# (hypo)coercive schemes for the Fokker-Planck equation

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# Introduction

We are interested in the study of the following inhomogeneous Fokker-Planck equation

$$\partial_t F + v\partial_x F - \partial_v(\partial_v + v)F = 0, \qquad F|_{t=0} = F^0,$$

where

$$0 \leq F = F(t,x,v), \qquad (t,x,v) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R} \qquad \iint F dx dv = 1.$$

We focus in this talk on the case $d = 1$.
The now rather standard hypocoercive methods give that

$$F(t,x,v) \underset{t\to+\infty}{\longrightarrow} \mathcal{M}(x,v),$$

exponentially fast (for a large family of similar kinetic equations), where here the Maxwellian is given by

$$\mathcal{M}(x,v) = \mu(v) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-v^2/2}.$$

A much simpler equation (homogeneous kinetic equation) is

$$\partial_t F - \partial_v(\partial_v + v)F = 0, \qquad F|_{t=0} = F^0, \tag{1}$$

where

$$0 \leq F = F(t,v), \qquad (t,v) \in \mathbb{R}^+ \times \mathbb{R}, \qquad \int F \mathrm{d}v = 1,$$

for which this is very easy to get (by "coercive" methods) that

$$F(t,x,v) \underset{t \to +\infty}{\longrightarrow} \mu(v).$$

(This is just the heat equation for the harmonic oscillator.)

Functional framework and proof for the homogeneous problem :

- set $F = \mu + \mu f$,
- the equation is $\partial_t f + (-\partial_v + v)\partial_v f = 0$ with $f|_{t=0} = f^0$,
- consider $f \in L^2(\mathrm{d}\mu) \subset L^1(\mathrm{d}\mu)$ (strictly smaller),
- note that $\langle f \rangle \stackrel{\text{def}}{=} \int f \mathrm{d}\mu = \int f_0 \mathrm{d}\mu = 0$,
- note that $f \in L^1(\mathrm{d}\mu) \Leftrightarrow F \in L^1(dv)$,
- compute $\frac{\mathrm{d}}{\mathrm{d}t} \|f\|^2_{L^2(\mathrm{d}\mu)} = -2 \langle(-\partial_v + v)\partial_v f, f\rangle_{L^2(\mathrm{d}\mu)} = -2 \|\partial_v f\|^2_{L^2(\mathrm{d}\mu)}$,
- use Poincaré inequality $\|f\|^2_{L^2(\mathrm{d}\mu)} \leq \|\partial_v f\|^2_{L^2(\mathrm{d}\mu)}$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|f\|^2_{L^2(\mathrm{d}\mu)} \leq -2 \|f\|^2_{L^2(\mathrm{d}\mu)} \,,$$

- use Gronwall inequality $\|f\|_{L^2(\mathrm{d}\mu)} \leq e^{-t} \left\|f^0\right\|_{L^2(\mathrm{d}\mu)}$,
- synthesis $\|F - \mathcal{M}\|_{L^1(\mathrm{d}v)} \leq \|f\|_{L^2(\mathrm{d}\mu)} \leq e^{-t} \left\|f^0\right\|_{L^2(\mathrm{d}\mu)}$.

Many ingredients were involved in the short previous proof :

Hilbertian framework, coercivity, Poincaré inequality, Gronwall lemma, existence of a Maxwellian ...

**Aim of this talk :**

- Explain how to adapt to the inhomogeneous case
  ▷ well understood and robust theory
- Explain how to discretize and numerically implement the problems
  ▷ new even in the homogeneous case

# The continuous inhomogeneous case

Perform the same change of variables

$$F = \mu + \mu f,$$

and work in $H^1(\mathrm{d}\mu dx) \hookrightarrow L^2(\mathrm{d}\mu \mathrm{d}x)$. The inhomogeneous equation reads

$$\partial_t f + v\partial_x f + (-\partial_v + v)\partial_v f = 0, \qquad f|_{t=0} = f^0,$$

$$\langle f \rangle \stackrel{\text{def}}{=} \iint f \, \mathrm{d}\mu dx = \langle f^0 \rangle$$

- very partial biblio (Guo, Villani-Desvillettes ... H. 06-07, Mouhot-Neumann 06, H.-Nier 04, Villani 07, Dolbeault-Mouhot-Schmeiser 15, *etc...*)
- robust proof (Boltzmann, Landau, fractional *etc...*) and methods (hypocoercivity, hypoellipticity)
- wide applications (hydro limits, perturbative NL solutions, VPFP, Landau damping, enlargement theory, statistical mechanics, low temperature, UQ *etc*

# The continuous inhomogeneous case

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- commutator identity $[\partial_v, v\partial_x] = \partial_x$ (hypoellipticity results by Hörmander, Kohn, developped by Helffer, Nourrigat, mention subunit balls by Fefferman and subelliptic geometry *etc...* ).
- how to discretize such an equality and equation ?
- fundamental point : have the simplest proofs and techniques in order to adapt them to the discretized cases.

# The modified entropy

We define the entropy functional for $C > D > E > 1$, to be defined later on

$$\mathcal{H} : f \mapsto C \|f\|^2 + D \|\partial_v f\|^2 + E \langle \partial_v f, \partial_x f \rangle + \|\partial_x f\|^2 . \qquad (2)$$

Then for $C, D, E$ well chosen, we will prove that $t \mapsto \mathcal{H}(f(t))$ is nonincreasing when $f$ solves the rescaled equation with initial datum $f^0 \in H^1(\mathrm{d}\mu)$.
First note that if $E^2 < D$, $\mathcal{H}$ is equivalent to the $H^1(\mathrm{d}\mu\mathrm{d}x)$-norm :

$$\frac{1}{2} \|f\|^2_{H^1} \leq \mathcal{H}(f) \leq 2C \|f\|^2_{H^1} \qquad (3)$$

We have modified the norm in $H^1$.

▷ First term

$$\frac{d}{dt}\|f\|^2 = 2\langle \partial_t f, f\rangle = -2\langle v\partial_x f, f\rangle - 2\langle (-\partial_v + v)\partial_v f, f\rangle = -2\|\partial_v f\|^2$$

▷ Second term

$$\begin{aligned}\frac{d}{dt}\|\partial_v f\|^2 &= 2\langle \partial_v(\partial_t f), \partial_v f\rangle \\ &= -2\langle \partial_v(v\partial_x f + (-\partial_v + v)\partial_v f), \partial_v f\rangle \\ &= -2\langle v\partial_x\partial_v f, \partial_v f\rangle - 2\langle [\partial_v, v\partial_x] f, \partial_v f\rangle - 2\langle \partial_v(-\partial_v + v)\partial_v f, \partial_v f\rangle . \\ &= -2\langle \partial_x f, \partial_v f\rangle - 2\|(-\partial_v + v)\partial_v f\|^2\end{aligned}$$

▷ Last term

$$\frac{d}{dt}\|\partial_x f\|^2 = -2\|\partial_v\partial_x f\|^2$$

▷ Third important term

$$
\begin{aligned}
\frac{d}{dt}\langle \partial_x f, \partial_v f\rangle
&= -\langle \partial_x (v\partial_x f + (-\partial_v + v)\partial_v f), \partial_v f\rangle - \langle \partial_x f, \partial_v (v\partial_x f + (-\partial_v + v)\partial_v f)\rangle \\
&= -\langle v\partial_x(\partial_x f), \partial_v f\rangle - \langle (-\partial_v + v)\partial_v f, \partial_x\partial_v f\rangle \\
&\quad - \langle \partial_x f, [\partial_v, v\partial_x]\, f\rangle - \langle \partial_x f, v\partial_x\partial_v f\rangle \\
&\quad - \langle \partial_x f, [\partial_v, (-\partial_v + v)]\,\partial_v f\rangle - \langle (-\partial_v + v)\partial_v f, \partial_x\partial_v f\rangle\,.
\end{aligned}
$$

we have

$$\langle v\partial_x\partial_x f, \partial_v f\rangle + \langle \partial_x f, v\partial_x\partial_v f\rangle = 0.$$

and

$$[\partial_v, (-\partial_v + v)] = 1$$

so that

$$\frac{d}{dt}\langle \partial_x f, \partial_v f\rangle = -\|\partial_x f\|^2 + 2\langle (-\partial_v + v)\partial_v f, \partial_x\partial_v f\rangle - \langle \partial_x f, \partial_v f\rangle\,.$$

▷ Entropy dissipation inequality

$$\frac{d}{dt}\mathcal{H}(f) = -2C\left\|\partial_v f\right\|^2 - 2D\left\|(-\partial_v + v)\partial_v f\right\|^2 - E\left\|\partial_x f\right\|^2 - 2\left\|\partial_x\partial_v f\right\|^2 - 2(D+E)\left\langle \partial_x f, \partial_v f\right\rangle - 2E\left\langle (-\partial_v + v)\partial_v f, \partial_x\partial_v f\right\rangle .$$

Therefore, using Cauchy-Schwartz : for $1 < E < D < C$ well chosen,

$$\frac{d}{dt}\mathcal{H}(f) \leq -C\left\|\partial_v f\right\|^2 - (E - 1/2)\left\|\partial_x f\right\|^2 \leq -\frac{E}{2}(\left\|\partial_v f\right\|^2 + \left\|\partial_x f\right\|^2).$$

Using the Poincaré inequality in space and velocity

$$\frac{d}{dt}\mathcal{H}(f) \leq -\frac{E}{4}(\left\|\partial_v f\right\|^2 + \left\|\partial_x f\right\|^2) - \frac{E}{4}c_p\left\|f\right\|^2 \leq -\frac{E}{4}\frac{c_p}{2C}\mathcal{H}(f).$$

□

▷ Synthesis

We pose $2\kappa = \dfrac{E}{4}\dfrac{c_p}{2C}$ and we get by Gronwall lemma

**Theorem**

*For all $f^0 \in H^1$, the solution $f$ to the rescaled inhomogeneous Fokker-Planck equation satisifies for all $t \geq 0$,*

$$\frac{1}{2}\|f\|_{L^2}^2 \leq \frac{1}{2}\|f\|_{H^1}^2 \leq \mathcal{H}(f(t)) \leq e^{-2\kappa t}\mathcal{H}(0) \leq 2Ce^{-2\kappa t}\left\|f^0\right\|_{H^1}^2 .$$

We want to discretize the equation in space, velocity and time, with preservation of the long time behavior, (hypo)coercivity, the notion of Maxwellian.

### Keywords and the discrete case

Equation ? derivative ? Hilbert space ? Maxwellian ? Gronwall ? Poincaré ? commutators ? local ?

Huge literature : (Herda, Bessemoulin, AP, Jin, etc etc...). In the spirit of hypocoercivity for short time : Poretta-Zuazua

# The semi-discrete homogeneous case

We want to discretize (and implement) the equation

$$\partial_t F - \partial_v(\partial_v + v)F = 0.$$

We look for a discretization only in velocity.

▷ The velocity derivative : for $F \in \ell^1(\mathbb{Z})$, define $D_v F \in \ell^1(\mathbb{Z}^*)$ by

$$(D_v F)_i = \frac{F_i - F_{i-1}}{h} \text{ for } i > 0, \qquad (D_v F)_i = \frac{F_{i+1} - F_i}{h} \text{ for } i < 0.$$

▷ The Maxwellian : solving equation $(D_v + v)\mu^h = 0$ yields

$$\mu_i^h = \frac{c_h}{\prod_{l=0}^{|i|}(1 + hv_l)}, \qquad i \in \mathbb{Z}.$$

Then $\mu^h$ is even, positive, in $\ell^1$.
Proof by direct computation : $(D_v + v)\mu^h = 0$ writes

$$\begin{cases} \dfrac{\mu_i^h - \mu_{i-1}^h}{h} + v_i\mu_i^h = 0 & \text{for } i > 0 \\ \dfrac{\mu_{i+1}^h - \mu_i^h}{h} + v_i\mu_i^h = 0 & \text{for } i < 0, \end{cases}$$

which gives the expression of $\mu^h \in \ell^1$.

▷ The "adjoint" : for $G \in \ell^1(\mathbb{Z}^*)$, define $D_v^\sharp F \in \ell^1(\mathbb{Z})$

$$(D_v^\sharp G)_i = \frac{G_{i+1} - G_i}{h} \text{ for } i > 0, \qquad (D_v^\sharp G)_i = \frac{G_i - G_{i-1}}{h} \text{ for } i < 0,$$

$$(D_v^\sharp G)_0 = \frac{G_1 - G_{-1}}{h}.$$

▷ The Hilbert spaces : we pose $F = \mu^h + \mu^h f$ and consider

$$f \in \ell^2(\mu^h) \Leftrightarrow \sum_i f_i^2 \mu_i^h < +\infty$$

then denoting $\mu_i^\sharp = \mu_{i-1}^h$ for $i > 0$ and $\mu_i^\sharp = \mu_{i+1}^h$ for $i < 0$

$$-D_v^\sharp((D_v + v)\mu^h f) = D_v^\sharp(\mu^\sharp (D_v f)_i) = \mu_i^h(-D_v^\sharp + v)D_v f.$$

The discrete equation is then

$$\partial_t F - D_v^\sharp (D_v + v) F = 0, \qquad F|_{t=0} = F^0.$$

With $F = \mu^h + \mu^h f$, we have

### Proposition

*The equation satisfied by $f$ is the following*

$$\partial_t f + (-D_v^\sharp + v) D_v f = 0, \qquad f|_{t=0} = f^0.$$

*The operator $(-D_v^\sharp + v)D_v$ is selfadjoint non-negative in $\ell^2(\mu^h)$ :*

$$\left\langle (-D_v^\sharp + v) D_v f, g \right\rangle = \langle D_v f, D_v g \rangle_\sharp = \left\langle f, (-D_v^\sharp + v) D_v g \right\rangle,$$

*where*

$$\varphi \in \ell^2(\mu^\sharp) \Leftrightarrow \|\varphi\|_\sharp^2 = \sum_{i \neq 0} \varphi_i^2 \mu_i^\sharp < \infty.$$

*Constant sequences are the equilibrium states of the equation.*

We can do the same proof as in the continuous case

- note that $\langle f \rangle \stackrel{\text{def}}{=} \sum f_i \mu_i^h = \sum f_i^0 \mu_i^h = 0$,
- compute $\frac{\mathrm{d}}{\mathrm{d}t} \|f\|^2 = -2 \left\langle (-D_v^\sharp + v) D_v f, f \right\rangle = -2 \|D_v f\|_\sharp^2$,
- use Poincaré inequality $\|f\|^2 \leq \|D_v f\|_\sharp^2$,
- use Gronwall inequality $\|f\| \leq e^{-t} \left\|f^0\right\|$,
- synthesis $\left\|F - \mu^h\right\|_{\ell^1} \leq \|f\| \leq e^{-t} \left\|f^0\right\|$.

? ? Poincare inequality ? ?

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- note that $\langle f \rangle \stackrel{\text{def}}{=} \sum f_i \mu_i^h = \sum f_i^0 \mu_i^h = 0$,
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- synthesis $\left\|F - \mu^h\right\|_{\ell^1} \leq \|f\| \leq e^{-t} \left\|f^0\right\|$.

?? Poincare inequality ??

# Poincaré inequality

**Lemma (adapted proof of that by H. Poincare (1912))**

*For all* $f \in H^1(\mu)$ *with* $\langle f \rangle = 0$, *we have* $\|f\|^2_{L^2(\mathrm{d}\mu)} \leq \|\partial_v f\|^2_{L^2(\mathrm{d}\mu)}$

*Proof.* Denote $f(v) = f$, $f(v') = f'$, $\mathrm{d}\mu = \mu(v)\mathrm{d}v$ and $\mathrm{d}\mu' = \mu(v')\mathrm{d}v'$.

$$\int_{\mathbb{R}} f^2 \mathrm{d}\mu = \frac{1}{2} \iint_{\mathbb{R}^2} (f' - f)^2 \mathrm{d}\mu \mathrm{d}\mu' = \frac{1}{2} \iint_{\mathbb{R}^2} \left( \int_v^{v'} \partial_v f(w) \mathrm{d}w \right)^2 \mathrm{d}\mu \mathrm{d}\mu'$$

From Cauchy Schwartz

$$\int_{\mathbb{R}} f^2 \mathrm{d}\mu \leq \frac{1}{2} \iint_{\mathbb{R}^2} \left( \int_v^{v'} |\partial_v f(w)|^2 \mathrm{d}w \right) (v' - v) \mathrm{d}\mu \mathrm{d}\mu'$$

Denote $F(v) = \int_a^v |\partial_v f(w)|^2 \,\mathrm{d}w$. Then

$$\int_{\mathbb{R}} f^2 \mathrm{d}\mu \leq \frac{1}{2} \iint_{\mathbb{R}^2} (F' - F)(v' - v) \mathrm{d}\mu \mathrm{d}\mu'$$
$$= \frac{1}{2} \left( \iint_{\mathbb{R}^2} F'v' \mathrm{d}\mu\mathrm{d}\mu' + \iint_{\mathbb{R}^2} Fv \,\mathrm{d}\mu\mathrm{d}\mu' - \iint_{\mathbb{R}^2} Fv' \mathrm{d}\mu\mathrm{d}\mu' - \iint_{\mathbb{R}^2} F'v \,\mathrm{d}\mu\mathrm{d}\mu' \right)$$
$$= \int_{\mathbb{R}} Fv \,\mathrm{d}\mu,$$

Note that $\partial_v \mu = -v\mu$ and perform an integration by parts

$$\int_{\mathbb{R}} f^2 \mathrm{d}\mu \leq \int_{\mathbb{R}} Fv\mu \mathrm{d}v = -\int_{\mathbb{R}} F \partial_v \mu \mathrm{d}v = \int_{\mathbb{R}} \partial_v F \mu \mathrm{d}v = \int_{\mathbb{R}} |\partial_v f|^2 \mathrm{d}\mu.$$

□

# Discrete Poincaré inequality

**Proposition (Discrete Poincaré inequality)**

*Let $f$ be a sequence in $H^1$. Then,*

$$\|f - \langle f \rangle\|^2_{\ell^2(\mu^h)} \leq \|D_v f\|^2_{\ell^2(\mu^\sharp)} .$$

*Proof.* Assume $\langle f \rangle = 0$ and write

$$\sum_i f_i^2 \mu_i = \frac{1}{2} \sum_{i,j} (f_j - f_i)^2 \mu_i \mu_j = \sum_{i<j} (f_j - f_i)^2 \mu_i \mu_j$$

$\cdots$ computations using an antiderivative of $f$ defined by

$$F_j = \sum_{l=-j_a}^{j} (f_l - f_{l-1})^2$$

and using the integration by part in the discrete weighted space

$$\sum_{i \neq 0} F_i i \mu_i = -\sum_{i>0} \frac{F_i - F_{i+1}}{h^2} \mu_i + \frac{F_1}{h^2} \mu_0 - \sum_{i<0} \frac{F_{i-1} - F_i}{h^2} \mu_i - \frac{F_{-1}}{h^2} \mu_0$$

# Poincaré inequality in space

The two preceding objects and proofs can be adapted to the inhomogeneous cases under the following assumption

## Hypothesis

*The operator $D_x$ is skew adjoint, commutes with velocity and satisfies the Poincaré inequality in space*

$$c_p \left\|\phi - \langle\phi\rangle\right\|^2_{L^2_x} \leq \left\|D_x\phi\right\|^2_{L^2_x} .$$

For example,

- centered discrete derivative

$$(D_x\phi)_j = \frac{\phi_{j+1} - \phi_{j-1}}{h}, \qquad j \in \mathbb{Z}/N\mathbb{Z},$$

- continuous derivative (on the torus).

# All cases

The two preceding objects and proofs can be adapted to the inhomogeneous cases (we give the $f$ version) on $\ell^2(\mu^h \mathrm{d}v\mathrm{d}x)$ :

- semi-discrete case

$$\partial_t f + vD_x f + (-D_v^\sharp + v)D_v f = 0, \qquad f|_{t=0} = f^0,$$

- the fully discrete Euler implicit case

$$\frac{f^{n+1} - f^n}{\delta t} + vD_x f^{n+1} + (-D_v^\sharp + v)D_v f^{n+1} = 0, \qquad f|_{t=0} = f^0,$$

- the fully discrete Euler explicit with Neumann on $v \in [-b, b]$

$$\frac{f^{n+1} - f^n}{\delta t} + vD_x f^n + (-D_v^\sharp + v)D_v f^n = 0, \quad f|_{t=0} = f^0, \quad D_v f_{\pm b} = 0.$$

An example of theorem

### Theorem

*Assume $C > D > E > 1$ well chosen. There exists $k, t_0, h_0 > 0$ such that for all $f^0 \in H^1$ with $\langle f^0 \rangle = 0$, all $\delta t \in (0, t_0)$, and all $h \in (0, h_0)$ the sequence defined by the implicit Euler scheme satisfies for all $n \in \mathbb{N}$,*

$$\frac{1}{2} \|f^n\|_{H^1}^2 \leq \mathcal{H}(f^n) \leq \mathcal{H}(f^0) e^{-kn\delta t} \leq 2C \left\|f^0\right\|_{H^1}^2 e^{-kn\delta t}.$$

# Elements of proof

Consider

$$\mathcal{H}(f) = C\left\|f\right\|^2 + D\left\|D_v f\right\|_\sharp^2 + E\left\langle D_v f, SD_x f\right\rangle_\sharp + \left\|D_x f\right\|^2. \qquad (4)$$

with $S = [D_v, v]$ and therefore $SD_x = [D_v, vD_x]$ :

$$(Sg)_i = g_{i-1} \text{ for } i \geq 1 \qquad (Sg)_i = g_{i+1} \text{ for } i \leq -1.$$

We have for example

$$D_v(-D_v^\sharp + v)S - S(-D_v^\sharp + v)D_v = S + \delta,$$

where $\delta$ is the singular operator from $\ell^2$ to $\ell^2_\sharp$ defined for $f \in \ell^2$ by

$$(\delta f)_j = 0 \text{ if } |j| \geq 2, \qquad (\delta f)_{-1} = \frac{f_1 - f_0}{h^2}, \qquad (\delta f)_1 = -\frac{f_0 - f_{-1}}{h^2}.$$

Especially for the singular term involving $\delta$, we have for all $\varepsilon > 0$,

$$\left|\langle \delta D_x f, D_v f\rangle_\sharp\right| \leq \frac{1}{\varepsilon}\left\|(-D_v^\sharp + v) D_v f\right\|^2 + \varepsilon \left\|D_v D_x f\right\|_\sharp^2 .$$

# Numerics

Thank you !