Chapter 6. Plasma kinetic models : the Fokker-Planck-Landau equation

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Summary. In this work, we present an approach for the Landau equation based on the relationship between entropy and entropy dissipation. Thanks to the same estimate, we recover on one hand an explicit bound on the long time behavior of the spatially homogeneous equation, and on the other hand the strong L^1 compactness of the solutions of the spatially inhomogeneous equation.

1 Introduction

1.1 Presentation of Landau's kernel

Different forms of the kernel

We study in this paper a quadratic collision kernel for plasmas, which models the binary grazing collisions between charged particles, usually called Landau's (or Fokker-Planck-Landau's) operator (Cf. [21]).

If $f \equiv f(v) \geq 0$ is the density of particles with velocity $v \in \mathbb{R}^N$, the evolution of f due to those collisions (sometimes denoted by $\left(\frac{\partial f}{\partial t}\right)_{coll}(v)$) is given by the kernel :

$$L_\Phi(f)(v) = \nabla_v \cdot \int_{v_* \in \mathbb{R}^N} \Phi(|v - v_*|^2) \left\{ |v - v_*|^2 \, Id - (v - v_*) \otimes (v - v_*) \right\}$$
$$\left\{ f(v_*) \, \nabla f(v) - f(v) \, \nabla f(v_*) \right\} dv_*, \tag{1}$$

with $N = 3$ and $\Phi(|x|^2) = |x|^{-3}$.

This kernel can also be rewritten as a parabolic operator :

$$L_\Phi(f)(v) = \nabla_v \cdot \left((a_\Phi * f) \, \nabla_v f - (b_\Phi * f) \, f \right),$$

$$= (a_\Phi * f) : \nabla_v \nabla_v f - (c_\Phi * f)\, f,$$

with

$$a_\Phi(x) = \Phi(|x|^2) \Big\{ |x|^2\, Id - x \otimes x \Big\},$$

$$b_\Phi(x) = \nabla \cdot a_\Phi(x) = -\,(N-1)\,\Phi(|x|^2)\,x,$$

$$c_\Phi(x) = \nabla \cdot b_\Phi(x) = -2\,(N-1)\,\Phi'(|x|^2)\,x^2 - N(N-1)\,\Phi(|x|^2).$$

Note that we used here (and we shall use in the sequel) the notation

$$A : B = \sum_{i,j} A_{ij}\, B_{ji}$$

when A and B are $N \times N$ matrices.

Under this form, the Landau operator is reminiscent of the linear Fokker-Planck kernel

$$FP(f)(v) = \nabla_v \cdot \Big(\nabla_v f(v) + v\, f(v) \Big). \tag{2}$$

However, under the form (1), its quadratic, non-local aspect is rather reminiscent of Boltzmann's kernel (Cf. [6]) :

$$Q_B(f)(v) = \int\int\int_{v_*, v', v'_* \in \mathbb{R}^N} \Big\{ f(v')\, f(v'_*) - f(v)\, f(v_*) \Big\}$$

$$\times B(|v - v_*|, (v - \widehat{v_*, v'} - v'_*))\, \delta_{v+v_* = v'+v'_*}\, \delta_{|v|^2+|v_*|^2 = |v'|^2+|v'_*|^2}\, dv' dv'_* dv_*$$

which can be parametrized by

$$Q_B(f)(v) = \int_{v_* \in \mathbb{R}^N} \int_{\sigma \in S^{N-1}} \Big\{ f\Big(\frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma \Big)\, f\Big(\frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma \Big)$$

$$- f(v)\, f(v_*) \Big\}\, B(|v - v_*|, \theta)\, d\sigma dv_*,$$

with $\cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma$.

Many weak forms of the kernel L_Φ are useful. We shall use in particular the following ones (valid when f, ϕ are smooth enough) :

$$\int L_\Phi(f)(v)\, \phi(v)\, dv = -\int_{v \in \mathbb{R}^N} \int_{v_* \in \mathbb{R}^N} \Phi(|v-v_*|^2) \Big\{ |v-v_*|^2\, Id - (v-v_*) \otimes (v-v_*) \Big\}$$

$$\Big(f(v_*)\, \nabla f(v) - f(v)\, \nabla f(v_*), \nabla \phi(v) \Big)\, dv dv_*$$

$$= -\frac{1}{2} \int_{v \in \mathbb{R}^N} \int_{v_* \in \mathbb{R}^N} \Phi(|v - v_*|^2) \Big\{ |v - v_*|^2\, Id - (v - v_*) \otimes (v - v_*) \Big\}$$

$$\Big(f(v_*)\, \nabla f(v) - f(v)\, \nabla f(v_*), \nabla \phi(v) - \nabla \phi(v_*) \Big)\, dv dv_*$$

$$= \int_{v \in \mathbb{R}^N} \Big\{ \nabla \nabla \phi(v) : (a_\Phi * f)(v)\, f(v) + 2\, \nabla \phi(v) \cdot (b_\Phi * f)(v)\, f(v) \Big\}\, dv.$$

In those formulas, we have used the notation $M\,(x, x)$ for $x^T\, M\, x$ when M is a symmetric matrix.

Relationship with other collision kernels

The link between the Boltzmann and the Landau collision kernels is described for example in [7]. One has (at least formally, that is, when $f \in C_c^2$) $L_\Phi(f) = \lim_{\varepsilon \to 0} Q_{B_\varepsilon}(f)$, when B_ε concentrates on the grazing collisions. This is obtained for example thanks to the scaling (Cf. [11]):

$$B_\varepsilon(|v - v_*|, \theta) = \varepsilon^{-3}\, B\bigg(|v - v_*|, \frac{|\theta|}{\varepsilon} \bigg).$$

The link between Φ and B is then given by

$$\Phi(|v - v_*|^2) = C \int_{\theta = -\pi}^{\pi} \theta^2\, B(|v - v_*|, |\theta|)\, d\theta,$$

where C is some strictly positive constant.

Another scaling, adapted to the Coulomb case, is explained in [10]. The two approaches are unified and generalized in [1].

A simple computation illustrating this link is made in dimension 2, and starts from the weak formulation of Boltzmann's kernel (written here with a slightly different parametrization) :

$$\int Q_{B_\varepsilon}(f)(v)\, \phi(v)\, dv = \int_{v \in \mathbb{R}^2} \int_{v_* \in \mathbb{R}^2} \int_{\theta = -\pi}^{\pi} f(v)\, f(v_*)$$

$$\times \bigg(\phi\bigg(\frac{v + v_*}{2} + R_{-\theta}\Big(\frac{v - v_*}{2}\Big) \bigg) - \phi(v) \bigg)\, B_\varepsilon(|v - v_*|, |\theta|)\, d\theta dv dv_*,$$

where $R_{-\theta}$ denotes the rotation of angle $-\theta$. It uses the following asymptotic formula (where $x^\perp$ denotes $R_{\pi/2} x$) :

$$\phi\bigg(\frac{v + v_*}{2} + \cos(\varepsilon\theta)\, \frac{v - v_*}{2} - \sin(\varepsilon\theta)\, \frac{(v - v_*)^\perp}{2} \bigg) - \phi(v)$$

$$= -\, \varepsilon\theta\, \frac{(v - v_*)^\perp}{2} \cdot \nabla \phi(v) - \frac{\varepsilon^2 \theta^2}{2}\, \frac{v - v_*}{2} \cdot \nabla \phi(v)$$

$$+ \frac{\varepsilon^2 \theta^2}{2}\, \frac{(v - v_*)^\perp}{2} \otimes \frac{(v - v_*)^\perp}{2} : \nabla \nabla \phi(v) + O(\varepsilon^3).$$

For more details, we refer to [7] and [11].

The link between Landau's kernel and the linear Fokker–Planck operator is described in [28]. One considers the important particular case when $\Phi = \frac{\Phi_0}{N-1}$, where $\Phi_0(|v - v_*|^2) = 1$ is the so-called Maxwellian molecules cross section. Then,

$$a_{ij,\Phi} = \frac{1}{N-1}\bigg(|v|^2\,\delta_{ij} - v_i v_j\bigg), \quad b_{i,\Phi} = -v_i, \quad c_\Phi = -N.$$

Supposing now that f is radially symmetric and that it satisfies the following normalization (those properties are propagated by the spatially homogeneous flow) :

$$\int f(v_*) \begin{pmatrix} 1 \\ v_* \\ |v_*|^2 \end{pmatrix} dv_* = \begin{pmatrix} 1 \\ 0 \\ N \end{pmatrix},$$

we get

$$a_{ij,\Phi} * f = \frac{1}{N-1}(|v|^2\,\delta_{ij} - v_i v_j) + \delta_{ij},$$

$$b_{i,\Phi} * f = -v_i, \quad c_\Phi * f = -N.$$

Noticing that $\nabla f(v)$ is parallel to v (remember that f is radially symmetric), we obtain

$$\sum_j (|v|^2\,\delta_{ij} - v_i v_j)\,\partial_j f = 0,$$

and finally

$$L_\Phi(f) = \nabla \cdot (\nabla f + f v).$$

Properties of Landau's kernel

As a limit of Boltzmann's kernel, Landau's kernel inherits its properties (that is, the properties which are independant of the cross section). In particular, the conservations of mass, momentum and energy hold :

$$\int Q_B(f)(v) \begin{pmatrix} 1 \\ v \\ |v|^2/2 \end{pmatrix} dv = \int L_\Phi(f)(v) \begin{pmatrix} 1 \\ v \\ |v|^2/2 \end{pmatrix} dv = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (3)$$

Moreover, the dissipation of entropy is nonnegative (first part of Boltzmann's H-theorem) :

$$D_\Phi(f) \equiv -\int L_\Phi(f)(v)\,\log f(v)\,dv \geq 0.$$

Finally, it is possible to prove that when $\Phi > 0$ a.e., the second part of Boltzmann's H-theorem also holds :

$$D_\Phi(f) = 0 \qquad \Longleftrightarrow \qquad \forall v \in \mathbb{R}^N, \quad L_\Phi(f)(v) = 0$$

$$\iff \qquad \exists \rho \geq 0, u \in \mathbb{R}^N, T > 0, \quad f(v) = \frac{\rho}{(2\pi\, T)^{N/2}} \exp\left(- \frac{|v-u|^2}{2T} \right)$$

as soon as f is smooth enough.

1.2 Presentation of Landau's equation

Landau's kinetic equation concerns the number density $f(t,x,v)$ of particles which at time t and point x move with velocity v. It writes

$$\partial_t f + v \cdot \nabla_x f = L_\Phi(f). \tag{4}$$

We add to this equation the initial datum $f(0,x,v) = f_{in}(x,v)$.

A particular case that we shall study in the sequel is that of the spatially homogeneous solutions, that is those solutions which do not depend on x. The equation then becomes

$$\partial_t f = L_\Phi(f),$$

together with its initial datum $f(0,v) = f_{in}(v)$.

The basic a priori estimates for equation (4) are consequences of the properties of Landau's kernel. We first notice that the solution of eq. (4) (formally) satisfies thanks to (3) :

$$\partial_t \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(t,x,v) \begin{pmatrix} 1 \\ |v|^2 \\ |x-vt|^2 \end{pmatrix} dvdx = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

whence the a priori estimate

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(t,x,v) \left(1 + |x|^2 + |v|^2 \right) dvdx$$

$$\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_{in}(x,v) \left(1 + 2\,|x|^2 + (2\,T^2+1)\,|v|^2 \right) dvdx, \tag{5}$$

and this quantity is finite as soon as

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_{in}(x,v) \left(1 + |x|^2 + |v|^2 \right) dvdx < +\infty.$$

Because the function $x \mapsto \log x$ is not nonnegative, it is not completely obvious to convert the H-theorem in an a priori estimate. The following computation is extracted from [14]. We first observe (still formally) that the solution of eq. (4) satisfies

$$\partial_t \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(t,x,v) \, \log f(t,x,v) \, dvdx = - \int_{x\in\mathbb{R}^N} D_\Phi(f)(t,x) \, dx \leq 0.$$

As a consequence,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(T,x,v)\, \log f(T,x,v)\, dvdx$$

$$+ \int_0^T \int_{x \in \mathbb{R}^N} D_\Phi(f)(t,x)\, dxdt = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_{in}(x,v)\, \log f_{in}(x,v)\, dxdv. \qquad (6)$$

Then, we observe that for all function f,

$$\int\int f\, |\log f| - \int\int f\, \log f = 2 \int\int_{f \le 1} -f\, \log f$$

$$= 2 \int\int_{\exp\left(-1-\frac{|x|^2+|v|^2}{2}\right) \le f \le 1} -f\, \log f + 2 \int\int_{f \le \exp\left(-1-\frac{|x|^2+|v|^2}{2}\right)} -f\, \log f$$

$$\le \int f \left(2 + |x|^2 + |v|^2\right) dvdx + (2\pi)^N\, (N+1)\, e^{-1}.$$

Finally,

$$\sup_{t \in [0,T]} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(t,x,v)\, |\log f(t,x,v)|\, dvdx$$

$$+ \int_0^T \int_{x \in \mathbb{R}^N} D_\Phi(f)(t,x)\, dxdt$$

$$\le \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_{in}(x,v) \Big(\log f_{in}(x,v) + 2 + 2\, |x|^2$$

$$+ (2\, T^2 + 1)\, |v|^2 \Big)\, dvdx + (2\pi)^N\, (N+1)\, e^{-1}, \qquad (7)$$

and this quantity is finite as soon as $f_{in} \in \mathcal{A} = \cup_k \mathcal{A}_k$, i.-e.

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_{in}(x,v) \left(\log f_{in}(x,v) + 1 + |x|^2 + |v|^2 \right) dvdx \le k.$$

We denote by $\mathcal{I}_{C,\Phi}$ the set of all functions $f : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ verifying

$$\sup_{t \in [0,T]} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(t,x,v) \left(1 + |x|^2 + |v|^2 + |\log f(t,x,v)|\right) dvdx$$

$$+ \int_0^T \int_{x \in \mathbb{R}^N} D_\Phi(f)(t,x)\, dxdt \le C.$$

Thanks to our computations, we see that (formally), a solution of Landau's equation (with cross section Φ) lies in $\mathcal{I}_{C,\Phi}$ for some well-chosen constant C (only depending on k) as soon as $f_{in} \in \mathcal{A}_k$.

1.3 Presentation of some tools for the study of Landau's equation

We now list some of the ideas and tools that will be used in the sequel.

Estimates for the dissipation of entropy

For the linear Fokker–Planck's equation (2), we know that the dissipation of free energy

$$D_{FP}(f) := -\int (\log f + |v|^2/2)\, \nabla_v \cdot (\nabla_v f + v\, f)\, dv$$

is equal to the relative Fisher information :

$$D_{FP}(f) = \int \left| \frac{\nabla_v f}{f}(v) + v \right|^2 f(v)\, dv. \tag{8}$$

Then, it is possible to use the logarithmic Sobolev inequality (Cf. [18], [19]) to get an estimate on the speed of return to equilibrium (Cf. [2], [24]). We shall detail in the sequel how to adapt those ideas to Landau's equation.

Criterions of compactness in L^p

Strong compactness will be the consequence of "Rellich-Kondrachov" type theorems (Cf. [3] for example) :

Proposition 1: Let $p \in [1, +\infty[$, and $\Omega \subset \mathbb{R}^Q$ be an open set. If $\mathcal{F}$ is a set of functions which is bounded in $W^{s,p}_{loc}(\Omega)$ for some $s > 0$, it is (strongly) compact in $L^p_{loc}(\Omega)$.

The following property (of uniform boundedness) will also be of constant use :

Proposition 2: Let $p \in [1, +\infty[$, and $\mathcal{F} = (f_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions of $L^p_{loc}(\Omega)$. We suppose that the following decomposition holds : for all $\varepsilon > 0$, $f_n = g_n^\varepsilon + h_n^\varepsilon$, with $g_n^\varepsilon \in \mathcal{K}_\varepsilon$ a (strong) compact set of $L^p_{loc}(\Omega)$, and $\lim_{\varepsilon\to 0} \sup_{n\in\mathbb{N}} ||h_n^\varepsilon||_{L^p(K)} = 0$ for all compact set K of Ω. Then, $\mathcal{F}$ is relatively (strongly) compact in $L^p_{loc}(\Omega)$.

Finally, weak ($L^1(\mathbb{R}^N)$) compactness will be a consequence of Dunford–Pettis criterion (Cf. also [23] for example) :

Proposition 3: Let $\mathcal{F} = (f_n)_{n\in\mathbb{N}}$ be a sequence of bounded functions of $L^1(\Omega)$. Then, the three following properties are equivalent :

1. $\mathcal{F}$ is weakly relatively compact in $L^1(\Omega)$,

2.
$$\lim_{|A|\to 0} \sup_{n\in\mathbb{N}} \int_A |f_n| + \lim_{R\to+\infty} \sup_{n\in\mathbb{N}} \int_{\Omega\cap B(0,R)^c} |f_n| = 0,$$

3.
$$\exists \phi_1, \phi_2 : \mathbb{R}_+ \to \mathbb{R}_+, \quad \text{such that} \quad \lim_{x\to+\infty} \frac{\phi_1(x)}{x}, \phi_2(x) = +\infty \tag{9}$$
$$\text{and} \quad \sup_{n\in\mathbb{N}} \int_\Omega \phi_1(|f_n|) + |f_n|\, \phi_2(|x|) < +\infty.$$

Averaging lemmas

Those are lemmas which yield an extra smoothness on the *averages* of f like $\int f(t,x,v)\,\phi(v)\,dv$ (for ϕ smooth enough) when f and $\partial_t f + v\cdot\nabla_x f$ are bounded in certain spaces (Cf. [16], [17], [15]).

Renormalized formulations

Quadratic kinetic equations of Boltzmann or Landau type cannot be written in the sense of distributions when only the natural a priori estimates are satisfied (that is, mass, energy, entropy and entropy dissipation are controled). This is due to the fact that they contain quadratic terms *which are local in x* whereas the a priori estimates at best yield an $L\log L$ estimate. One then has to find a more complicated way of writing the equation, using nonlinear functions of the solution, which has a sense as soon as mass, energy, entropy and entropy dissipation are controled. This is called a renormalized formulation and was first introduced by R. DiPerna and P.-L. Lions in [14].

1.4 Plan of the sequel

In section 2, we first present a proof of an entropy/entropy dissipation estimate extracted from [13]. Then, this estimate is used to get a quantitative explicit estimate (also extracted from [13]) of exponentially fast convergence towards equilibrium for the spatially homogeneous Landau equation.

In section 3, we transform our entropy/entropy dissipation estimate in a smoothness estimate in the v-variable. Then, using variants of the proofs devised by P.-L. Lions in [34] (and also of C. Villani in [27] and R. Alexandre, C. Villani in [1]), we recover a strong compactness result first obtained in this work. In particular, we use the notions of averaging lemmas and renormalized formulations.

2 Large time behavior

2.1 Entropy dissipation estimate

We begin with an estimate which relates the entropy dissipation of Landau's kernel (with Maxwellian molecules) and the relative Fisher information. The proof that we propose here is a variant of one of the proofs of [13]. The linearization of the result of this section is very close to the results of [9].

Definition 1 : We denote by $\Phi_0 = 1$ the "Maxwellian" cross section. For $f \equiv f(v)$, we also denote the macroscopic quantities

$$\rho_f = \int f\, dv, \quad \rho_f\, u_f = \int f\, v\, dv, \quad N \rho_f\, T_f = \int f\, |v - u_f|^2\, dv,$$

the pressure tensor $K_{ij}^f = \int f\,(v_i - u_i^f)\,(v_j - u_j^f)\, dv$, the Maxwellian

$$M_f = \frac{\rho_f}{(2\pi T_f)^{N/2}}\, e^{-\frac{|v-u_f|^2}{2T_f}},$$

the relative Fisher information

$$I(f|M_f) = \int \left| \frac{\nabla f}{f} - \frac{\nabla M_f}{M_f} \right|^2 f,$$

and finally the quantity

$$q_f = \inf_{e \in S^{N-1}} \int ((v - u_f) \cdot e)^2\, f(v)\, dv.$$

Proposition 4 : The following functional estimate holds for all (smooth enough) function f :

$$I(f|M_f) \le \frac{2}{N-1}\, q_f^{-1}\, D_{\Phi_0}(f). \tag{10}$$

Proof : It is enough to prove (10) when $\rho_f = 1$, $u_f = 0$, $T_f = 1$, $K_{ij}^f = T_i^f\, \delta_{ij}$. The estimate then becomes

$$\int \frac{|\nabla f|^2}{f} - N \le \frac{2}{N-1}\, \frac{D_{\Phi_0}(f)}{\inf_k T_k^f}. \tag{11}$$

This is a consequence of the invariance of $I(f|M_f)$ and $D_{\Phi_0}(f)$ with respect to rotations on one hand, and of the following laws of transformation of $I(f|M_f)$, $D_{\Phi_0}(f)$, q_f with respect to the dilations $(d_\lambda f)(v) = f(\lambda\, v)$ on the other hand :

$$D_{\Phi_0}(d_\lambda f) = \lambda^{-2N}\, D_{\Phi_0}(f), \quad I(d_\lambda f|M_{d_\lambda f}) = \lambda^{2-N}\, I(f|M_f), \quad q_{d_\lambda f} = \lambda^{-2-N}\, q_f.$$

Those laws are applied when changing f in $T_f^{-N/2}\,\rho_f^{-1} f\left(R\,\frac{v-u_f}{\sqrt{T_f}}\right)$, where R is a rotation.

We now prove (11). For $i \neq j$, we use the notation :

$$S_{ij}^f(v,v_*) = (v-v_*)_i\left(\frac{\partial_j f}{f}(v)-\frac{\partial_j f}{f}(v_*)\right)-(v-v_*)_j\left(\frac{\partial_i f}{f}(v)-\frac{\partial_i f}{f}(v_*)\right). \quad (12)$$

Noticing that

$$\left(|x|^2 Id - x\otimes x\right)(a,a) = |x|^2|a|^2 - (x\cdot a)^2$$

$$= \frac{1}{2}\sum\sum_{i\neq j} |x_i a_j - x_j a_i|^2,$$

we see that

$$D_{\Phi_0}(f) = \frac{1}{2}\sum\sum_{i\neq j}\int\int |S_{ij}^f(v,v_*)|^2 f(v) f(v_*)\,dv dv_*.$$

Integrating (12) against $f(v_*)\,\phi(v_*)$, using the (classical) shorthand $f_* = f(v_*)$, $\phi_* = \phi(v_*)$ and dropping the index f whenever this is possible (like in S_{ij} instead of S_{ij}^f for example), we get when $i \neq j$:

$$\left[v_i\,\frac{\partial_j f}{f}(v) - v_j\,\frac{\partial_i f}{f}(v)\right]\int f_*\phi_* + \int f_*\phi_* v_j^*\,\frac{\partial_i f}{f}(v) - \int f_*\phi_* v_i^*\,\frac{\partial_j f}{f}(v)$$

$$= \int \partial_i f_*\phi_* v_j^* - \int \partial_j f_*\phi_* v_i^* - v_j\int \partial_i f_*\phi_* + v_i\int \partial_j f_*\phi_* + \int S_{ij} f_*\phi_*$$

$$= -\int \partial_i \phi_* f_* v_j^* + \int \partial_j\phi_* f_* v_i^* + v_j\int \partial_i\phi_* f_* - v_i\int \partial_j\phi_* f_* + \int S_{ij} f_*\phi_*.$$

Taking $\phi(v) = v_i$, we see that

$$\frac{\partial_j f}{f}\,T_i = -v_j + \int S_{ij} f_* v_i^*.$$

Thanks to the Cauchy–Schwarz inequality, for $i \neq j$,

$$\int \left|\frac{\partial_j f}{f} + \frac{v_j}{T_i}\right|^2 f \leq \frac{1}{T_i}\int\int |S_{ij}|^2 f f_*.$$

Therefore,

$$\sum\sum_{i\neq j}\int \frac{|\partial_j f|^2}{f} + \frac{T_j}{|T_i|^2} - \frac{2}{T_i} \leq \frac{1}{\inf_k T_k}\sum\sum_{i\neq j}\int\int |S_{ij}|^2 f f_*.$$

But

$$\sum\sum_{i\neq j}\int\frac{|\partial_j f|^2}{f}=(N-1)\int\frac{|\nabla f|^2}{f},$$

$$\sum\sum_{i\neq j}\frac{T_j}{|T_i|^2}-\frac{2}{T_i}=\sum_i\frac{N}{T_i^2}-\sum_i\frac{1}{T_i}-2(N-1)\sum_i\frac{1}{T_i}$$

$$=-N(N-1)+(N-1)\sum_i\left(\frac{1}{T_i}-1\right)^2+\sum_i\frac{1}{T_i^2}-\sum_i\frac{1}{T_i}.$$

Then, we notice that

$$\left(\sum_i\frac{1}{T_i}\right)^2\leq N\sum_i\frac{1}{T_i^2},$$

so that

$$\sum_i\frac{1}{T_i}\leq\frac{N}{\sum_i\frac{1}{T_i}}\sum_i\frac{1}{T_i^2}.$$

But

$$\sum_i\frac{1}{T_i}\geq N,$$

so that finally :

$$\sum_i\frac{1}{T_i}\leq\sum_i\frac{1}{T_i^2}.$$

Then,

$$(N-1)\int\frac{|\nabla f|^2}{f}-N(N-1)\leq 2\,\frac{D_{\Phi_0}(f)}{\inf_k T_k},$$

whence the desired inequality (that is, (11)).

□

2.2 Return to equilibrium

We begin by recalling Gross' logarithmic Sobolev inequality (Cf. [18], [19]).

Proposition 5 (Logarithmic Sobolev inequality) : Let $f:\mathbb{R}^N\to\mathbb{R}_+$ such that

$$\int f(v)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}dv=\begin{pmatrix}1\\ 0\\ N\end{pmatrix}.$$

Then

$$I(f|M_f)\geq 2\,H(f|M_f),$$

where the relative Fisher information $I(f|M_f)$ has been defined in def. 1 and the relative entropy $H(f|M_f)$ is given by

$$H(f|M_f) = \int f \,\log \frac{f}{M_f} = \int f \,\log \frac{f}{(2\pi)^{-N/2}\exp(-|v|^2/2)}.$$

When f does not satisfy the previous normalization, this proposition becomes

Proposition 6 : Let $f : \mathbb{R}^N \to \mathbb{R}_+$. Then,

$$I(f|M_f) \geq \frac{2}{T_f}\, H(f|M_f).$$

Proof : We use the translations and dilations $d_\lambda f(v) = f(\lambda\, v)$. The quantities I and H are transformed in the following way :

$$I(d_\lambda f|M_{d_\lambda f}) = \lambda^{2-N}\, I(f|M_f), \quad H(d_\lambda f|M_{d_\lambda f}) = \lambda^{-N}\, H(f|M_f).$$

Moreover, the temperature becomes

$$T_{d_\lambda f} = \lambda^{-2} T_f.$$

$\square$

Then, we state our main theorem (first proven in [13]) on the large time behavior of the spatially homogeneous Landau equation :

Theorem 1 : Let f_{in} be an initial datum with finite mass, energy and entropy. Then, any (smooth enough) solution of the spatially homogeneous Landau equation with Maxwellian molecules and initial datum f_{in} converges *exponentially* rapidly (and with constants that can be explicitly estimated) in L^1 towards its associated Maxwellian :

$$M_{f_{in}}(v) = \frac{\rho_{f_{in}}}{(2\pi T_{f_{in}})^{N/2}} e^{-\frac{|v-u_{f_{in}}|}{2T_{f_{in}}}}.$$

Proof : We know that

$$\partial_t H(f|M_f) = -\frac{1}{2}\, D_{\Phi_0}(f),$$

and (thanks to the use of propositions 4 and 6)

$$D_{\Phi_0}(f) \geq \frac{N-1}{2}\, q_f\, I(f|M_f) \geq (N-1)\, \frac{q_f}{T_f}\, H(f|M_f).$$

Note then that T_f is constant. Supposing that q_f is bounded below, the exponential convergence of the (relative) entropy towards 0 becomes a simple consequence of Gronwall's lemma.

We now prove that q_f is bounded below. We suppose (without loss of generality) that $\rho_f = 1, u_f = 0, T_f = 1$. Then

$$q_f = \inf_{e \in S^{N-1}} \int f (v \cdot e)^2 \geq \delta^2 \varepsilon^2 \inf_{e \in S^{N-1}} \int_{\varepsilon \leq |v| \leq R, |v \cdot e| \geq \delta |v|} f$$

$$\geq \delta^2 \varepsilon^2 \left(1 - \int_{|v| \geq R} f - \int_{|v| \leq \varepsilon} f - \int_{|v \cdot e| \leq \delta |v|, |v| \leq R} f \right).$$

Denoting now $A = \{v \in \mathbb{R}^N, |v| \leq \varepsilon \text{ or } (|v \cdot e| \leq \delta |v| \text{ and } |v| \leq R)\}$, we see that

$$|A| \leq (2\varepsilon)^N + Cte\, \delta\, R^N,$$

so that (for any $S > 1$)

$$\int_A f = \int_A f \frac{|\log f|}{|\log f|} 1_{f \geq S} + \int_A f\, 1_{f \leq S}$$

$$\leq \frac{1}{\log S} \int f\, |\log f| + S \left((2\varepsilon)^N + Cte\, \delta\, R^N \right).$$

Then,

$$\delta^2 \varepsilon^2 \left(1 - \int_{|v| \geq R} f - \int_{|v| \leq \varepsilon} f - \int_{|v \cdot e| \leq \delta |v|, |v| \leq R} f \right)$$

is strictly positive (with a lower bound independant of time) when ε, δ are small enough and R, S are large enough.

We now know that the (relative) entropy converges exponentially rapidly. The exponential convergence in L^1 is then a simple consequence of Csiszar-Kullback's inequality (Cf. [8], [20]):

$$H(f|M_f) \geq \frac{1}{2} \, ||f - M_f||_{L^1}^2$$

(under the assumption that $\rho_f = 1$).

□

Remark: The same theorem holds in the so-called "overMaxwellian" case, that is when the cross section is larger than some constant. It can be somehow extended to hard potential cross sections (Cf. [13]). The situation is much more complex in the case of the Boltzmann equation. After the pioneering works of [4] and [5], this problem was almost completely solved by G. Toscani and C. Villani in [25], [26] and by C. Villani in [29]. We also refer to [9] for an interesting result in the linearized setting.

3 Compactness for the Landau equation

3.1 Smoothness in the space of velocities

We show here how the entropy dissipation estimate (10) can be converted in a smoothness estimate for the velocity variable.

Proposition 7: We consider cross sections which satisfy

$$\varPhi(|v-v_*|^2)\, e^{\frac{|v|^2+|v_*|^2}{2}} \geq C_\varPhi. \tag{13}$$

Then, for $f \equiv f(v)$,

$$\int |\nabla \sqrt{f}|^2\, e^{-\frac{|v|^2}{2}} dv \leq \left(\frac{D_\varPhi(f)}{(N-1)\, C_\varPhi} + N\, \rho_f^2 \right) q^{-1}_{f e^{-|v|^2/2}} + 2\, e^{-1}\, \rho_f .$$

In other words, a weighted variant of the Fisher information is bounded by the entropy dissipation (provided that ρ_f and $q^{-1}_{f e^{-|v|^2/2}}$ are also bounded).

Proof: We begin by the estimate

$$D_\varPhi(f) \geq C_\varPhi \int\int e^{-\frac{|v|^2+|v_*|^2}{2}} \Big(|v-v_*|^2 Id - (v-v_*) \otimes (v-v_*) \Big)$$

$$\left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(v_*), \frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(v_*) \right) f(v)\, f(v_*)\, dv_* dv$$

$$= C_\varPhi\, D_{\varPhi_0}(f\, e^{-|v|^2/2})$$

(with $\varPhi_0 = 1$).

Then, using the estimate of entropy dissipation (10),

$$D_\varPhi(f) \geq \frac{N-1}{2}\, C_\varPhi\, q_{f e^{-|v|^2/2}} \int \left| \frac{\nabla f}{f} + \frac{v - u_{f e^{-|v|^2/2}}}{T_{f e^{-|v|^2/2}}} - v \right|^2 f\, e^{-|v|^2/2}\, dv.$$

Thanks to the elementary inequality $(a+b)^2 \geq \frac{1}{2}\, a^2 - b^2$,

$$\int \frac{|\nabla f|^2}{f}\, e^{-\frac{|v|^2}{2}} dv \leq \frac{4}{(N-1)\, C_\varPhi}\, q^{-1}_{f e^{-|v|^2/2}}\, D_\varPhi(f)$$

$$+ 4 \int \left(\left| \frac{v - u_{f e^{-|v|^2/2}}}{T_{f e^{-|v|^2/2}}} \right|^2 + |v|^2 \right) f\, e^{-|v|^2/2}\, dv$$

$$\leq \frac{4}{(N-1)\, C_\varPhi}\, q^{-1}_{f e^{-|v|^2/2}}\, D_\varPhi(f) + 4N\, \frac{\rho_{f e^{-|v|^2/2}}}{T_{f e^{-|v|^2/2}}} + 8\, e^{-1}\, \rho_f .$$

Noticing that (for all g smooth enough)

$$T_g^{-1} \le \rho_g \, q_g^{-1},$$

we see that

$$\int |\nabla \sqrt{f}|^2 \, e^{-\frac{|v|^2}{2}} dv \le \left(\frac{D_\Phi(f)}{(N-1)\, C_\Phi} + N\, \rho_f^2 \right) q^{-1}_{f e^{-|v|^2/2}} + 2\, e^{-1} \rho_f. \qquad (14)$$

□

We now consider a function $f \equiv f(t,x,v) \ge 0$ (which will be in the sequel a solution of the Landau equation). For $\eta \le 1$, we denote the set of "bad" points $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N$ by

$$A_\eta(f) = \left\{ (t,x), \quad \rho_f \ge \eta^{-1} \right\} \cup \left\{ (t,x), \quad q_{f e^{-|v|^2/2}} \le \eta \right\}.$$

As a consequence of estimate (14), we see that (for $f \equiv f(t,x,v) \ge 0$), the following inequality holds when $(t,x) \in A_\eta(f)^c$:

$$\int |\nabla \sqrt{f}|^2 \, e^{-\frac{|v|^2}{2}} \, dv \le \eta^{-3} \left[\frac{D_\Phi(f)}{(N-1)\, C_\Phi} + N + 1 \right]. \qquad (15)$$

We now show that the "bad" points (t,x), that is those points which lie in $A_\eta(f)$, constitute a set of small measure when η is itself small enough, and $f \in \mathcal{I}_{C,\Phi}$. More precisely, we prove the

Proposition 8: For all $\varepsilon > 0$, $C > 0$,

$$\lim_{\eta \to 0} \sup_{f \in \mathcal{I}_{C,\Phi}} |A_\eta(f) \cap \{\rho_f \ge \varepsilon\}| = 0.$$

Proof: We define $\lambda_k(\nu) = \inf_{\delta > 0} [\delta\, \nu + k\, (\log \delta)^{-1}]$. We observe that $\lim_{\nu \to 0} \lambda_k(\nu) = 0$. Then, if $B \subset \mathbb{R}^N_v$ is such that $|B| \le \nu$ ($|B|$ denoting the Lebesgue measure de B) and if $\int f\, |\log f| \le k$,

$$\begin{aligned} \int_B f &\le \int_{B \cap \{f \le \delta\}} f + \int_{B \cap \{f \ge \delta\}} f \\ &\le \delta\, |B| + (\log \delta)^{-1} \int f\, |\log f| \\ &\le \lambda_k(\nu). \end{aligned}$$

Assume now that $(t,x) \in [0,T] \times \mathbb{R}^N$ is such that $\rho_f(t,x) \ge \varepsilon$. Then, we observe that (for some $e \in S^{N-1}$) for all $\theta, R > 0$,

$$q_{fe^{-v^2/2}}(t,x) = \int f\, e^{-v^2/2} \left((v - u_{fe^{-v^2/2}}) \cdot e \right)^2$$

$$\geq e^{-R^2/2}\, \theta^2 \int_{\{|v| \leq R, |(v-u_{fe^{-v^2/2}})\cdot e| \geq \theta\}} f$$

$$\geq e^{-R^2/2}\, \theta^2 \left(\varepsilon - \frac{1}{R^2} \int f\, |v|^2 - \sup_{D \subset \mathbb{R}^N_v, |D| \leq 2^N R^{N-1} \theta} \int_D f \right).$$

We now denote by B_k the set of $(t,x) \in [0,T] \times \mathbb{R}^N$ such that $\int f(t,x,v)\,(1 + |v|^2 + |\log f(t,x,v)|)\, dv \leq k$. Then, for $(t,x) \in B_k$ such that $\rho_f(t,x) \geq \varepsilon$, taking $R = \sqrt{2k/\varepsilon}$,

$$q_{fe^{-v^2/2}}(t,x) \geq e^{-k/\varepsilon}\, \theta^2 \left(\varepsilon/2 - \lambda_k(2^N\, (2k/\varepsilon)^{(N-1)/2}\, \theta) \right).$$

Choosing now $\theta = \theta(k,\varepsilon) > 0$ in such a way that

$$\lambda_k(2^N\, (2k/\varepsilon)^{(N-1)/2}\, \theta(k,\varepsilon)) \leq \varepsilon/4$$

(this is possible because $\lim_{\nu \to 0} \lambda_k(\nu) = 0$), we get the estimate

$$q_{fe^{-v^2/2}}(t,x) \geq e^{-k/\varepsilon}\, \theta(k,\varepsilon)^2\, \varepsilon/4.$$

Moreover (still for $(t,x) \in B_k$),

$$\rho_f(t,x) \leq k.$$

Now since $f \in \mathcal{I}_{C,\Phi}$, we know that

$$k\, |B_k^c| \leq \int_{B_k^c} \left[\int f(t,x,v)\, (1 + |v|^2 + |\log f(t,x,v)|)\, dv \right] dxdt$$

$$\leq T \sup_{t \in [0,T]} \int_{x \in \mathbb{R}^N} \int_{v \in \mathbb{R}^N} f(t,x,v)\, (1 + |v|^2 + |x|^2 + |\log f(t,x,v)|)\, dvdx$$

$$\leq C\, T,$$

so that $|B_k^c| \leq C\, T/k$. Finally,

$$\lim_{\eta \to 0} \sup_{f \in \mathcal{I}_{C,\Phi}} \left| A_\eta(f) \cap \{\rho_f \geq \varepsilon\} \right| = 0.$$

□

Heuristically, propositions 7 and 8 can be summarized in this way : if a function $f \equiv f(t,x,v) \geq 0$ satisfies the natural a priori estimates of the Landau equation, then $\sqrt{f}$ lies in a weighted H^1 space in the v variable, except for a set of (t,x) of arbitrarily small measure.

We now introduce in the two next sections two analytical tools that we shall use when we state the theorem of strong compactness that we intend to prove (that is, theorem 2).

3.2 The renormalized formulation of the equation

The concept of renormalized solutions was introduced by R. DiPerna and P.-L. Lions in 1989 in order to prove the existence of global solutions of the Boltzmann equation (Cf. [14]). It enables to define solutions belonging to L^1 (or $L\log L$) to quadratic equations. We describe in this subsection the computation corresponding to this concept.

Let β be a function of class C^2 on $\mathbb{R}_+$, concave, and γ, ζ defined in such a way that

$$\forall x \in \mathbb{R}_+, \qquad \gamma'(x)^2 = -\beta''(x), \quad \zeta(x) = \beta(x) - x\,\beta'(x).$$

When f is a (smooth) solution of Landau's equation, one has (with a, b, c defined in section 1, without mentioning the dependance with respect to Φ except when it is necessary, and the convolution being with respect to v) :

$$(\partial_t + v\cdot\nabla_x) f = \nabla_v \cdot \Big((a*f)\,\nabla_v f - (b*f)\,f \Big). \tag{16}$$

Then,

$$(\partial_t + v\cdot\nabla_x)\beta(f) = \beta'(f)\,\nabla_v\cdot\Big((a*f)\,\nabla_v f - (b*f)\,f\Big)$$

$$= \nabla_v\cdot\Big(\beta'(f)\,(a*f)\,\nabla_v f\Big) - (a*f)\,\beta''(f) : \nabla_v f \nabla_v f$$

$$- \beta'(f)\,(b*f)\,\nabla_v f - \beta'(f)\,(c*f)\,f$$

$$= \nabla_v\cdot\Big((a*f)\,\nabla_v\beta(f)\Big) - (a*f)\,\beta''(f) : \nabla_v f\nabla_v f$$

$$- (b*f)\,\nabla_v\beta(f) - \beta'(f)\,f\,(c*f)$$

$$= \nabla_v\nabla_v : \Big((a*f)\,\beta(f)\Big) - \nabla_v\Big((b*f)\,\beta(f)\Big) + (a*f) : \nabla_v\gamma(f)\nabla_v\gamma(f)$$

$$- \nabla_v\cdot\Big((b*f)\,\beta(f)\Big) + (c*f)\,\beta(f) - \beta'(f)\,f\,(c*f)$$

$$= \nabla_v\nabla_v : \Big((a*f)\,\beta(f)\Big) + (c*f)\,\zeta(f)$$

$$- 2\,\nabla_v\cdot\Big((b*f)\,\beta(f)\Big) + (a*f) : \nabla_v\gamma(f)\nabla_v\gamma(f). \tag{17}$$

While there is no hope of defining in the sense of distributions the quantity $\nabla_v\cdot\Big((a*f)\,\nabla_v f - (b*f)\,f\Big)$ appearing in (16) when $f\in\mathcal{I}_{C,\Phi}$, it is possible to define (still when $f\in\mathcal{I}_{C,\Phi}$, and in the sense of distributions) the three first terms of (17) provided that β and $x \mapsto x\,\beta'(x)$ are bounded, and under the (reasonable) condition on Φ :

$$\forall R \geq 0, \quad K_{R,\Phi} \equiv \sup_z \int_{w \in B(z,R)} \Big[\Phi(|w|^2) + |w|\, \Phi'(|w|^2) \Big] < +\infty. \tag{18}$$

This is due to the fact that $a_\Phi * f$, $b_\Phi * f$ and $c_\Phi * f$ lie in L^1_{loc} as soon as $f \in \mathcal{I}_{C,\Phi}$. More precisely, we state the

Lemma 1 : For $f \equiv f(v) \geq 0$, one has

$$\int_{|v| \leq R} |(a_\Phi * f)(v)|\, dv \leq 2\, K_{R,\Phi} \left(R^2 \int f\, dv + \int f |v|^2\, dv \right), \tag{19}$$

$$\int_{|v| \leq R} |(b_\Phi * f)(v)|\, dv \leq \; K_{R,\Phi} \left((R + \frac{1}{2}) \int f\, dv + \frac{1}{2} \int f |v|^2\, dv \right), \tag{20}$$

$$\int_{|v| \leq R} |(c_\Phi * f)(v)|\, dv \leq 2\, K_{R,\Phi} \int f\, dv. \tag{21}$$

Proof : We treat only the first term, since the other ones lead to the same kind of computations :

$$\int_{|v| \leq R} |(a_\Phi * f)(v)|\, dv \leq \int_{v_*} f(v_*) \int_{|w + v_*| \leq R} \Phi(|w|^2)\, |w|^2\, dw dv_*$$

$$\leq \int (R + |v_*|)^2\, f(v_*)\, dv_* \;\times\; \sup_z \int_{B(z,R)} \Phi(|\cdot|)^2.$$

□

Finally, the last term in (17) cannot be easily bounded (under the assumption that $f \in \mathcal{I}_{C,\Phi}$) but this will not be a problem in the sequel because this term is nonnegative.

A typical example of function β that can be used is $\beta(x) = x/(1+x)$. Then, $\beta'(x) = (1+x)^{-2}$, $\beta''(x) = -2\,(1+x)^{-3}$, $\gamma(x) = \sqrt{2}\,(1+x)^{-3/2}$, and $\zeta(x) = (x/(1+x))^2$.

We shall not use directly in this work the notion of renormalized solutions of the Landau equation, and therefore we shall not try to give a precise definition of this concept. We shall however use eq. (17) and lemma 1 for sequences of smooth solutions of the Landau equation (in other words, we shall use the renormalized formulation of the equation for solutions which are smooth) in the proof of the theorem of strong compactness (theorem 2).

3.3 Averaging lemmas

Those lemmas were introduced at the beginning of the eighties in [16] and [17] in order to treat transport problems. They turned out to be a key tool in the general theory of kinetic equations. The theorem given here is a variant of lemmas which can be found in [15].

Proposition 9 : Let $p > 1$. We suppose that $g \in C([0,T];\mathcal{D}'(\mathbb{R}^N_x \times \mathbb{R}^N_v))$, $g \in L^p_{loc}([0,T] \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$, and that

$$\partial_t g + v \cdot \nabla_x g \in W^{\alpha,p}(]0,T[; W^{\alpha,p}_{loc}(\mathbb{R}^N_x; W^{\beta,p}_{loc}(\mathbb{R}^N_v))),$$

with $p \in]1,+\infty[$, $\alpha > -1$ and $\beta \in \mathbb{R}$. Finally, we suppose that $g(0) \in L^p_{loc}(\mathbb{R}^N_x \times \mathbb{R}^N_v)$. Then, there exists $s(p,\alpha,\beta,N) > 0$ such that for all $\phi \in \mathcal{D}(\mathbb{R}^N)$, $M_\phi(g) := \int g\,\phi\,dv \in W^{s,p}(]0,T[; W^{s,p}_{loc}(\mathbb{R}^N_x))$. Moreover, for all $R > 0$, there exists $R' > 0$ such that (for some function F),

$$||M_\phi(g)||_{W^{s,p}(]0,T[\times B^x_R)} \leq F(\phi, ||g||_{L^p(]0,T[\times B^x_{R'} \times B^v_{R'})},$$

$$||g(0)||_{L^p(B^x_{R'} \times B^v_{R'})}, ||\partial_t g + v \cdot \nabla_x g||_{W^{\alpha,p}([0,T]\times B^x_{R'}; W^{\beta,p}(B^v_{R'}))}).$$

3.4 Strong compactness

We prove here a variant of a theorem due to P.-L. Lions (Cf. [34]). The proof presented here is itself a variant of that of [34].

Theorem 2 : Let Φ be a cross section satisfying (13) and (18). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^N \times \mathbb{R}^N))$ verifying

1. For some $k > 0$, $f_n(0) \in \mathcal{A}_k$, i.-e.

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_n(0,x,v)\,(1 + |x|^2 + |v|^2 + |\log f_n(0,x,v)|)\,dvdx \leq k,$$

2. Each f_n is smooth and bounded below by a Gaussian function in x, v locally uniformly in t, but *not uniformly in n*,
3. Each f_n is a (strong) solution of Landau's equation.

Then, it is possible to extract from $(f_n)_{n\in\mathbb{N}}$ a subsequence which converges *strongly* in $L^1_{loc}([0,T] \times \mathbb{R}^N \times \mathbb{R}^N)$ towards some function f (for all $T > 0$).

Remark : Note that in this theorem, one does not suppose that $f_n(0)$ converges strongly in L^1. This means that the Landau equation has a regularizing effect in all variables. This behavior is at variance with that of the Boltzmann equation with angular cutoff.

Proof: We begin by writing down the a priori estimates (5) and (7). Thanks to the first hypothesis of theorem 2, there exists C ($= C(k)$, independant of n) such that $f_n \in \mathcal{I}_{C,\Phi}$.

We get therefore the existence (thanks to Dunford-Pettis theorem) of a subsequence converging weakly in L^1 towards some function f. This subsequence will still be denoted by $(f_n)_{n\in\mathbb{N}}$.

We write down the renormalized equation (17) on $\beta_q(f_n)$ with $\beta_q = \sqrt{\cdot}\wedge\sqrt{q}$ (or on a smooth approximation of this function).

Then, $\beta_q'(x) = \frac{1}{2}\,x^{-1/2}\,1_{x\le q}$, $\zeta_q(x) = \frac{1}{2}\,x^{1/2}\,1_{x\le q} + \sqrt{q}\,1_{x\ge q}$. The terms in $(a*f_n)\,\beta_q(f_n)$, $(b*f_n)\,\beta_q(f_n)$, $(c*f_n)\,\zeta_q(f_n)$ of the right-hand side of the equation are bounded (for a given q, and uniformly in n) in $L^1_{loc}([0,T]\times\mathbb{R}^N_x; W^{-2,1}_{loc}(\mathbb{R}^N_v))$, thanks to the estimates (19), (20), (21).

Moreover, for any cutoff function $\chi : \mathbb{R}^N \to \mathbb{R}_+$,

$$\int\int \beta_q(f_n)(T)\,\chi(x)\,\chi(v) - \int\int \beta_q(f_n)(0)\,\chi(x)\,\chi(v)$$

$$= \int\int v\cdot\nabla_x\chi\,\chi(v)\,\beta_q(f_n)$$

$$+\int\int \chi(x)\,\nabla_v\nabla_v\chi\,(a*f_n)\,\beta_q(f_n) + 2\int\int \chi(x)\,\nabla_v\chi\,(b*f_n)\,\beta_q(f_n)$$

$$+\int\int \chi(x)\,\chi(v)\,(c*f_n)\,\zeta_q(f_n) + \int\int \chi(x)\,\chi(v)\,(a*f_n) \,:\, \nabla_v\gamma_q(f_n)\nabla_v\gamma_q(f_n).$$

Noticing that $0 \le \beta_q(f_n) \le f_n\wedge 1$, we use again estimates (19), (20), (21), and obtain that $(a*f_n) \,:\, \nabla_v\gamma_q(f_n)\nabla_v\gamma_q(f_n) \in L^1_{loc}([0,T]\times\mathbb{R}^N_x\times\mathbb{R}^N_v)$. Note that here, the nonnegativity of

$$(a*f_n) \,:\, \nabla_v\gamma_q(f_n)\nabla_v\gamma_q(f_n)$$

plays a decisive role.

Finally, all the terms in the right-hand side of the equation satisfied by $\beta_q(f_n)$ (that is, (17)) are bounded in $L^1_{loc}([0,T]\times\mathbb{R}^N_x; W^{-2,1}_{loc}(\mathbb{R}^N_v))$. Because of the Sobolev embeddings, they are also bounded in $W^{-\varepsilon,p(\varepsilon)}(]0,T[; W^{-\varepsilon,p(\varepsilon)}_{loc}(\mathbb{R}^N_x; W^{-2-\varepsilon,p(\varepsilon)}_{loc}(\mathbb{R}^N_v)))$ for ε small enough and some $p(\varepsilon) > 1$ verifying $p(\varepsilon)\to 1$ when $\varepsilon\to 0$.

Then, according to the averaging lemma (that is, proposition 9), the quantity $\int \beta_q(f_n)\,\phi(v)\,dv$ is bounded in $W^{s(p(\varepsilon),-\varepsilon,-2-\varepsilon,N),p(\varepsilon)}(]0,T[; W^{s(p(\varepsilon),-\varepsilon,-2-\varepsilon,N)}_{loc}(\mathbb{R}^N))$ for ε small enough, and $\phi\in\mathcal{D}(\mathbb{R}^N)$.

In particular, thanks to Rellich–Kondrachov characterization (that is, proposition 1), the strong compactness holds in $L^{p(\varepsilon)}_{loc}([0,T]\times\mathbb{R}^N)$ (for $\varepsilon > 0$ small enough), and consequently in $L^1_{loc}([0,T]\times\mathbb{R}^N)$, for $\int \beta_q(f_n)\,\phi(v)\,dv$.

We write down

$$\int \sqrt{f_n}\,\phi(v)\,dv = \int \beta_q(f_n)\,\phi(v)\,dv + \int \Big(\sqrt{f_n} - \beta_q(f_n)\Big)\,\phi(v)\,dv,$$

and

$$\left| \int_0^T \int\int \Big(\sqrt{f_n} - \beta_q(f_n)\Big)\,\phi(v)\,dvdxdt \right| \le ||\phi||_{L^\infty} \int_0^T \int\int 2\sqrt{f_n}\,1_{f_n \ge q}$$

$$\le 2\,||\phi||_{L^\infty}\,q^{-1/2}\,C\,T.$$

Then, for all $\phi \in \mathcal{D}(\mathbb{R}^N)$, $\int \sqrt{f_n}\,\phi(v)\,dv$ is (strongly) compact in $L^1_{loc}([0,T] \times \mathbb{R}^N)$ (thanks to proposition 2, as sum of a sequence which is compact for all q and a sequence which tends to 0 with q uniformly in n).

We now want to show that $\sqrt{f_n}$ is strongly compact in L^1. Since we know that its averages in v are strongly compact (in t,x), it remains to use a property of smoothness in v of $\sqrt{f_n}$. This smoothness holds thanks to proposition 7, except on a set (in t,x) of arbitrarily small measure thanks to proposition 8.

We now introduce the decomposition which enables to perform in a precise way the program described above :

$$\sqrt{f_n} = \sqrt{f_n} *_v \chi_\delta + \Big(\sqrt{f_n} - \sqrt{f_n} *_v \chi_\delta\Big),$$

where χ_δ is a mollifying sequence (Cf. [3] for example).

The first term converges strongly in $L^1_{loc}([0,T] \times \mathbb{R}^N \times \mathbb{R}^N)$ for all $\delta \in]0,1]$ thanks to the previous estimates (the whole sequence converges thanks to the uniqueness of the weak limit).

Therefore, according to proposition 2, it is sufficient (in order to get the strong compactness of $\sqrt{f_n}$ in L^1_{loc}) to prove that the second term tends to 0 (in L^1_{loc}, uniformy with respect to n) when δ goes to 0.

For any compact set $K \subset [0,T] \times \mathbb{R}^N$, one has

$$Q_{n,\delta} = \left| \int_K \int_{B(0,R)} \Big(\sqrt{f_n} - \sqrt{f_n} *_v \chi_\delta\Big) dvdxdt \right|$$

$$\le \int_{K \cap \{\rho_{f_n} \le \varepsilon\}} \int_{B(0,R)} \sqrt{f_n}\,dvdxdt$$

$$+ \int_{K \cap \{\rho_{f_n} \le \varepsilon\}} \Big(\int_{B(0,1)} \chi_\delta\,dv\Big) \Big(\int_{B(0,R+1)} \sqrt{f_n}\,dv\Big)\,dxdt$$

$$+ \left| \int_{K \cap \{\rho_{f_n} \ge \varepsilon\}} \int_{B(0,R)} \Big(\sqrt{f_n} - \sqrt{f_n} *_v \chi_\delta\Big) dvdxdt \right|$$

$$\le 2\,|K|\,|B(0,R+1)|^{1/2}\,\varepsilon^{1/2}$$
$$+\left|\int_{K\cap\{\rho_{f_n}\ge\varepsilon\}\cap A_\eta(f_n)}\int_{B(0,R)}\left(\sqrt{f_n}-\sqrt{f_n}*_v\chi_\delta\right)dvdxdt\right|$$
$$+\left|\int_{K\cap\{\rho_{f_n}\ge\varepsilon\}\cap A_\eta(f_n)^c}\int_{B(0,R)}\left(\sqrt{f_n}-\sqrt{f_n}*_v\chi_\delta\right)dvdxdt\right|$$
$$\le 2\,|K|\,|B(0,R+1)|^{1/2}\varepsilon^{1/2}$$
$$+\,2\,|\{\rho_{f_n}\ge\varepsilon\}\cap A_\eta(f_n)|^{1/2}\,C^{1/2}\,|B(0,R+1)|^{1/2}$$
$$+\int_{K\cap A_\eta(f_n)^c}\left|\int_{B(0,R)}\left(\sqrt{f_n}-\sqrt{f_n}*_v\chi_\delta\right)dv\right|dxdt.$$

This last quantity can be bounded thanks to proposition 7 (or, more precisely, thanks to estimate (95)) by

$$\int_{K\cap A_\eta(f_n)^c}\int_{B(0,R)}\int_{\mathbb{R}^N}\left|\int_{\theta=0}^1 v_*\cdot\nabla_v\sqrt{f_n}(v-\theta\,v_*)\,d\theta\right|\chi_\delta(v_*)\,dv_*dvdxdt$$
$$\le\int_{K\cap A_\eta(f_n)^c}\left(\int_{\mathbb{R}^N}|v_*|\,\chi_\delta(v_*)\,dv_*\right)\left(\int_{B(0,R+1)}|\nabla\sqrt{f_n}(v)|\,dv\right)dxdt$$
$$\le\delta\,e^{(R+1)^2/4}\,|B(0,R+1)|^{1/2}\,|K|^{1/2}\eta^{-3/2}\left(\frac{C}{(N-1)\,C_\Phi}+(N+1)\,|K|\right)^{1/2}.$$

Taking ε and η small enough, and using proposition 8, we see that $\lim_{\delta\to0}\sup_{n\in\mathbb{N}}Q_{n,\delta}=0$. As we previously noticed, we get in this way the (strong) compactness in L^1_{loc} of $\sqrt{f_n}$. This immediately ensures the (strong) compactness in L^1_{loc} of f_n, and concludes the proof of theorem 2.

$\square$

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