A REMARK CONCERNING
THE CHAPMAN-ENSKOG ASYMPTOTICS

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We present in this work a remark precising some properties of the solutions of the linearized Boltzmann equation coming out of the Chapman–Enskog asymptotics.

1. Introduction

In order to understand the Chapman-Enskog asymptotics of the Boltzmann equation (Cf. [Ch, Co], [Ce], [Ba], [Ka, Ma, Ni]), one has to study the solutions h_i and g_{ij} of the following equations (when $v \in \mathbb{R}^3$):

$$(Lh_i)(v) = A_i(v), \tag{1.1}$$

$$\int_{v \in \mathbb{R}^3} h_i(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv = 0, \tag{1.2}$$

and

$$(Lg_{ij})(v) = B_{ij}(v), \tag{1.3}$$

$$\int_{v\in I\!R^3} g_{ij}(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv = 0, \tag{1.4}$$

where A_i and B_{ij} are the Sonine polynomials defined for $i, j \in \{1, 2, 3\}$ by

$$A_i(v) = \{\frac{|v|^2}{2} - \frac{5}{2}\} v_i, \tag{1.5}$$

$$B_{ij}(v) = v_i v_j - \frac{|v|^2}{3} \delta_{ij}, \tag{1.6}$$

and L is the linearized Boltzmann operator:

$$(Lf)(v) = \int_{v_*\in I\!R^3} \int_{\sigma\in S^2} e^{-\frac{|v_*|^2}{2}} \{f(\frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma) + f(\frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma)$$

$$-f(v) - f(v_*)\} B(|v-v_*|, \sigma\cdot\frac{v-v_*}{|v-v_*|}) d\sigma dv_*$$

$$= \int_{v_*\in I\!R^3} \int_{\sigma\in S^2} e^{-\frac{|v_*|^2}{2}} \{2f(\frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma) - f(v) - f(v_*)\}$$

$$B(|v-v_*|, \sigma\cdot\frac{v-v_*}{|v-v_*|}) d\sigma dv_*. \tag{1.7}$$

The cross section B occuring in (1.7) can be written under the form

$$B(x, y) = x^\alpha \beta_\alpha(y), \tag{1.8}$$

with $\alpha \in [-3, 1[$ when the interactions between particles satisfy the inverse power law of order α with or without angular cut-off (Cf. [Gr]).

The theory of the linearized Boltzmann equation yields the following theorem (Cf. [Ba], [Ce]) (at least when $\alpha > 0$ in (1.8) and with the hypothesis of angular cut–off):

Theorem 1.1: *There exists a unique solution h_i, $g_{i,j}$ to equations (1.1)-(1.2) and (1.3)-(1.4) which belongs to*

$$\mathcal{D}(L) = \{f(v) \in L^2(I\!R^3, e^{-\frac{|v|^2}{2}}dv), \quad \nu(v)f(v) \in L^2(I\!R^3, e^{-\frac{|v|^2}{2}}dv)\}, \tag{1.9}$$

where

$$\nu(v) = \int_{v_*\in I\!R^3} \int_{\sigma\in S^2} e^{-\frac{|v_*|^2}{2}} B(|v-v_*|, \sigma\cdot\frac{v-v_*}{|v-v_*|}) d\sigma dv_*. \tag{1.10}$$

Note that this theorem holds because the vector A with components A_i (for $i \in \{1,2,3\}$) and the tensor B with components B_{ij} (for $i,j \in \{1,2,3\}$) satisfy the orthogonality relations

$$\int_{v\in \mathbb{R}^3} A(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv = 0, \tag{1.11}$$

$$\int_{v\in \mathbb{R}^3} B(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv = 0. \tag{1.12}$$

We shall denote from now on

$$h(v) = (h_i(v))_{i\in\{1,2,3\}} \tag{1.13}$$

the vector whose components are the solutions of (1.1)-(1.2) and

$$g(v) = (g_{i,j}(v))_{i,j\in\{1,2,3\}} \tag{1.14}$$

the tensor whose components are solutions of (1.3)-(1.4).

In the second section of this work, we prove rigorously that h et g are of the form given in [Ch, Co]. Then, in a third section, we introduce a unique equation replacing the three equations (1.1), and another unique equation replacing (1.3).

2. Isometric Invariance of the Solutions of the Linearized Boltzmann Equation

We want to prove in this section the following property of the solutions of (1.1)-(1.2) and (1.3)-(1.4):

Theorem 2.1: *The solutions of (1.1)-(1.2) can be written under the form*

$$h_i(v) = a(|v|)v_i, \tag{2.1}$$

and the solutions of (1.3)-(1.4) can be written under the form

$$g_{ij}(v) = b(|v|)B_{ij}(v), \tag{2.2}$$

where $a, b : \mathbb{R}_+ \longrightarrow \mathbb{R}$ are such that

$$\forall i \in \{1,2,3\}, \qquad a(|v|)v_i \in \mathcal{D}(L), \tag{2.3}$$

and

$$\forall i,j \in \{1,2,3\}, \qquad b(|v|)v_iv_j \in \mathcal{D}(L). \tag{2.4}$$

This theorem is given without any proof in [Ni] and [Ba], with the reference [Ch, Co]. In this book, it is physically justified. We prove in this work that it is a consequence of the following lemmas (some of which are well-known).

Lemma 1: *We introduce for each isometry $R \in O(\mathbb{R}^3)$ and for each function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ the operator T_R defined by*

$$T_R f(v) = f(Rv). \tag{2.5}$$

Then,

$$L \circ T_R = T_R \circ L. \tag{2.6}$$

Proof: We consider

$$[(L \circ T_R)f](v) = (L(T_R f))(v)$$

$$= \int_{v_* \in \mathbb{R}^3} \int_{\sigma \in S^2} e^{-\frac{|v_*|^2}{2}} \{2f(R\{\frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma\}) - f(Rv) - f(Rv_*)\}$$

$$B(|v-v_*|, \sigma \cdot \frac{v-v_*}{|v-v_*|}) d\sigma dv_*$$

$$= \int_{v_* \in \mathbb{R}^3} \int_{\sigma \in S^2} e^{-\frac{|Rv_*|^2}{2}} \{2f(\{\frac{Rv+Rv_*}{2} + \frac{|Rv-Rv_*|}{2}R\sigma\}) - f(Rv) - f(Rv_*)\}$$

$$B(|Rv-Rv_*|, R\sigma \cdot \frac{Rv-Rv_*}{|Rv-Rv_*|}) d\sigma dv_*. \tag{2.7}$$

We introduce now the change of variables

$$\omega = R\sigma, \tag{2.8}$$

$$w_* = Rv_*, \tag{2.9}$$

and we get

$$[(L \circ T_R)f](v) = \int_{w_* \in \mathbb{R}^3} \int_{\omega \in S^2} e^{-\frac{|w_*|^2}{2}} \{2f(\{\frac{Rv+w_*}{2} + \frac{|Rv-w_*|}{2}\omega\})$$

$$-f(Rv) - f(w_*)\} B(|Rv-w_*|, \omega \cdot \frac{Rv-w_*}{|Rv-w_*|}) d\omega dw_*$$

$$= (Lf)(Rv)$$

$$= [(T_R \circ L)f](v). \tag{2.10}$$

Lemma 2: *For all isometry $R \in O(I\!R^3)$, The function h defined by (1.13) satisfies:*

$$(T_R h)(v) = Rh(v). \tag{2.11}$$

Moreover, the function g defined by (1.14) satisfies the following properties:
i) for all v in $I\!R^3$, $g(v)$ is a symetric tensor with zero trace,
ii) for all isometry $R \in O(I\!R^3)$,

$$(T_R g)(v) = Rg(v)R^{-1} \tag{2.12}$$

in the sense of the products of matrices.

Proof: We note that, according to lemma 1,

$$\begin{aligned} L(T_R h) &= T_R(Lh) \\ &= T_R A \\ &= R \circ A, \end{aligned} \tag{2.13}$$

and that

$$\begin{aligned} L(R \circ h) &= R \circ (Lh) \\ &= R \circ A. \end{aligned} \tag{2.14}$$

Moreover,

$$\int_{v \in I\!R^3} (T_R h)(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv = 0 \iff \int_{v \in I\!R^3} h(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv = 0$$

$$\iff R\{\int_{v \in I\!R^3} h(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv\} = 0$$

$$\iff \int_{v \in I\!R^3} (R \circ h)(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv = 0. \tag{2.15}$$

Using now the uniqueness of the solutions p of the system

$$Lp = R \circ A, \tag{2.16}$$

$$\int_{v\in I\!R^3} p(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv = 0, \tag{2.17}$$

we get (2.11).

We now turn to the tensor g. In order to get i), we note that

$$L(Trg) = Tr(Lg) = TrB = 0, \tag{2.18}$$

$$L(g + g^T) = Lg + (Lg)^T = B + B^T = 0, \tag{2.19}$$

$$\int_{v\in I\!R^3} (Trg)(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv = Tr\{\int_{v\in I\!R^3} g(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv\} = 0, \tag{2.20}$$

$$\int_{v\in I\!R^3} (g + g^T)(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv = \int_{v\in I\!R^3} g(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv +$$

$$\{\int_{v\in I\!R^3} g(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv\}^T = 0, \tag{2.21}$$

and we get

$$Trg = 0, \tag{2.22}$$

$$g + g^T = 0, \tag{2.23}$$

by a uniqueness argument.

Finally, we prove ii). We note that

$$\begin{aligned} L(T_R g) &= T_R(Lg) \\ &= T_R B \\ &= RBR^{-1}, \end{aligned} \tag{2.24}$$

and that

$$\begin{aligned} L(RgR^{-1}) &= R(Lg)R^{-1} \\ &= RBR^{-1}. \end{aligned} \tag{2.25}$$

Moreover,

$$\int_{v\in \mathbb{R}^3} (T_R g)(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv = 0 \iff \int_{v\in \mathbb{R}^3} g(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv = 0$$

$$\iff \int_{v\in \mathbb{R}^3} Rg(v)R^{-1} \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv = 0. \tag{2.26}$$

Therefore, the uniqueness of the solutions q of the system

$$Lq = RBR^{-1}, \tag{2.27}$$

$$\int_{v\in \mathbb{R}^3} q(v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} e^{-\frac{|v|^2}{2}} dv = 0, \tag{2.28}$$

ensures that ii) holds.

Lemma 3: *Let $N \geq 2$ and $s : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a function such that for all isometry R of $O(\mathbb{R}^N)$, one has:*

$$s \circ R = R \circ s. \tag{2.29}$$

Then, there exists $t : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that

$$\forall x \in \mathbb{R}^N, \quad s(x) = t(|x|)x. \tag{2.30}$$

Preuve: We begin with the case when $N = 2$. Note that $|s| : \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ satisfies

$$|s| \circ R = |s|. \tag{2.31}$$

Therefore, $|s|$ depends only on $|x|$.

We write

$$|s(x)| = w(|x|), \tag{2.32}$$

and

$$s(x) = w(|x|)u(x), \tag{2.33}$$

where $u : \mathbb{R}^2 \longrightarrow S^1$ still satisfies

$$u \circ R = R \circ u. \tag{2.34}$$

Identifying $I\!R^2$ to $\mathbb{C}$ and S^1 to the set of complex numbers of modulus 1, we get a function $\phi : I\!R^+ \longrightarrow I\!R$, such that

$$u(re^{i\theta}) = e^{i\{\theta+\phi(r)\}}. \qquad (2.35)$$

Using now (2.34) when R is the symetry with real axis, we get

$$u(re^{-i\theta}) = e^{i\{-\theta+\phi(r)\}} = e^{i\{-\theta-\phi(r)\}}. \qquad (2.36)$$

Finally,

$$\phi(r) = 0 \text{ or } \pi, \qquad (2.37)$$

and therefore

$$u(x) = v(|x|)x, \qquad (2.38)$$

with $v : I\!R^+ \longrightarrow \{-1, 1\}$. Denoting

$$t = wv, \qquad (2.39)$$

we get (2.30).

We are now interested by the case when $N = 3$ (the proof is the same when $N > 3$). We write once again (2.33) and (2.34) with $u : I\!R^3 \longrightarrow S^2$.

Let $x \in I\!R^3$, and $\mathcal{P}$ be the plane containing $0, x$, and $u(x)$. The intersection of $\mathcal{P}$ and of the sphere of center 0 and of radius $|x|$ is a cercle C. The intersection of $\mathcal{P}$ with the sphere of center 0 and radius 1 is also a cercle C'. We denote by $\mathcal{R}$ the set of all rotations whose axis is orthogonal to $\mathcal{P}$ and of planar symetries whose symetry plane is orthogonal to $\mathcal{P}$. According to (2.34), used in the case when $R \in \mathcal{R}$, we see that u maps $\mathcal{C}$ into $\mathcal{C}'$. Therefore, we can apply the result obtained in dimension 2, and get (2.30) (because $\mathcal{R}$ restricted to $\mathcal{P}$ contains all the isometries of $\mathcal{P}$).

Remark: One can prove in fact that for $N \geq 3$, it is enough to have (2.29) when $R \in SO(I\!R^N)$ in order to get (2.30), since every isometries of $O(I\!R^2)$ can be obtained as restrictions to planes of isometries of $SO(I\!R^N)$.

Lemma 4: *Let $N \geq 2$ and $m : I\!R^N \longrightarrow M_n(I\!R)$ be a function such that for all isometry R of $O(I\!R^N)$, one has:*

$$\forall x \in I\!R^N, \quad m(Rx) = Rm(x)R^{-1}. \qquad (2.40)$$

We suppose moreover that for all x in $I\!R^N$, $m(x)$ is a symetric matrix with zero trace.

Then, there exists $n : I\!R^+ \longrightarrow I\!R$ such that

$$\forall x \in I\!R^N, \quad m(x) = n(|x|)\{x \otimes x - \frac{|x|^2}{N} Id\}. \qquad (2.41)$$

Proof: We only look to the case when $N = 3$, since the proof is the same for $N > 3$. We compute $m(r \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix})$ for $r \in I\!R$. For each isometry $R \in O(I\!R^3)$ mapping the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ on itself, we have

$$m(r \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = Rm(r \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix})R^{-1}. \tag{2.42}$$

Suppose now that

$$m(r \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = \begin{pmatrix} x & \alpha & \beta \\ \alpha & y & \gamma \\ \beta & \gamma & z \end{pmatrix}. \tag{2.43}$$

Using (2.42) with

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.44}$$

we get

$$\alpha = \gamma = 0. \tag{2.45}$$

Using now (2.42) with

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.46}$$

we also get

$$\beta = 0. \tag{2.47}$$

Finally, using (2.42) with

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \tag{2.48}$$

we get

$$y = z. \tag{2.49}$$

Since we assumed that

$$x + y + z = 0, \tag{2.50}$$

we obtain $n' : I\!R \longrightarrow I\!R$ such that

$$m(r \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = n'(r) \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.51}$$

Finally,

$$\forall x \in I\!\!R^3, \quad m(x) = m(-x), \tag{2.52}$$

and therefore

$$m(r\begin{pmatrix}1\\0\\0\end{pmatrix}) = n'(|r|)\begin{pmatrix}-2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = -3n'(|r|)\{\begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{1}{3}Id\}. \tag{2.53}$$

We define now $n(r)$ by

$$n(r) = -3\frac{n'(r)}{r^2}. \tag{2.54}$$

For all isometry R of $O(I\!\!R^3)$, we get:

$$f(rR\begin{pmatrix}1\\0\\0\end{pmatrix}) = Rf(r\begin{pmatrix}1\\0\\0\end{pmatrix})R^{-1}$$

$$= r^2 n(|r|)R\{\begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{1}{3}Id\}R^{-1}$$

$$= n(|r|)\{rR\begin{pmatrix}1\\0\\0\end{pmatrix} \otimes rR\begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{r^2}{3}Id\}. \tag{2.55}$$

But every vectors y of $I\!\!R^N$ can be put under the form

$$y = R(r\begin{pmatrix}1\\0\\0\end{pmatrix}), \tag{2.56}$$

and therefore lemma 4 holds.

The proof of theorem 2.1 is then a straightforward consequence of lemmas 2 et 3 on one hand, and 2 and 4 on the other hand.

3. The Equation Giving a and b

In this section, we give a unique equation allowing us to get the functions a and b of theorem 2. Such an equation is necessary to study the asymptotic properties of a and b when $|v|$ tends to infinity. Note that such a study is useful for the rigourous analysis of the incompressible fluid dynamics limit of the Boltzmann equation (Cf [Ba, Go, Le]).

We prove here the

Theorem 3.1: *The function a of theorem 2.1 is solution of*

$$(Ma)(y) = \{\frac{y^2}{2} - \frac{5}{2}\}y, \tag{3.1}$$

$$\int_{y=0}^{+\infty} a(y)y^4 e^{-\frac{y^2}{2}} dy = 0, \tag{3.2}$$

where

$$(Ma)(y) = 2\pi \int_{r=0}^{+\infty} \int_{\theta=0}^{\pi} \int_{k=0}^{\pi} \int_{\phi=0}^{2\pi}$$
$$\{a(\sqrt{\frac{r^2+y^2}{2} + \frac{1}{2}\sqrt{r^2+y^2-2ry\cos\theta}[y\cos k + rU]})$$
$$(y + r\cos\theta + \sqrt{r^2+y^2-2ry\cos\theta}\cos k) - a(r)r\cos\theta - a(y)y\}$$
$$B(\sqrt{r^2+y^2-2ry\cos\theta}, y\cos k - rU)\ \sin\theta\sin k\ r^2 e^{-\frac{|r|^2}{2}} d\phi dk d\theta dr, \tag{3.3}$$

and

$$U = \cos\theta\cos k + \sin\theta\sin k\cos\phi. \tag{3.4}$$

Proof: Equation (3.2) comes clearly out of (1.2).

In order to get (3.1) and (3.3), we write (1.1) for $i = 1$ and we introduce the following notations:

$$v = y\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad v_* = r\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\end{pmatrix}, \quad \sigma = \begin{pmatrix}\cos k\\ \sin k\cos l\\ \sin k\sin l\end{pmatrix}. \tag{3.5}$$

we get

$$\int_{r=0}^{+\infty}\int_{\theta=0}^{\pi}\int_{k=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{l=0}^{2\pi}\{a(\sqrt{\frac{r^2+y^2}{2} + \frac{1}{2}\sqrt{r^2+y^2-2ry\cos\theta}[y\cos k + rV]})$$
$$(y\begin{pmatrix}1\\0\\0\end{pmatrix} + r\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\end{pmatrix} + \sqrt{r^2+y^2-2ry\cos\theta}\begin{pmatrix}\cos k\\ \sin k\cos l\\ \sin k\sin l\end{pmatrix})$$
$$-a(r)r\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\end{pmatrix} - a(y)y\begin{pmatrix}1\\0\\0\end{pmatrix}\}$$
$$B(\sqrt{r^2+y^2-2ry\cos\theta}, y\cos k - rV)\ \sin\theta\sin k\ r^2 e^{-\frac{|r|^2}{2}} dl d\phi dk d\theta dr$$
$$= \{\frac{y^2}{2} - \frac{5}{2}\}y\begin{pmatrix}1\\0\\0\end{pmatrix}, \tag{3.6}$$

with

$$V = \cos\theta\cos k + \sin\theta\sin k\cos(\phi - l). \tag{3.7}$$

In order to get (3.1) and (3.3), it is enough to note that for all functions γ,

$$\int_{\phi=0}^{2\pi}\int_{l=0}^{2\pi}\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\end{pmatrix}\gamma(\cos(\phi-l))d\phi dl = 2\pi\cos\theta\int_{\phi=0}^{2\pi}\gamma(\cos\phi)d\phi\begin{pmatrix}1\\0\\0\end{pmatrix}, \tag{3.8}$$

$$\int_{\phi=0}^{2\pi}\int_{l=0}^{2\pi}\begin{pmatrix}\cos k\\ \sin k\cos l\\ \sin k\sin l\end{pmatrix}\gamma(\cos(\phi-l))d\phi dl = 2\pi\cos k\int_{\phi=0}^{2\pi}\gamma(\cos\phi)d\phi\begin{pmatrix}1\\0\\0\end{pmatrix}, \tag{3.9}$$

$$\int_{\phi=0}^{2\pi}\int_{l=0}^{2\pi}\begin{pmatrix}1\\0\\0\end{pmatrix}\gamma(\cos(\phi-l))d\phi dl = 2\pi\int_{\phi=0}^{2\pi}\gamma(\cos\phi)d\phi\begin{pmatrix}1\\0\\0\end{pmatrix}. \tag{3.10}$$

We now give the

Theorem 3.2: *The function b of theorem 2.1 is solution of*

$$(Nb)(y) = y^2, \tag{3.11}$$

where

$$(Nb)(y) = 2\pi\int_{r=0}^{+\infty}\int_{\theta=0}^{\pi}\int_{k=0}^{\pi}\int_{\phi=0}^{2\pi}$$

$$\{\frac{1}{2}b(\sqrt{\frac{r^2+y^2}{2}+\frac{1}{2}\sqrt{r^2+y^2-2ry\cos\theta}[y\cos k + rU]})(y^2 + r^2(\frac{3}{2}\cos^2\theta - \frac{1}{2})$$

$$+(r^2+y^2-2ry\cos\theta)(\frac{3}{2}\cos^2 k - \frac{1}{2}) + 2yr\cos\theta + 2y\sqrt{r^2+y^2-2ry\cos\theta}\cos\theta$$

$$+2r\sqrt{r^2+y^2-2ry\cos\theta}(\cos k\cos\theta - \frac{1}{2}\sin k\sin\theta\cos\phi)) - b(r)r^2(\frac{3}{2}\cos^2\theta - \frac{1}{2})$$

$$-b(y)y^2\}B(\sqrt{r^2+y^2-2ry\cos\theta}, y\cos k - rU)\ \sin\theta\sin k\ r^2 e^{-\frac{|r|^2}{2}}d\phi dk d\theta dr. \tag{3.12}$$

Proof: In order to get (3.11) et (3.12), we write (1.3) for $i=1, j=1$ and we use the notations (3.5). We get

$$\int_{r=0}^{+\infty}\int_{\theta=0}^{\pi}\int_{k=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{l=0}^{2\pi}\{\frac{1}{2}b(\sqrt{\frac{r^2+y^2}{2}+\frac{1}{2}\sqrt{r^2+y^2-2ry\cos\theta}[y\cos k + rV]})$$

$$(y^2\{\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{1}{3}Id\} + r^2\{\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\end{pmatrix}\otimes\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\end{pmatrix} - \frac{1}{3}Id\}$$

$$+(r^2+y^2-2ry\cos\theta)\{\begin{pmatrix}\cos k\\ \sin k\cos l\\ \sin k\sin l\end{pmatrix}\otimes\begin{pmatrix}\cos k\\ \sin k\cos l\\ \sin k\sin l\end{pmatrix}-\frac{1}{3}Id\}$$

$$+2yr\{\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\end{pmatrix}-\frac{\cos\theta}{3}Id\}$$

$$+2y\sqrt{r^2+y^2-2ry\cos\theta}\{\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes\begin{pmatrix}\cos k\\ \sin k\cos l\\ \sin k\sin l\end{pmatrix}-\frac{\cos k}{3}Id\}$$

$$+2r\sqrt{r^2+y^2-2ry\cos\theta}\{\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\end{pmatrix}\otimes\begin{pmatrix}\cos k\\ \sin k\cos l\\ \sin k\sin l\end{pmatrix}-\frac{V}{3}Id\})$$

$$-b(r)r^2\{\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\end{pmatrix}\otimes\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\end{pmatrix}-\frac{1}{3}Id\}-b(y)y^2\{\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\\0\end{pmatrix}-\frac{1}{3}Id\}\}$$

$$B(\sqrt{r^2+y^2-2ry\cos\theta},y\cos k-rV)\ \sin\theta\sin kr^2\ e^{-\frac{|r|^2}{2}}dld\phi dkd\theta dr$$

$$=y^2\{\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\\0\end{pmatrix}-\frac{1}{3}Id\}. \tag{3.13}$$

In order to get (3.11) and (3.12), it is enough to note that for all functions γ,

$$\int_{\phi=0}^{2\pi}\int_{l=0}^{2\pi}\{\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\\0\end{pmatrix}-\frac{1}{3}Id\}\gamma(\cos(\phi-l))d\phi dl$$

$$=2\pi\int_{\phi=0}^{2\pi}\gamma(\cos\phi)d\phi\begin{pmatrix}2/3&0&0\\0&-1/3&0\\0&0&-1/3\end{pmatrix}, \tag{3.14}$$

$$\int_{\phi=0}^{2\pi}\int_{l=0}^{2\pi}\{\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\end{pmatrix}\otimes\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\end{pmatrix}-\frac{1}{3}Id\}\gamma(\cos(\phi-l))d\phi dl$$

$$=2\pi(\frac{3}{2}\cos^2\theta-\frac{1}{2})\int_{\phi=0}^{2\pi}\gamma(\cos\phi)d\phi\begin{pmatrix}2/3&0&0\\0&-1/3&0\\0&0&-1/3\end{pmatrix}, \tag{3.15}$$

$$\int_{\phi=0}^{2\pi}\int_{l=0}^{2\pi}\{\begin{pmatrix}\cos k\\ \sin k\cos l\\ \sin k\sin l\end{pmatrix}\otimes\begin{pmatrix}\cos k\\ \sin k\cos l\\ \sin k\sin l\end{pmatrix}-\frac{1}{3}Id\}\gamma(\cos(\phi-l))d\phi dl$$

$$=2\pi(\frac{3}{2}\cos^2 k-\frac{1}{2})\int_{\phi=0}^{2\pi}\gamma(\cos\phi)d\phi\begin{pmatrix}2/3&0&0\\0&-1/3&0\\0&0&-1/3\end{pmatrix}, \tag{3.16}$$

$$\int_{\phi=0}^{2\pi}\int_{l=0}^{2\pi}\{\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\end{pmatrix}-\frac{\cos\theta}{3}Id\}\gamma(\cos(\phi-l))d\phi dl$$

$$=2\pi\cos\theta\int_{\phi=0}^{2\pi}\gamma(\cos\phi)d\phi\begin{pmatrix}2/3&0&0\\0&-1/3&0\\0&0&-1/3\end{pmatrix}, \tag{3.17}$$

$$\int_{\phi=0}^{2\pi}\int_{l=0}^{2\pi}\{\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes\begin{pmatrix}\cos k\\ \sin k\cos l\\ \sin k\sin l\end{pmatrix}-\frac{\cos k}{3}Id\}\gamma(\cos(\phi-l))d\phi dl$$

$$=2\pi\cos k\int_{\phi=0}^{2\pi}\gamma(\cos\phi)d\phi\begin{pmatrix}2/3&0&0\\0&-1/3&0\\0&0&-1/3\end{pmatrix}, \tag{3.18}$$

$$\int_{\phi=0}^{2\pi}\int_{l=0}^{2\pi}\{\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\end{pmatrix}\otimes\begin{pmatrix}\cos k\\ \sin k\cos l\\ \sin k\sin l\end{pmatrix}-\frac{V}{3}Id\}\gamma(\cos(\phi-l))d\phi dl$$

$$=2\pi\int_{\phi=0}^{2\pi}(\cos k\cos\theta-\frac{1}{2}\sin k\sin\theta\cos\phi)\gamma(\cos\phi)d\phi\begin{pmatrix}2/3&0&0\\0&-1/3&0\\0&0&-1/3\end{pmatrix}. \tag{3.19}$$

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