

Exponential Decay toward Equilibrium via Entropy Methods for Reaction-Diffusion Equations

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Abstract

In this work, we show how the entropy method enables to get in an elementary way (and without linearization) estimates of exponential decay towards equilibrium for solutions of reaction-diffusion equations corresponding to a reversible reaction. Explicit rates of convergence combining the dissipative effects of diffusion and reaction are given.

Key words: Reaction-Diffusion, Entropy method, Exponential Decay

AMS subject classification: 35B40, 35K57

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1 Introduction

The entropy method for the study of the long-time asymptotics of a dissipative PDE consists in looking for a nonnegative Lyapounov functional $H \equiv H(f)$ and its nonnegative dissipation $D \equiv D(f)$ (i.e. functionals which satisfy

$$\frac{d}{dt}H(f(t)) = -D(f(t))$$

along the flow of the PDE), which are well-behaved in the following sense: first,

$$H(f) = 0 \iff f = f_\infty$$

for some equilibrium f_∞ (usually, such a result is true only when all the conserved quantities have been taken into account), and secondly,

$$D(f) \geq \Phi(H(f))$$

for some nonnegative function Φ such that $\Phi(x) = 0 \iff x = 0$.

If $\Phi'(0) \neq 0$, one usually gets exponential convergence toward f_∞ with a rate which can be explicitly estimated. This method, which is an alternative to the linearization around the equilibrium, has the advantage of being quite robust. This is due to the fact that it mainly relies on functional inequalities which have no direct link with the original PDE.

The entropy method has lately been used in many situations: nonlinear diffusion equations (such as fast diffusions [10, 9], equations of fourth order [5], Landau equation [13], etc.), integral equations (such as the spatially homogeneous Boltzmann equation [41, 42, 43]), or kinetic equations ([6], [14, 15], [17]).

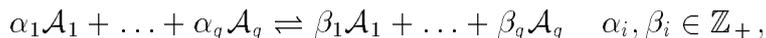
We propose here to use the entropy method in the context of systems of reaction-diffusion equations. Several previous results on the long-time behavior of reaction-diffusion systems have been obtained by different (for instance, by linearization) methods (e.g. [7, 33, 2]).

In [7], exponential convergence to equilibrium for systems of reaction-diffusion equations (for which the solution trajectories remain in invariant domains) was shown provided that the diffusion term dominates over the reaction- (as well as convection-) terms. More precisely, the first non-zero eigenvalue of the diffusion term (with boundary conditions) multiplied by the minimal diffusion constant has to be bigger than the linearized effects of reaction (and convection) estimated within the invariant domain. The obtained convergence rate is then simply the difference of the two according values.

Lyapunov functionals were previously considered by many authors, see, for instance, [46, 33, 26, 45, 25, 27, 29] and the references therein. In particular, [35] presents nicely how Lyapunov functionals are used (for the system (14)–(18) below) to prove the ω -limit set to consist only of the steady states (see (22) below). Moreover, we emphasize [34, 20, 21] for how generalized Lyapunov structures of reaction-diffusion systems yield a-priori estimates to establish global existence of solutions.

The works which are closest to our approach are [22, 23, 24], where reaction-diffusion systems including drift and modelling the transport of electrically charged species are considered. A lower bound of the entropy dissipation in terms of the entropy was established there, but in a non-constructive way, i.e. via a contradiction argument with no control on the constants.

Our aim is to provide quantitative exponential convergence to equilibrium with explicit rates and constants for reversible reaction processes of species $\mathcal{A}_i, i = 1, 2, \dots, q$ of the type



in a bounded box $\Omega \subset \mathbb{R}^N$ ($N \geq 1$). More precisely, we consider a system of PDE's whose unknowns are $a_i \equiv a_i(t, x) \geq 0, i = 1, \dots, q$, where $t \geq 0$ and $x \in \Omega$. This system writes

$$\partial_t a_i - d_i \Delta_x a_i = (\beta_i - \alpha_i) \left(l \prod_{i=1}^q a_i^{\alpha_i} - k \prod_{i=1}^q a_i^{\beta_i} \right) \quad (1)$$

with the homogeneous Neumann boundary condition $\nabla_x a_i \cdot n = 0$ (on $\partial\Omega$, with n the outward normal to Ω). Here, d_i are constant diffusion rates, α_i, β_i the stoichiometric coefficients, and $k > 0, l > 0$ are strictly positive reaction rates corresponding to a reversible reaction. In [23], systems quite more general than (1) are proven to have a unique asymptotically stable steady state.

Applications of systems like (1) have been stated to model reactions of chemical substances (see e.g. [35, 18] for the system (14)–(18) below and [19, 16, 44, 36] more generally).

They can be obtained by a suitable scaling, either starting from microscopic systems (Cf. for example [11] and [39] in simplified situations) or from mesoscopic (kinetic) equations, see [37], [38] and [3].

In particular, we shall consider two typical situations. The first one corresponds to a system of two equations :

$$\partial_t a - d_a \Delta_x a = -2(a^2 - b), \quad (2)$$

$$\partial_t b - d_b \Delta_x b = a^2 - b. \quad (3)$$

They satisfy the homogeneous Neumann conditions

$$n(x) \cdot \nabla_x a = 0, \quad n(x) \cdot \nabla_x b = 0 \quad x \in \partial\Omega, \quad (4)$$

and the nonnegative initial condition

$$a(0, x) = a_0(x) \geq 0, \quad b(0, x) = b_0(x) \geq 0. \quad (5)$$

We remark that compared to (1) and thanks to the rescaling $t \rightarrow \frac{1}{k}t$, $x \rightarrow |\Omega|^{\frac{1}{N}}x$, $(a, b) \rightarrow \frac{k}{l}(a, b)$, it is - without loss of generality - convenient to assume that

$$l = k = 1, \quad |\Omega| = 1. \quad (6)$$

The flow of equations (2) – (5) conserves the total L^1 -norm

$$0 < M \equiv \int_{\Omega} (a(t, x) + 2b(t, x)) dx = \int_{\Omega} (a_0(x) + 2b_0(x)) dx, \quad (7)$$

which we assume strictly positive and determines (at least formally) the unique equilibrium states (a_{∞}, b_{∞}) as the nonnegative constants satisfying $a_{\infty} + 2b_{\infty} = M$ and $a_{\infty}^2 = b_{\infty}$, i.e.

$$a_{\infty} = -\frac{1}{4} + \frac{1}{4}\sqrt{1 + 8M}, \quad b_{\infty} = \frac{M - a_{\infty}}{2} = a_{\infty}^2. \quad (8)$$

Finally, we introduce the entropy functional (which has the physical meaning of a free energy) associated to (2) – (5)

$$E(a, b) \equiv \int_{\Omega} (a(\ln a - 1) + b(\ln b - 1)) dx \quad (9)$$

to state our main result for this system:

Theorem 1.1 *Let Ω be a bounded, connected, and regular open set of \mathbb{R}^N ($N \geq 1$), and d_a, d_b be two strictly positive diffusivity constants. Let the initial data a_0, b_0 be two nonnegative functions of $L^{\infty}(\Omega)$ with strictly positive mass $\int_{\Omega} a_0 + 2b_0 dx = M > 0$ and denote $L_1 \equiv \|a_0\|_{\infty} + 2\|b_0\|_{\infty}$. Then, the unique nonnegative global solution $t \in \mathbb{R}_+ \mapsto (a(t), b(t))$ in $L^{\infty}(\Omega)$ to equations (2) – (6) obeys the following exponential decay toward equilibrium:*

$$\begin{aligned} & \frac{1}{2} \|a(t, \cdot) - a_{\infty}\|_{L^1(\Omega)}^2 + \|b(t, \cdot) - b_{\infty}\|_{L^1(\Omega)}^2 \\ & \leq \frac{(6 + 2\sqrt{2})M}{3 + 2\sqrt{2}} (E(a_0, b_0) - E(a_{\infty}, b_{\infty})) e^{-\frac{4t}{K_1} \min\{1, \frac{d_a}{P(\Omega)K_2}\}}, \end{aligned} \quad (10)$$

where $P(\Omega)$ is the Poincaré constant of Ω , and $K_1(L_1, M), K_2(M, d_a/d_b)$ are constants defined as follows: we introduce the function $\Phi : (0, \infty)^2 \rightarrow \mathbb{R}$ defined by

$$\Phi(x, y) = \frac{x(\ln(x) - \ln(y)) - (x - y)}{(\sqrt{x} - \sqrt{y})^2}. \quad (11)$$

Then

$$K_1(L_1, M) = \max \left\{ \frac{\Phi(L_1, a_\infty)}{a_\infty}, \Phi\left(\frac{L_1}{2}, b_\infty\right) \right\} = O(\ln(L_1)) \quad \text{for large } L_1, \quad (12)$$

$$K_2(M, d_a/d_b) = \frac{d_a \sqrt{1+8M}}{d_b} + \sqrt{\frac{d_a^2}{d_b^2} \frac{1+8M}{4} + \frac{d_a \sqrt{1+8M-1}}{d_b}}. \quad (13)$$

The second situation we wish to investigate corresponds to a system of three equations:

$$\partial_t a - d_a \Delta_x a = -ab + c, \quad (14)$$

$$\partial_t b - d_b \Delta_x b = -ab + c, \quad (15)$$

$$\partial_t c - d_c \Delta_x c = ab - c, \quad (16)$$

with a, b, c satisfying homogeneous Neumann conditions

$$n(x) \cdot \nabla_x a = 0, \quad n(x) \cdot \nabla_x b = 0, \quad n(x) \cdot \nabla_x c = 0 \quad x \in \partial\Omega, \quad (17)$$

and the nonnegative initial condition

$$a(0, x) = a_0(x) \geq 0, \quad b(0, x) = b_0(x) \geq 0, \quad c(0, x) = c_0(x) \geq 0. \quad (18)$$

As above, due to the rescaling $t \rightarrow \frac{1}{k}t$, $x \rightarrow |\Omega|^{\frac{1}{N}}x$, $(a, b, c) \rightarrow \frac{k}{l}(a, b, c)$, it means no restriction for (14) – (16) to assume that

$$l = k = 1, \quad |\Omega| = 1. \quad (19)$$

The following conservation laws hold for solutions of (14) – (18):

$$0 < M_1 \equiv \int_{\Omega} (a(t, x) + c(t, x)) dx = \int_{\Omega} (a_0(x) + c_0(x)) dx, \quad (20)$$

$$0 < M_2 \equiv \int_{\Omega} (b(t, x) + c(t, x)) dx = \int_{\Omega} (b_0(x) + c_0(x)) dx, \quad (21)$$

where we assume strictly positive masses M_1, M_2 characterizing the unique equilibrium $(a_\infty, b_\infty, c_\infty)$ as the unique nonnegative constants satisfying $a_\infty + c_\infty = M_1$, $b_\infty + c_\infty = M_2$, and $a_\infty b_\infty = c_\infty$, i.e.

$$\begin{aligned} c_\infty &= \frac{1}{2}(1 + M_1 + M_2) - \frac{1}{2}\sqrt{(1 + M_1 + M_2)^2 - 4M_1M_2}, \\ a_\infty &= M_1 - c_\infty, \quad b_\infty = M_2 - c_\infty. \end{aligned} \quad (22)$$

Introducing the entropy functional (or physically free energy) associated to (14) – (18)

$$E(a, b, c) \equiv \int_{\Omega} (a(\ln(a) - 1) + b(\ln(b) - 1) + c(\ln(c) - 1)) dx, \quad (23)$$

our main theorem in this case writes:

Theorem 1.2 *Let Ω be a bounded, connected, and regular (C^3 if $N > 5$) open set of \mathbb{R}^N ($N \geq 1$), and d_a, d_b, d_c be three strictly positive diffusivity constants. Let the initial data a_0, b_0, c_0 be three nonnegative functions of $L^\infty(\Omega)$ with strictly positive masses $0 < M_1, 0 < M_2$ (if $N > 5$, we suppose moreover that a_0, b_0, c_0 are $C^3(\overline{\Omega})$). Then, the unique nonnegative global solution $t \in \mathbb{R}_+ \mapsto (a(t), b(t), c(t))$ in $L^\infty(\Omega)$ to equations (14) – (19) satisfies the following estimate of exponential decay toward equilibrium:*

$$\begin{aligned} \frac{1}{2M_1} \|a(t, \cdot) - a_\infty\|_{L^1(\Omega)} + \frac{1}{2M_2} \|b(t, \cdot) - b_\infty\|_{L^1(\Omega)} + \frac{1}{M_1 + M_2} \|c(t, \cdot) - c_\infty\|_{L^1(\Omega)} \\ \leq \frac{9+2\sqrt{2}}{3+2\sqrt{2}} (E(a_0, b_0, c_0) - E(a_\infty, b_\infty, c_\infty)) e^{-K_1 t}, \end{aligned} \quad (24)$$

with

$$K_1 = \frac{1}{K_2} \min \left\{ 4, \frac{4d_a}{P(\Omega)(\frac{b_\infty}{4} + K_3)}, \frac{4d_b}{P(\Omega)(\frac{a_\infty}{4} + K_4)}, \frac{4d_c}{P(\Omega)(2 + K_5)} \right\}, \quad (25)$$

where $P(\Omega)$ is the Poincaré constant of Ω , and K_2, \dots, K_5 are constants (depending only on d_a, d_b, d_c , and M_1, M_2 , and the global L^∞ bound L_2 (see (43) below)), whose complicated expressions are given in (47) and (51) – (53).

We strongly believe that the presented method should still work whenever some uniform in time L^∞ -bounds are available for the concentrations and one unique, asymptotically stable, steady state exists. Other natural extensions when one of the diffusion constants can be zero or when the L^∞ -bounds grow as a polynomial in time will be studied in [12].

Among the open problems for which extra ideas are probably necessary, we would like to quote:

- cases when L^p -bounds ($p \in [2, \infty)$) for the concentrations are available, but no L^∞ -bounds (that happens, for instance, for four species $\mathcal{A}_1 + \mathcal{A}_2 \rightleftharpoons \mathcal{A}_3 + \mathcal{A}_4$ in dimension $N \geq 2$ (see [12])).
- cases with large number of species (this number can even be infinite, like in coagulation/fragmentation problems).

- cases when the reaction terms give rise to steady states which are not asymptotically stable (like in predator-prey type models).

Notations: In the formulas for K_1, \dots, K_5 as well as in all the following, we introduce capital letters as a short notation for square roots of lower case concentrations

$$A \equiv \sqrt{a}, \quad A_\infty \equiv \sqrt{a_\infty}, \quad B \equiv \sqrt{b}, \quad B_\infty \equiv \sqrt{b_\infty}, \quad C \equiv \sqrt{c}, \quad C_\infty \equiv \sqrt{c_\infty},$$

and overlines for spatial averaging (remember that $|\Omega| = 1$): $\bar{A} = \int_\Omega A dx, \dots$. Though we prefer different letters for different unknowns, there are some points where an index notation is more convenient: $a_1 \equiv a$, $a_2 \equiv b$, $a_3 \equiv c$. There will be no confusion with K_i with i integer denoting various constants. Moreover, we denote $\|f\|_2^2 = \int_\Omega f^2 dx$ for a given function $f : \Omega \rightarrow \mathbb{R}$.

Outline: In section 2, we prove theorem 1.1 and make some remarks. Next, in section 3, we state the proof of theorem 1.2.

2 The case of two equations

We begin with an elementary lemma that will be useful in sections 2 and 3:

Lemma 2.1 *We consider the function $\Phi : (0, \infty)^2 \rightarrow \mathbb{R}$ defined by (11). Then, Φ is continuous on $(0, +\infty)^2$. For all $y > 0$, $\Phi(\cdot, y)$ is strictly increasing on $(0, +\infty)$, and satisfies $\lim_{x \rightarrow 0} \Phi(x, y) = 1$, $\Phi(y, y) = 2$, and $\Phi(x, y) \sim \ln x$ for $x \rightarrow \infty$. Finally, for all $x > 0$, $\Phi(x, \cdot)$ is strictly decreasing.*

Proof of the lemma 2.1: We notice that $\partial_x \Phi(x, y) > 0$ if and only if

$$1 > \ln \left(\frac{x}{y} \right) \left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}} \right)^{-1}. \quad (26)$$

Then, remembering that $\ln a < \sqrt{a} - \frac{1}{\sqrt{a}}$ for $a > 1$, we see that $\partial_x \Phi(x, y) > 0$ for all $x \in (0, +\infty) \setminus \{y\}$. Similarly, we notice that $\partial_y \Phi(x, y) < 0$ if and only if (26) holds and therefore $\partial_y \Phi(x, y) < 0$ for all $y \in (0, +\infty) \setminus \{x\}$. \square

Before we start to prove the theorem, we note that the system (2) – (5) has a unique solution such that

$$0 \leq a(t) \leq L_1 \equiv \|a_0\|_\infty + 2\|b_0\|_\infty, \quad 0 \leq b(t) \leq \frac{L_1}{2}, \quad \text{for } t \geq 0, \quad (27)$$

as can be shown by a direct application of the maximum principle or by comparison with the diffusionless system (see e.g. [30, 4]).

Proof of theorem 1.1: We recall the entropy for equation (2) – (5)

$$E(a, b) \equiv \int_{\Omega} (a (\ln a - 1) + b (\ln b - 1)) dx ,$$

and introduce the entropy dissipation

$$D(a, b) = d_a \int_{\Omega} \frac{|\nabla_x a|^2}{a} dx + d_b \int_{\Omega} \frac{|\nabla_x b|^2}{b} dx + \int_{\Omega} (a^2 - b) \ln \frac{a^2}{b} dx. \quad (28)$$

It is clear that (for nonnegative functions a, b such that identity $\int_{\Omega} (a + 2b) = M$ holds) $D(a, b) = 0$ if and only if $(a, b) = (a_{\infty}, b_{\infty})$. In the following, we prove a quantitative lower bound of the entropy dissipation in terms of the relative entropy with respect to the equilibrium - called sometimes the entropy/entropy-dissipation estimate. Note that this estimate is valid for functions which may have nothing to do with the solutions of eq. (2) – (6).

Lemma 2.2 *For all (measurable) functions $a, b : \Omega \rightarrow \mathbb{R}$, which satisfy that $0 \leq a \leq L_1$, $0 \leq b \leq \frac{L_1}{2}$, and $\int_{\Omega} (a + 2b) = M$,*

$$D(a, b) \geq \frac{4}{K_1} \min \left\{ 1, \frac{d_a}{P(\Omega)K_2} \right\} (E(a, b) - E(a_{\infty}, b_{\infty})), \quad (29)$$

where $P(\Omega)$ is the Poincaré constant of Ω , a_{∞}, b_{∞} are given by (8), and the explicit constants $K_1(L_1, M)$, $K_2(M, d_a/d_b)$ are defined by the formulas (12) and (13).

Proof of lemma 2.2: Recalling the notation $A = \sqrt{a}$, we start with the identity $|\nabla_x a|^2/a = 4|\nabla_x A|^2$, and apply Poincaré's inequality. Using then the inequality $(a - b)(\ln(a) - \ln(b)) \geq 4(A - B)^2$, we get

$$D(a, b) \geq 4 \|A^2 - B\|_2^2 + \frac{4d_a}{P(\Omega)} \|A - \bar{A}\|_2^2 + \frac{4d_b}{P(\Omega)} \|B - \bar{B}\|_2^2. \quad (30)$$

We shall prove in the sequel that the r.h.s. of (30) is bounded below by (some constant times) the relative entropy $E(a, b) - E(a_{\infty}, b_{\infty})$.

Firstly, we use the conservation law (7) to rewrite the relative entropy as

$$\begin{aligned} E(a, b) - E(a_{\infty}, b_{\infty}) &= \int_{\Omega} \left(a \ln \left(\frac{a}{a_{\infty}} \right) - (a - a_{\infty}) \right. \\ &\quad \left. + b \ln \left(\frac{b}{b_{\infty}} \right) - (b - b_{\infty}) \right) dx , \end{aligned}$$

and use lemma 2.1 as well as the global bound (27) to obtain

$$E(a, b) - E(a_\infty, b_\infty) \leq K_1(L_1, M) (A_\infty^2 \|A - A_\infty\|_2^2 + \|B - B_\infty\|_2^2), \quad (31)$$

with $K_1(L_1, M)$ given in (12).

Defining now (for some $\gamma > 0$)

$$K_2(\gamma) = A_\infty \gamma, \quad K_3(\gamma) = 4B_\infty + 1 + \frac{A_\infty}{\gamma}, \quad (32)$$

we prove that the quantity Γ defined below is nonnegative:

$$\begin{aligned} 0 \leq \Gamma \equiv & \|A(A - A_\infty)\|_2^2 + 2A_\infty \int_{\Omega} A(A - A_\infty)^2 dx + K_2 \|A - \bar{A}\|_2^2 \\ & + K_3 \|B - \bar{B}\|_2^2 + 2 \underbrace{\int_{\Omega} (A^2 - A_\infty^2)(B_\infty - B) dx}_{*}. \end{aligned} \quad (33)$$

Note that in (33) only $*$ may be nonpositive. We distinguish three cases:

1. We suppose that $B_\infty - \bar{B} > 0$ and $A_\infty - \bar{A} > 0$. Then, the conservation law (7), i.e. $\int_{\Omega} (A^2 - A_\infty^2) dx = 2 \int_{\Omega} (B_\infty^2 - B^2) dx$ yields

$$* = 2(B_\infty - \bar{B})^2 \int_{\Omega} (B_\infty + B) dx - 2(B_\infty - \bar{B}) \|B - \bar{B}\|_2^2 \quad (34)$$

$$\begin{aligned} & + \int_{\Omega} (A(A - A_\infty)(\bar{B} - B) + A_\infty(A - \bar{A})(\bar{B} - B)) dx \\ & \geq -2B_\infty \|B - \bar{B}\|_2^2 - \frac{1}{2} \|A(A - A_\infty)\|_2^2 - \frac{1}{2} \|B - \bar{B}\|_2^2 \quad (35) \\ & \quad - \frac{A_\infty \gamma}{2} \|A - \bar{A}\|_2^2 - \frac{A_\infty}{2\gamma} \|B - \bar{B}\|_2^2, \end{aligned}$$

thanks to Young's inequality (and for all $\gamma > 0$). By comparing (35) with (33), we obtain the constants (32).

2. We now suppose that $B_\infty - \bar{B} > 0$ and $A_\infty - \bar{A} < 0$. We observe that

$$\begin{aligned} & 2(B_\infty - \bar{B})^2 \int_{\Omega} (B_\infty + B) dx - 2(B_\infty - \bar{B}) \|B - \bar{B}\|_2^2 \\ & \quad = (B_\infty - \bar{B}) \int_{\Omega} (A^2 - A_\infty^2) dx \\ & \quad = (B_\infty - \bar{B}) \left(\|A - \bar{A}\|_2^2 + (\bar{A} - A_\infty)(\bar{A} + A_\infty) \right) \geq 0. \end{aligned}$$

As a consequence, according to (34),

$$* \geq \int_{\Omega} (A(A - A_\infty)(\bar{B} - B) + A_\infty(A - \bar{A})(\bar{B} - B)) dx \quad (36)$$

and (35) still holds.

3. Finally, if $B_\infty - \bar{B} < 0$, then $A_\infty - \bar{A} > 0$ because of (7) and the line (34) is obviously nonnegative (as in the second case), so that (35) holds again.

Therefore, using (33) and (31) yields

$$\begin{aligned} \frac{1}{K_1} (E(a, b) - E(a_\infty, b_\infty)) &\leq A_\infty^2 \|A - A_\infty\|_2^2 + \|B - B_\infty\|_2^2 + \Gamma \\ &= \|A^2 - A_\infty^2 + B_\infty - B\|_2^2 + K_2 \|A - \bar{A}\|_2^2 + K_3 \|B - \bar{B}\|_2^2, \end{aligned} \quad (37)$$

by recalling that $A_\infty^2 = B_\infty$. To conclude the proof of the lemma, it remains to compare (37) with (30), which gives (29) after choosing γ in order to set the fraction $K_2/K_3 = d_a/d_b$ according to (30), i.e. by taking

$$\gamma = \frac{d_a}{d_b} \left(2A_\infty + \frac{1}{2A_\infty} \right) + \sqrt{\frac{d_a^2}{d_b^2} \left(2A_\infty + \frac{1}{2A_\infty} \right)^2 + \frac{d_a}{d_b}}, \quad (38)$$

so that (13) follows (32) and $a_\infty = -\frac{1}{4} + \frac{1}{4}\sqrt{1 + 8M}$ in (8). \square

We now turn to another lemma, which plays here the same role as the Csiszar-Kullback-Pinsker inequality ([8] and [31]) in information theory. That is, we show that the relative entropy $E(a, b) - E(a_\infty, b_\infty)$ controls (from above) the squares of the L^1 -distances to the equilibrium.

Lemma 2.3 *For all (measurable) functions $a, b : \Omega \rightarrow \mathbb{R}$ such that $0 \leq a$, $0 \leq b$ and $\int_\Omega (a + 2b) = M$,*

$$E(a, b) - E(a_\infty, b_\infty) \geq \frac{3 + 2\sqrt{2}}{(6 + 2\sqrt{2})M} \left(\frac{1}{2} \|a - a_\infty\|_1^2 + \|b - b_\infty\|_1^2 \right), \quad (39)$$

where a_∞ and b_∞ are defined by (8).

Proof of lemma 2.3: Recalling $\bar{a} = \int_\Omega a \, dx$ and $\bar{b} = \int_\Omega b \, dx$, we define $q(x) \equiv x \ln x - x$ to write

$$\begin{aligned} E(a, b) - E(a_\infty, b_\infty) &= \int_\Omega a \ln\left(\frac{a}{\bar{a}}\right) dx + \int_\Omega b \ln\left(\frac{b}{\bar{b}}\right) dx \\ &\quad + (q(\bar{a}) - q(a_\infty)) + (q(\bar{b}) - q(b_\infty)). \end{aligned}$$

We first note that thanks to the Csiszar-Kullback-Pinsker inequality

$$\int_\Omega a \ln\left(\frac{a}{\bar{a}}\right) dx \geq \frac{1}{2\bar{a}} \|a - \bar{a}\|_1^2, \quad \int_\Omega b \ln\left(\frac{b}{\bar{b}}\right) dx \geq \frac{1}{2\bar{b}} \|b - \bar{b}\|_1^2,$$

and moreover $\bar{a} \leq M$ and $\bar{b} \leq M/2$ by the conservation of mass (7). Then, we consider $Q(\bar{a}) \equiv q(\bar{a}) + q(\frac{M-\bar{a}}{2})$ for $\bar{a} \in (0, M)$ and $R(\bar{b}) \equiv q(\bar{b}) + q(M-2\bar{b})$ for $\bar{b} \in (0, M/2)$. Since

$$Q''(\bar{a}) = \frac{1}{\bar{a}} + \frac{1}{2} \frac{1}{M-\bar{a}} \geq \frac{3+2\sqrt{2}}{2M}, \quad (40)$$

and

$$R''(\bar{b}) = \frac{1}{\bar{b}} + \frac{4}{M-2\bar{b}} \geq \frac{6+4\sqrt{2}}{M}, \quad (41)$$

we combine 2/3 of (40) and 1/3 of (41) to Taylor-expand

$$(q(\bar{a}) - q(a_\infty)) + (q(\bar{b}) - q(b_\infty)) \geq \frac{3+2\sqrt{2}}{6M} |\bar{a} - a_\infty|^2 + \frac{3+2\sqrt{2}}{3M} |\bar{b} - b_\infty|^2.$$

Finally, we conclude the proof of the lemma by observing that

$$\|a - a_\infty\|_1^2 \leq \frac{6+2\sqrt{2}}{3+2\sqrt{2}} \left(\|a - \bar{a}\|_1^2 + \frac{3+2\sqrt{2}}{3} |\bar{a} - a_\infty|^2 \right),$$

and

$$\|b - b_\infty\|_1^2 \leq \frac{6+2\sqrt{2}}{3+2\sqrt{2}} \left(\|b - \bar{b}\|_1^2 + \frac{3+2\sqrt{2}}{3} |\bar{b} - b_\infty|^2 \right),$$

by Young's inequality. \square

End of the proof of theorem 1.1: We observe that

$$\frac{d}{dt}(E(a(t), b(t)) - E(a_\infty, b_\infty)) = -D(a(t), b(t)).$$

Using lemma 2.2 and Gronwall's lemma, we see that

$$E(a(t), b(t)) - E(a_\infty, b_\infty) \leq (E(a_0, b_0) - E(a_\infty, b_\infty)) e^{-\frac{4t}{K_1} \min(1, \frac{d_a}{P(\Omega)K_2})}, \quad (42)$$

and we obtain theorem 1.1 by combining lemma 2.3 and estimate (42). \blacksquare

Remark 2.1 (Decay rate)

The result of theorem 1.1 express, up to our knowledge, the first explicit rates of convergence to equilibrium for reaction-diffusion systems. The rate $4/K_1 \min\{1, d_a/P(\Omega)K_2\}$ obtained in lemma 2.2 via the entropy method reflects the combined dissipative effects of reaction (i.e. 1 due to the rescaling (6)) and the diffusion (i.e. $d_a/P(\Omega)K_2$).

This is an improvement compared to classical linearization results like [7], where the diffusion term had to dominate over the reaction, which was estimated like a perturbation within a invariant region.

Nevertheless, the obtained rate is not sharp (which is obvious, for instance, in the estimate of case 1 in lemma 2.2).

Remark 2.2 (Example)

We give a numerical example of the rate of exponential decay in theorem 1.1, in order to show that the rates obtained by our method are of order 1 when the data also are of order 1. For $L = 3M$, $M = 3$, $a_\infty = 1 = b_\infty$, $d_a = d_b$, we get

$$\approx \min \left\{ 1.36; 0.26 \frac{d_a}{P(\Omega)} \right\}.$$

3 The case of three equations

Proof of theorem 1.2: Under the assumptions of theorem 1.2, the system (14), (15), (16) with boundary condition (17) and initial data (18) has a unique nonnegative globally bound solution (See e.g. [1, 28, 34, 20, 32] and the references therein for general results. Especially for (14)–(16), see e.g. [35] for dimensions $d \leq 5$ and [18] for all dimensions under the additional assumptions of $C^{2+\alpha}$ -boundaries ($0 < \alpha < 1$) and correspondingly smooth initial data (18)). We denote by L_2 the global bound for this system :

$$L_2 \equiv \sup_{t \geq 0} \{ \|a(t, \cdot)\|_\infty, \|b(t, \cdot)\|_\infty, \|c(t, \cdot)\|_\infty \} < \infty. \quad (43)$$

We recall the entropy functional $E(a, b, c)$ associated to (14) – (19)

$$E(a, b, c) \equiv \int_\Omega (a(\ln a - 1) + b(\ln b - 1) + c(\ln c - 1)) dx,$$

and introduce the corresponding entropy dissipation

$$\begin{aligned} D(a, b, c) &= d_a \int_\Omega \frac{|\nabla_x a|^2}{a} dx + d_b \int_\Omega \frac{|\nabla_x b|^2}{b} dx + d_c \int_\Omega \frac{|\nabla_x c|^2}{c} dx \\ &\quad + \int_\Omega (ab - c)(\ln(ab) - \ln c) dx. \end{aligned}$$

Note that $D(a, b, c) = 0$ if and only if $(a, b, c) = (a_\infty, b_\infty, c_\infty)$ (provided that the conservation laws (20) and (21) hold).

We now state the entropy/entropy-dissipation lemma for our model. Note once again that this lemma applies for functions which are not necessarily solutions of system (14) – (19).

Lemma 3.1 *Let a, b, c be (measurable) functions from Ω to \mathbb{R} such that $0 \leq a \leq L_2$, $0 \leq b \leq L_2$, $0 \leq c \leq L_2$ and $\int_\Omega a + c = M_1$, $\int_\Omega b + c = M_2$. Then,*

$$D(a, b, c) \geq K_1(E(a, b, c) - E(a_\infty, b_\infty, c_\infty)) \quad (44)$$

with K_1 defined by (25) (and (47), (51) – (53)), and $a_\infty, b_\infty, c_\infty$ defined by (22).

Proof of lemma 3.1: Let still square roots be denoted by capital letters $A = \sqrt{a}$, $B = \sqrt{b}$, $C = \sqrt{c}$. Using Poincaré's inequality and $(ab - c)(\ln(ab) - \ln c) \geq 4(AB - C)^2$, we obtain the estimate

$$D(a, b, c) \geq 4 \|AB - C\|_2^2 + \frac{4d_a}{P(\Omega)} \|A - \bar{A}\|_2^2 + \frac{4d_b}{P(\Omega)} \|B - \bar{B}\|_2^2 + \frac{4d_c}{P(\Omega)} \|C - \bar{C}\|_2^2. \quad (45)$$

Analog to the proof of lemma 2.2, we show in the sequel that the r.h.s. of (45) is bounded below by the relative entropy $E(a, b, c) - E(a_\infty, b_\infty, c_\infty)$.

First, we use the conservation laws (20), (21) to rewrite the relative entropy as

$$\begin{aligned} E(a, b, c) - E(a_\infty, b_\infty, c_\infty) &= \int_{\Omega} \left(a \ln \left(\frac{a}{a_\infty} \right) - (a - a_\infty) \right. \\ &\quad \left. + b \ln \left(\frac{b}{b_\infty} \right) - (b - b_\infty) + c \ln \left(\frac{c}{c_\infty} \right) - (c - c_\infty) \right) dx, \end{aligned}$$

and we use lemma 2.1 as well as the global bound (43) to estimate

$$\begin{aligned} E(a, b, c) - E(a_\infty, b_\infty, c_\infty) &\leq K_2 \left(\frac{B_\infty^2}{4} \|A - A_\infty\|_2^2 + \frac{A_\infty^2}{4} \|B - B_\infty\|_2^2 \right. \\ &\quad \left. + \|C - C_\infty\|_2^2 \right), \end{aligned} \quad (46)$$

with

$$K_2(L_2, M_1, M_2) = \max \left\{ \frac{4}{b_\infty} \Phi(\|a\|_\infty, a_\infty), \frac{4}{a_\infty} \Phi(\|b\|_\infty, b_\infty), \Phi(\|c\|_\infty, c_\infty) \right\}. \quad (47)$$

The statement of lemma 3.1 with the constant K_1 given by (25) follows from the following lemma, which provides an upper bound for the r.h.s. of (46) in terms of the r.h.s. of (45).

Lemma 3.2 *Let A , B , and C be (measurable) functions from Ω to \mathbb{R}_+ such that*

$$\overline{A^2} + \overline{C^2} = M_1 \quad (48)$$

and

$$\overline{B^2} + \overline{C^2} = M_2. \quad (49)$$

Then, the estimate

$$\begin{aligned} \frac{B_\infty^2}{4} \|A - A_\infty\|_2^2 + \frac{A_\infty^2}{4} \|B - B_\infty\|_2^2 + \|C - C_\infty\|_2^2 &\leq \|AB - C\|_2^2 \\ + \left(\frac{B_\infty^2}{4} + K_3 \right) \|A - \bar{A}\|_2^2 + \left(\frac{A_\infty^2}{4} + K_4 \right) \|B - \bar{B}\|_2^2 + (2 + K_5) \|C - \bar{C}\|_2^2 &(50) \end{aligned}$$

holds, with the constants

$$K_3 = \max \left\{ \begin{array}{l} C_\infty + \frac{A_\infty B_\infty}{2} + \frac{2C_\infty^2}{M_1}, \\ \sqrt{\min\{M_1, M_2\}} + \frac{A_\infty B_\infty}{2}, \\ C_\infty + \frac{A_\infty \sqrt{M_2}}{2} + \frac{A_\infty (\sqrt{M_2} + B_\infty) C_\infty}{M_1} + \frac{A_\infty^2}{2} + \frac{B_\infty (\sqrt{M_2} - B_\infty)}{2}, \\ C_\infty + \frac{\sqrt{M_1} B_\infty}{2} - \frac{B_\infty^2}{2} - \frac{A_\infty (\sqrt{M_1} - A_\infty)}{2}, \\ C_\infty + \frac{A_\infty (\sqrt{M_2} - B_\infty)}{4} + \frac{(\sqrt{M_1} - A_\infty) B_\infty}{4} \end{array} \right\}, \quad (51)$$

$$K_4 = \max \left\{ \begin{array}{l} C_\infty + \frac{A_\infty B_\infty}{2} + \frac{2C_\infty^2}{M_2}, \\ \sqrt{\min\{M_1, M_2\}} + \frac{A_\infty B_\infty}{2}, \\ C_\infty + \frac{A_\infty \sqrt{M_2}}{2} - \frac{A_\infty^2}{2} - \frac{B_\infty (\sqrt{M_2} - B_\infty)}{2}, \\ C_\infty + \frac{\sqrt{M_1} B_\infty}{2} + \frac{(\sqrt{M_1} + A_\infty) B_\infty C_\infty}{M_2} + \frac{B_\infty^2}{2} + \frac{A_\infty (\sqrt{M_1} - A_\infty)}{2}, \\ C_\infty + \frac{A_\infty (\sqrt{M_2} - B_\infty)}{4} + \frac{(\sqrt{M_1} - A_\infty) B_\infty}{4} \end{array} \right\}, \quad (52)$$

and

$$K_5 = \max \left\{ 2C_\infty^2 \left(\frac{1}{M_1} + \frac{1}{M_2} \right), \frac{A_\infty (\sqrt{M_2} + B_\infty) C_\infty}{M_1}, \frac{(\sqrt{M_1} + A_\infty) B_\infty C_\infty}{M_2} \right\}. \quad (53)$$

Proof of lemma 3.2: In a first step, we reformulate (50) in order to isolate the “bad” terms (i.e. the analogs to $*$ in equation (33) in section 2). In a second step, we shall control these terms using the conservations laws $\overline{A^2} + \overline{C^2} = M_1$ and $\overline{B^2} + \overline{C^2} = M_2$.

We start with

$$\|AB - C\|_2^2 = \|AB - A_\infty B_\infty\|_2^2 + \|C - C_\infty\|_2^2 \quad (54)$$

$$+ 2 \int_{\Omega} (AB - A_\infty B_\infty)(C_\infty - C) dx, \quad (55)$$

where we split (55) using $C_\infty - C = (C_\infty - \overline{C}) + (\overline{C} - C)$ and calculate for the first part

$$2 \int_{\Omega} (AB - A_\infty B_\infty)(C_\infty - \overline{C}) = (C_\infty - \overline{C}) \left(2 \int_{\Omega} (A - \overline{A})(B - \overline{B}) dx \right) \quad (56)$$

$$+ (\overline{A} - A_\infty)(\overline{B} + B_\infty) + (\overline{B} - B_\infty)(\overline{A} + A_\infty), \quad (57)$$

while we estimate by Young’s inequality for the second part

$$2 \int_{\Omega} (AB - A_\infty B_\infty)(\overline{C} - C) dx \geq -\frac{1}{2} \|AB - A_\infty B_\infty\|_2^2 - 2 \|C - \overline{C}\|_2^2. \quad (58)$$

After inserting (58) into (55), there remains $\frac{1}{2}\|AB - A_\infty B_\infty\|_2^2$, which we split again as the sum of two halves. Expanding the first half using $-A_\infty B + A_\infty B$ and the second half using $-AB_\infty + AB_\infty$ yields

$$\begin{aligned} & \frac{1}{2}\|AB - A_\infty B_\infty\|_2^2 = \\ & \frac{1}{4}\|(A - A_\infty)B\|_2^2 + \frac{1}{2}\int_\Omega (A - A_\infty)BA_\infty(B - B_\infty) dx + \frac{A_\infty^2}{4}\|B - B_\infty\|_2^2 \quad (59) \\ & + \frac{1}{4}\|A(B - B_\infty)\|_2^2 + \frac{1}{2}\int_\Omega A(B - B_\infty)(A - A_\infty)B_\infty dx + \frac{B_\infty^2}{4}\|A - A_\infty\|_2^2 \quad (60) \end{aligned}$$

Next, the integrals in (59) and (60) are expanded using $B - B_\infty = (B - \bar{B}) + (\bar{B} - B_\infty)$ (respectively $A - A_\infty = (A - \bar{A}) + (\bar{A} - A_\infty)$) and the first of these further parts are estimated thanks to

$$\frac{1}{2}\int_\Omega (A - A_\infty)BA_\infty(B - \bar{B}) dx \geq -\frac{1}{4}\|(A - A_\infty)B\|_2^2 - \frac{A_\infty^2}{4}\|B - \bar{B}\|_2^2, \quad (61)$$

$$\frac{1}{2}\int_\Omega A(B - B_\infty)(A - \bar{A})B_\infty dx \geq -\frac{1}{4}\|A(B - B_\infty)\|_2^2 - \frac{B_\infty^2}{4}\|A - \bar{A}\|_2^2. \quad (62)$$

Altogether, we obtain from (59)–(62) that

$$\begin{aligned} & \frac{1}{2}\|AB - A_\infty B_\infty\|_2^2 \geq \quad (63) \\ & \frac{B_\infty^2}{4}\|A - A_\infty\|_2^2 + \frac{A_\infty^2}{4}\|B - B_\infty\|_2^2 - \frac{A_\infty^2}{4}\|B - \bar{B}\|_2^2 - \frac{B_\infty^2}{4}\|A - \bar{A}\|_2^2 \\ & + \frac{1}{2}(A_\infty(\bar{B} - B_\infty) + (\bar{A} - A_\infty)B_\infty)\int_\Omega (A - \bar{A})(B - \bar{B}) dx \\ & + \frac{1}{2}(\bar{A} - A_\infty)(\bar{B} - B_\infty)(A_\infty\bar{B} + B_\infty\bar{A}). \quad (64) \end{aligned}$$

After inserting (54)–(64) into (50), it remains (as second step) to show that

$$\begin{aligned} & K_3\|A - \bar{A}\|_2^2 + K_4\|B - \bar{B}\|_2^2 + K_5\|C - \bar{C}\|_2^2 + \\ & (2(C_\infty - \bar{C}) + \frac{1}{2}(A_\infty(\bar{B} - B_\infty) + (\bar{A} - A_\infty)B_\infty))\int_\Omega (A - \bar{A})(B - \bar{B}) dx \\ & + (C_\infty - \bar{C})((\bar{A} - A_\infty)(\bar{B} + B_\infty) + (\bar{B} - B_\infty)(\bar{A} + A_\infty)) \\ & + \frac{1}{2}(\bar{A} - A_\infty)(\bar{B} - B_\infty)(A_\infty\bar{B} + B_\infty\bar{A}) \geq 0, \quad (65) \end{aligned}$$

for which we are going to distinguish five cases:

1. For the case $\bar{A} < A_\infty$, $\bar{B} < B_\infty$, and $\bar{C} < C_\infty$, it is sufficient to show

$$\begin{aligned} & K_3\|A - \bar{A}\|_2^2 + K_4\|B - \bar{B}\|_2^2 + K_5\|C - \bar{C}\|_2^2 \\ & - (2C_\infty + A_\infty B_\infty)\left(\frac{1}{2}\|A - \bar{A}\|_2^2 + \frac{1}{2}\|B - \bar{B}\|_2^2\right) \quad (66) \end{aligned}$$

$$- (C_\infty - \bar{C})(A_\infty - \bar{A})(\bar{B} + B_\infty) \quad (67)$$

$$- (C_\infty - \bar{C})(B_\infty - \bar{B})(\bar{A} + A_\infty) \geq 0. \quad (68)$$

Since (67) and (68) are symmetric in A and B , we choose (68) to show how the conservation (21) - rewritten in the form

$$\|B - \bar{B}\|_2^2 + \|C - \bar{C}\|_2^2 = (B_\infty - \bar{B})(\bar{B} + B_\infty) + (C_\infty - \bar{C})(\bar{C} + C_\infty), \quad (69)$$

allows to prove that (remember that $\bar{A} < A_\infty$, $\bar{B} < B_\infty$, $\bar{C} < C_\infty$)

$$\begin{aligned} & \frac{2C_\infty^2}{M_2} (\|B - \bar{B}\|_2^2 + \|C - \bar{C}\|_2^2) - (C_\infty - \bar{C})(B_\infty - \bar{B})(\bar{A} + A_\infty) \geq \\ & \frac{2C_\infty^2}{M_2} ((B_\infty - \bar{B})B_\infty + (C_\infty - \bar{C})C_\infty) - 2A_\infty(C_\infty - \bar{C})(B_\infty - \bar{B}). \end{aligned}$$

This last expression is a linear function of \bar{B} (which will be denoted by $\Psi(\bar{B})$) which is nonnegative on $0 \leq \bar{B} \leq B_\infty$ because

$$\Psi(B_\infty) \geq \frac{2C_\infty^2}{M_2} (C_\infty - \bar{C})C_\infty \geq 0, \quad (70)$$

$$\Psi(0) \geq \frac{2A_\infty B_\infty}{\frac{B_\infty^2}{C_\infty - \bar{C}} + C_\infty} (B_\infty^2 + (C_\infty - \bar{C})C_\infty) \quad (71)$$

$$-2A_\infty B_\infty (C_\infty - \bar{C}) = 0, \quad (72)$$

where we have used $A_\infty B_\infty = C_\infty$ and $C_\infty \geq C_\infty - \bar{C}$.

For (67), there is a symmetric estimate based on the conservation (20). Adding these two estimates together with the coefficient of (66) gives the first lines for the constants K_3 (51) and K_4 (52) and the first expression for K_5 (53).

2. For the second case $\bar{A} < A_\infty$, $\bar{B} < B_\infty$, and $\bar{C} > C_\infty$, we proceed in a similar way to (66)–(68), but instead of line (66), we find here

$$-(2\bar{C} + A_\infty B_\infty) \left(\frac{1}{2} \|A - \bar{A}\|_2^2 + \frac{1}{2} \|B - \bar{B}\|_2^2 \right),$$

while the lines (67) and (68) are nonnegative in this case and thus neglected. Using the estimate $\bar{C}^2 \leq \min\{M_1, M_2\}$ (due to (20) and (21)), we get the second line of (51) and (52).

3. In the third case $\bar{A} < A_\infty$, $\bar{B} > B_\infty$, the latter hypothesis implies $\bar{C} < C_\infty$ by the conservation law (21). As above, we estimate

$$K_3 \|A - \bar{A}\|_2^2 + K_4 \|B - \bar{B}\|_2^2 + K_5 \|C - \bar{C}\|_2^2 \quad (73)$$

$$-(2C_\infty + A_\infty \sqrt{M_2}) \left(\frac{1}{2} \|A - \bar{A}\|_2^2 + \frac{1}{2} \|B - \bar{B}\|_2^2 \right) \quad (74)$$

$$-(C_\infty - \bar{C})(A_\infty - \bar{A})(\bar{B} + B_\infty) \quad (75)$$

$$+\frac{1}{2}(\bar{A} - A_\infty)(\bar{B} - B_\infty)(A_\infty \bar{B} + B_\infty \bar{A}) \geq 0 \quad (76)$$

where we have used the conservation law (21) to estimate

$$\bar{B} \leq \sqrt{\bar{B}^2} \leq \sqrt{M_2},$$

in the coefficient of (74). An analog argument to (69)–(72) shows for the term (75) that

$$\begin{aligned} & \frac{A_\infty(\sqrt{M_2} + B_\infty)C_\infty}{M_1} (\|A - \bar{A}\|_2^2 + \|C - \bar{C}\|_2^2) \\ & - (C_\infty - \bar{C})(A_\infty - \bar{A})(\bar{B} + B_\infty) \geq 0. \end{aligned} \quad (77)$$

For (76), we use

$$\|A - \bar{A}\|_2^2 - \|B - \bar{B}\|_2^2 = (A_\infty - \bar{A})(\bar{A} + A_\infty) - (B_\infty - \bar{B})(\bar{B} + B_\infty)$$

in order to calculate

$$\begin{aligned} & (\bar{A} - A_\infty)(\bar{B} - B_\infty)(A_\infty\bar{B} + A_\infty B_\infty - 2A_\infty B_\infty + A_\infty B_\infty + B_\infty\bar{A}) \\ & = A_\infty(\bar{A} - A_\infty)^2(\bar{A} + A_\infty) + B_\infty(\bar{B} - B_\infty)^2(\bar{B} + B_\infty) \\ & \quad - 2A_\infty B_\infty(\bar{A} - A_\infty)(\bar{B} - B_\infty) \end{aligned} \quad (78)$$

$$+ (A_\infty(\bar{A} - A_\infty) - B_\infty(\bar{B} - B_\infty)) (\|A - \bar{A}\|_2^2 - \|B - \bar{B}\|_2^2), \quad (79)$$

where the line (78) is nonnegative in the considered case. For (79), we observe that in the present case, $\|A - \bar{A}\|_2^2 - \|B - \bar{B}\|_2^2$ is nonnegative and hence that (79) is bounded below by

$$-(A_\infty^2 + B_\infty(\sqrt{M_2} - B_\infty)) (\|A - \bar{A}\|_2^2 - \|B - \bar{B}\|_2^2). \quad (80)$$

Hence, combining (76) and (77)–(80) yields the third contributions to K_3 (51), K_4 (52) and the second to K_5 (53).

4. Concerning the fourth case $\bar{A} > A_\infty$ implying $\bar{C} < C_\infty$ and $\bar{B} < B_\infty$ (by (20)), we proceed in a symmetric way compared to case three, which leads to the fourth contributions to K_3 (51), K_4 (52) and the third to K_5 (53).
5. In the final case, we consider $\bar{A} > A_\infty$, $\bar{B} > B_\infty$ implying $\bar{C} < C_\infty$. Therefore, (65) is bounded below by

$$-(C_\infty + \frac{A_\infty(\sqrt{M_2} - B_\infty)}{4} + \frac{(\sqrt{M_1} - A_\infty)B_\infty}{4}) (\|A - \bar{A}\|_2^2 + \|B - \bar{B}\|_2^2),$$

which completes the formulas for K_3 (51) and K_4 (52).

This ends the proof of lemma 3.2. \square

According to estimate (46) and lemma 3.2, we obtain lemma 3.1. \square

We now write down the lemma which plays the role of Csiszar-Kullback-Pinsker inequality in information theory.

Lemma 3.3 *For all (measurable) functions $a, b, c : \Omega \rightarrow \mathbb{R}$ such that $0 \leq a$, $0 \leq b$, $0 \leq c$ and $\int_{\Omega}(a + c) = M_1$, $\int_{\Omega}(b + c) = M_2$,*

$$\begin{aligned} E(a, b, c) - E(a_{\infty}, b_{\infty}, c_{\infty}) &\geq \frac{3+2\sqrt{2}}{2M_1(9+2\sqrt{2})} \|a - a_{\infty}\|_1^2 \\ &+ \frac{3+2\sqrt{2}}{2M_2(9+2\sqrt{2})} \|b - b_{\infty}\|_1^2 + \frac{3+2\sqrt{2}}{(M_1+M_2)(9+2\sqrt{2})} \|c - c_{\infty}\|_1^2, \end{aligned} \quad (81)$$

where a_{∞} , b_{∞} and c_{∞} are defined by (22).

Proof of lemma 3.3: As in lemma 2.3, we define $q(a_i) = a_i \ln a_i - a_i$ for $a_1 \equiv a$, $a_2 \equiv b$, and $a_3 \equiv c$, and rewrite

$$E(a, b, c) - E(a_{\infty}, b_{\infty}, c_{\infty}) = \int_{\Omega} a \ln \frac{a}{a_{\infty}} dx + \int_{\Omega} b \ln \frac{b}{b_{\infty}} dx + \int_{\Omega} c \ln \frac{c}{c_{\infty}} dx \quad (82)$$

$$+ q(\bar{a}) - q(a_{\infty}) + q(\bar{b}) - q(b_{\infty}) + q(\bar{c}) - q(c_{\infty}). \quad (83)$$

Using the conservations (20) and (21), and defining

$$Q(M_i, \bar{a}_i) = q(\bar{a}_i) + \frac{1}{2}q(M_i - \bar{a}_i) \quad \text{for } \bar{a}_i \in [0, M_i], \quad i = 1, 2,$$

$$R(M_i, \bar{a}_i) = \frac{1}{2}q(\bar{a}_i) + q(M_i - \bar{a}_i) \quad \text{for } \bar{a}_i \in [0, M_i], \quad i = 1, 2,$$

we can write the line (83) in two ways as

$$\begin{aligned} &Q(M_1, \bar{a}) - Q(M_1, a_{\infty}) + Q(M_2, \bar{b}) - Q(M_2, b_{\infty}) \\ &= R(M_1, \bar{c}) - R(M_1, c_{\infty}) + R(M_2, \bar{c}) - R(M_2, c_{\infty}). \end{aligned} \quad (84)$$

Since

$$Q'(M_1, a_{\infty}) + Q'(M_2, b_{\infty}) = -R'(M_1, c_{\infty}) - R'(M_2, c_{\infty}) = \ln \frac{a_{\infty} b_{\infty}}{c_{\infty}} = 0$$

and (for $\bar{a}_i \in [0, M_i], i = 1, 2$)

$$Q''(M_i, \bar{a}_i), R''(M_i, \bar{a}_i) \geq \frac{K_6}{2M_i} \quad \text{with} \quad K_6 = 3 + 2\sqrt{2},$$

we bound (84) from below by

$$\frac{K_6}{4M_1} (\theta |\bar{a} - a_\infty|^2 + (1 - \theta) |\bar{c} - c_\infty|^2) + \frac{K_6}{4M_2} (\theta |\bar{b} - b_\infty|^2 + (1 - \theta) |\bar{c} - c_\infty|^2) ,$$

for all $0 \leq \theta \leq 1$. Using the Csiszar-Kullback-Pinsker inequality and choosing $\theta = \frac{1}{3}$, we obtain

$$\begin{aligned} E(a, b, c) - E(a_\infty, b_\infty, c_\infty) &\geq \frac{1}{2M_1} (\|a - \bar{a}\|_1^2 + \frac{K_6}{6} |\bar{a} - a_\infty|^2) \\ &+ \frac{1}{2M_2} (\|b - \bar{b}\|_1^2 + \frac{K_6}{6} |\bar{b} - b_\infty|^2) + \frac{1}{M_1 + M_2} (\|c - \bar{c}\|_1^2 + \frac{K_6}{6} |\bar{c} - c_\infty|^2) . \end{aligned}$$

Then,

$$\|a - a_\infty\|_1^2 \leq \left(1 + \frac{6}{K_6}\right) \left(\|a - \bar{a}\|_1^2 + \frac{K_6}{6} |\bar{a} - a_\infty|^2\right) ,$$

and the same holds for b and c , so that we get

$$\begin{aligned} E(a, b, c) - E(a_\infty, b_\infty, c_\infty) &\geq \frac{1}{2M_1} \frac{K_6}{K_6 + 6} \|a - a_\infty\|_1^2 + \frac{1}{2M_2} \frac{K_6}{K_6 + 6} \|b - b_\infty\|_1^2 \\ &+ \frac{1}{M_1 + M_2} \frac{K_6}{K_6 + 6} \|c - c_\infty\|_1^2 . \end{aligned}$$

This ends the proof of lemma 3.3. \square

End of the proof of theorem 1.2: Noting that the solution to the system (14) – (19) satisfies the entropy equality

$$\frac{d}{dt} (E(a(t), b(t), c(t)) - E(a_\infty, b_\infty, c_\infty)) = -D(a(t), b(t), c(t)) ,$$

we see that theorem 1.2 is a direct consequence of lemma 3.1, lemma 3.3 and Gronwall's lemma. \blacksquare

References

- [1] H. Amann, *Global existence for semilinear parabolic problems*. J. Reine Angew. Math. **360**, (1985), pp. 47–83.
- [2] A. Barabanova, *On the global existence of solutions of a reaction-diffusion equation with exponential nonlinearity*. Proc. Amer. Math. Soc. **122**, no. 3 (1994), pp. 827–831.
- [3] M. Bisi, L. Desvillettes, *From Reactive Boltzmann Equations to Reaction-Diffusion Systems*. preprint 2005-11 of the CMLA, ENS de Cachan.

- [4] D. Bothe, D. Hilhorst, *A reaction-diffusion system with fast reversible reaction*. J. Math. Anal. Appl., **286**, no. 1 (2003), pp. 125–135.
- [5] M. Caceres, J. Carrillo, G. Toscani, *Long-time behavior for a nonlinear fourth order parabolic equation* Trans. Amer. Math. Soc. **357**, no. 3 (2005), pp. 1161–1175 (electronic).
- [6] M. Caceres, J. Carrillo, T. Goudon, *Equilibration rate for the linear inhomogeneous relaxation-time Boltzmann equation for charged particles*. Comm. Partial Differential Equations **28**, no. 5-6 (2003), pp. 969–989.
- [7] E. Conway, D. Hoff, J. Smoller, *Large time behaviour of solutions of systems of nonlinear reaction-diffusion equations*. SIAM J. Appl. Math. **35**, no. 1 (1978), pp. 1–16.
- [8] I. Csiszár, *Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis von Markoffschen Ketten*. Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), pp. 85–108.
- [9] J. Carrillo, J.L. Vazquez, *Fine asymptotics for fast diffusion equations*. Comm. Partial Differential Equations **28**, no. 5-6 (2003), pp. 1023–1056.
- [10] M. Del Pino, J. Dolbeault, *Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions*. J. Math. Pures Appl. **81**, no. 9 (2002), pp. 847–875.
- [11] A. De Masi, E. Presutti, *Mathematical methods for hydrodynamic limits*, Springer-Verlag, Berlin, 1991.
- [12] L. Desvillettes, K. Fellner, *Entropy Methods for Reaction-Diffusion Equations: Degenerate Diffusion and Slowly Growing A-priori Bounds*. work in preparation.
- [13] L. Desvillettes, C. Villani, *On the spatially homogeneous Landau equation for hard potentials. II. H-theorem and applications*. Comm. Partial Differential Equations **25**, no. 1-2 (2000), pp. 261–298.
- [14] L. Desvillettes, C. Villani, *On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation*. Comm. Pure Appl. Math. **54**, no. 1 (2001), pp. 1–42.
- [15] L. Desvillettes, C. Villani, *On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation*. Invent. Math. **159** (2005), no. 2, pp. 245–316.

- [16] P. Érdi, J. Tóth, Mathematical models of chemical reactions. Nonlinear Science: theory and applications, Manchester University Press (1989).
- [17] K. Fellner, L. Neumann, C. Schmeiser, *Convergence to global equilibrium for spatially inhomogeneous kinetic models of non-micro-reversible processes*, Monatsh. Math. **141** (2004), no. 4, pp. 289–299.
- [18] W. Feng, *Coupled system of reaction-diffusion equations and applications in carrier facilitated diffusion*. Nonlinear Analysis, Theory, Methods & Applications **17**, no. 3 (1991), pp. 285–311.
- [19] P. C. Fife, Mathematical Aspects of Reacting and Diffusing Systems. Lecture Notes in Biomathematics, Vol. 28, Springer, Berlin (1979).
- [20] W. E. Fitzgibbon, J. Morgan, R. Sanders, *Global existence and boundedness for a class of inhomogeneous semilinear parabolic systems*. Nonlinear Analysis, Theory, Methods & Application, **19**, no.9 (1992), pp. 885-899.
- [21] W. Fitzgibbon, S. Hollis, J. Morgan, *Stability and Lyapunov functions for reaction-diffusion systems*. SIAM J. Math. Anal. **28** (1997), no. 3, pp. 595–610.
- [22] K. Gröger, *Free energy estimates and asymptotic behaviour of reaction-diffusion processes*. Preprint 20, Institut für Angewandte Analysis und Stochastik, Berlin, 1992.
- [23] A. Glitzky, K. Gröger, R. Hünlich, *Free energy and dissipation rate for reaction-diffusion processes of electrically charged species*. Appl. Anal. **60**, no. 3-4 (1996), pp. 201–217.
- [24] A. Glitzky, R. Hünlich, *Energetic estimates and asymptotics for electro-reaction-diffusion systems*. Z. Angew. Math. Mech. **77** (1997), pp. 823–832.
- [25] A. Haraux, A. Youkana, *On a result of K. Masuda concerning reaction-diffusion equations*. Tôhoku Math. J. **4** (1988), pp. 159–163.
- [26] S. Hollis, R. Martin, M. Pierre, *Global existence and boundedness in reaction-diffusion systems*. SIAM J. Math. Appl. **18** (1987), pp. 744–761.
- [27] H. Hoshino, Y. Yamada, *Asymptotic behavior of global solutions for some reaction-diffusion systems*. Nonlinear Anal. **23** (1994), no. 5, pp. 639–650.

- [28] Y. I. Kanel, *Cauchy's problem for semilinear parabolic equations with balance conditions*. translation from Differ. Uravn. **20**, No.10, (1984) pp. 1753–1760; Differ. Equations **20**, (1984) pp. 1260-1266.
- [29] Y. I. Kanel, M. Kirane, *Global solutions of Reaction-Diffusion Systems with a Balance Law and Nonlinearities of Exponential Growth*. J. Diff. Equ. **165**, (2000), pp. 24–41.
- [30] M. Kirane, *On stabilization of solutions of the system of parabolic differential equations describing the kinetics of an autocatalytic reversible chemical reaction*. Bull. Inst. Mat. Acad. Sin. **18**, no. 4 (1990), pp. 369–377.
- [31] S. Kullback, Information Theory and Statistics, John Wiley, (1959).
- [32] R. H. Martin, M. Pierre, *Nonlinear Reaction-Diffusion Systems*. in Non-linear Equations in Applied Sciences, W. F. Ames and C. Rogers , eds., Math. Sci. Engrg. 185, Academic Press, New York (1991).
- [33] K. Masuda, *On the global existence and asymptotic behavior of solution of reaction-diffusion equations*. Hokkaido Math. J. **12**, (1983), pp. 360–370.
- [34] J. Morgan, *Global existence for semilinear parabolic systems*. SIAM J. Math. Ana. **20**, (1989), pp. 1128–1144.
- [35] F. Rothe, Global Solutions of Reaction-Diffusion Systems. Lecture Notes in Mathematics, Springer, Berlin, (1984).
- [36] J. Smoller, Shock Waves and Reaction-Diffusion Equations. Grundlehren der mathematischen Wissenschaften, Vol. 258, Springer, New-York (1983).
- [37] R. Spigler, D.H. Zanette, “Reaction–diffusion models from the Fokker–Planck formulation of chemical processes”, *IMA J. Appl. Math.*, **49** (1992), 217–229.
- [38] R. Spigler, D.H. Zanette, “A BGK model for chemical processes: The reaction–diffusion approximation”, *Math. Mod. Meth. Appl. Sci.*, **4** (1994), 35–47.
- [39] A.-S. Sznitman, “Propagation of Chaos for a System of Annihilating Brownian Spheres”, *Comm. Pure Appl. Math.*, **40** (1987), 663–690.

- [40] M. E. Taylor, Partial Differential Equation III – Nonlinear Equations. Springer Series Applied Mathematical Sciences Vol. 117, New York, (1996).
- [41] G. Toscani, C. Villani, *Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation.* Comm. Math. Phys. **203**, no. 3 (1999), pp. 667–706.
- [42] G. Toscani, C. Villani, *On the trend to equilibrium for some dissipative systems with slowly increasing a priori bounds.* J. Statist. Phys. **98**, no. 5-6 (2000), pp. 1279–1309.
- [43] C. Villani, *Cercignani’s conjecture is sometimes true and always almost true.* Comm. Math. Phys. **234**, no. 3 (2003), pp. 455–490.
- [44] A. Volpert, Vitaly A. Volpert, Vladimir A. Volpert, Traveling Wave Solutions of Parabolic Systems. Translations of Mathematical Monographs, Vol. 140, AMS, (1994).
- [45] G. F. Webb, Theory of nonlinear age-dependent population dynamics. Monographs and Textbooks in Pure and Applied Mathematics, 89. Marcel Dekker, Inc., New York, 1985.
- [46] T. Zelenyak, *Stabilization of solutions to boundary value problems for second order parabolic equations in one space variable.* Differential’nye Uravneniya **4** (1968), pp. 34–45.