),

$$\Delta_\phi(f) \geq \left[\inf_{\lambda^2+\mu^2+\nu^2=1} \int_{\mathbb{R}^3} \phi(|v|^2/2)\,f(v)\,|\lambda+\mu\,v_i+\nu\,v_j|^2\,dv\right]^3$$

$$\geq \varepsilon^6 \left[\inf_{\lambda^2+\mu^2+\nu^2=1} \int_{B(0,R)} \phi(|v|^2/2)\,f(v)\,1_{\{|\lambda+\mu\,v_i+\nu\,v_j|\geq\varepsilon\}}\,dv\right]^3$$

$$\geq \varepsilon^6 \inf_{B(0,R)} \phi\left(\frac{|\cdot|^2}{2}\right)^3 \left(\int_{B(0,R)} f(v)\,dv - \sup_{\lambda^2+\mu^2+\nu^2=1} \int_{B(0,R)} f(v)\,1_{\{|\lambda+\mu\,v_i+\nu\,v_j|\leq\varepsilon\}}\,dv\right)^3.$$

□

Proof of Proposition 4 : Thanks to Proposition 3, we know that for all $R > 1$, $\varepsilon \in]0,1/2[$, $A > 1$,

$$\Delta_\phi(f) \geq \varepsilon^6 \inf_{B(0,R)} \phi\left(\frac{|\cdot|^2}{2}\right)^3 \left(\int_{B(0,R)} f(v)\,dv - \sup_{\lambda^2+\mu^2+\nu^2=1} \int_{B(0,R)} f(v)\,1_{\{|\lambda+\mu\,v_i+\nu\,v_j|\leq\varepsilon\}}\,dv\right)^3.$$

$$\geq \varepsilon^6 \inf_{B(0,R)} \phi\left(\frac{|\cdot|^2}{2}\right)^3 \left(\int_{\mathbb{R}^3} f(v)\,dv - R^{-2}\int_{\mathbb{R}^3} f(v)\,|v|^2\,dv - \bar{H}\,(\ln A)^{-1} - A \sup_{\lambda^2+\mu^2+\nu^2=1} Y_{\{\lambda,\mu,\nu,R,\varepsilon\}}\right)^3,$$

where $Y_{\{\lambda,\mu,\nu,R,\varepsilon\}}$ is the Lebesgue measure of the set

$$\{v\in\mathbb{R}^3, \quad |\lambda+\mu\,v_i+\nu\,v_j|\leq\varepsilon\}\cap B(0,R),$$

and $\bar{H}$ is any constant larger than $\int f(v)\,|\ln f(v)|\,dv$.

Using a rotation, we see that $\sup_{\lambda^2+\mu^2+\nu^2=1} Y_{\{\lambda,\mu,\nu,R,\varepsilon\}} = \sup_{\lambda^2+\mu^2=1} Z_{\{\lambda,\mu,R,\varepsilon\}}$, where $Z_{\{\lambda,\mu,R,\varepsilon\}}$ is the Lebesgue measure of the set

$$\{v\in\mathbb{R}^3, \quad |\lambda+\mu\,v_1|\leq\varepsilon\}\cap B(0,R).$$

Then $Z_{\{\lambda,\mu,R,\varepsilon\}} \leq 4\,R^2\,W_{\{\lambda,\mu,R,\varepsilon\}}$, where $W_{\{\lambda,\mu,R,\varepsilon\}}$ is the one-dimensional Lebesgue measure of the set

$$\{v_1\in\mathbb{R}, \quad |\lambda+\mu\,v_1|\leq\varepsilon\}\cap B(0,R).$$

As a consequence, for any $\mu_0 > 0$, and $|\mu| \le \mu_0$, $|v_1| \le R$, λ, μ such that $\lambda^2 + \mu^2 = 1$,

$$\begin{aligned}|\lambda + \mu\, v_1| &\ge |\lambda| - |\mu_0| R \\ &\ge \sqrt{1-\mu_0^2} - \mu_0 R \\ &\ge 1 - \mu_0 - \mu_0 R \\ &\ge 1 - 2R\,\mu_0.\end{aligned}$$

Taking $\mu_0 = \frac{1-\varepsilon}{2R}$, we see that $W_{\{\lambda,\mu,R,\varepsilon\}} = 0$ if $|\mu| \le \mu_0$.

Then, for $|\mu| \ge \mu_0$, $W_{\{\lambda,\mu,R,\varepsilon\}} \le \frac{2\varepsilon}{\mu_0}$, so that finally

$$W_{\{\lambda,\mu,R,\varepsilon\}} \le \frac{4\varepsilon R}{1-\varepsilon} \le 8\varepsilon R,$$

and

$$Z_{\{\lambda,\mu,R,\varepsilon\}} \le 32 R^3\, \varepsilon.$$

Taking

$$R = \sup\left(1, 2\left(\frac{\int f(v)\, v^2\, dv}{\int f(v)\, dv}\right)^{1/2}\right),$$

we see that

$$R^{-2}\int f(v)\, v^2\, dv \le \frac{1}{4}\int f(v)\, dv.$$

Then choosing

$$A = \exp\left(\frac{4\bar{H}}{\int f(v)\, dv}\right),$$

we also see that

$$\frac{\bar{H}}{\ln A} = \frac{1}{4}\int f(v)\, dv.$$

Finally, considering

$$\varepsilon = \inf\left[2^{-1}, 2^{-7}\int f(v)\, dv\, \exp\left(-\frac{4\bar{H}}{\int f(v)\, dv}\right)\, \sup\left(1, 2^{-3}\left(\frac{\int f(v)\, dv}{\int f(v)\, v^2\, dv}\right)^{3/2}\right)\right],$$

we obtain the inequality

$$32 R^3\, \varepsilon A \le \frac{1}{4}\int f(v)\, dv.$$

We end up with estimate (24).

This ends the proof of Proposition 4. □

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