Hypocoercivity based sensitivity analysis for multiscale kinetic equations with random inputs and their numerical approximations

Shi Jin

University of Wisconsin-Madison, USA
Shanghai Jiao Tong University, China

Uncertainty Quantification for Kinetic Equations

Co-authors

Jingwei Hu

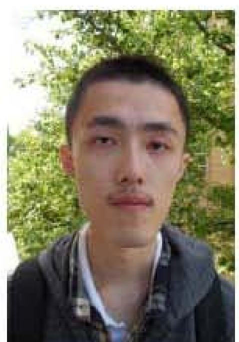
Jian-Guo Liu

Liu Liu

Hanqing Lu

Dongbin Xiu

Zheng Ma



Ruiwen Shu

Xueyu Zhu

Yuhua Zhu

Where do kinetic equations sit in physics

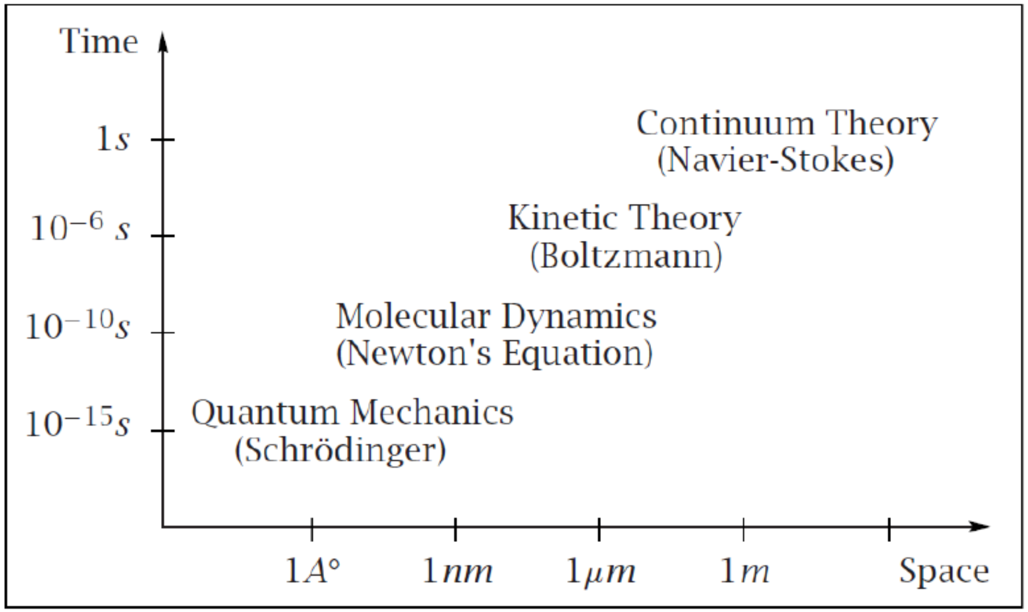


Figure 1. Different laws of physics are required to describe properties and processes of fluids at different scales.

- E & Engquist, AMS Notice (2003)

Kinetic equations with applications

- Rarefied gas—astronautics (Boltzmann equation)
- Plasma (Vlasov-Poisson, Landau, Fokker-Planck,…)
- Semiconductor device modeling
- Microfluidics
- Nuclear reactor (nentron transport)
- Astrophysics, medical imaging (radiative transfer)
- Multiphase flows
- Environmental science, energy, social science, neuronal networks, biology, …

Challenges in kinetic computation

- High dimension (phase space, 6d for Boltzmann)
- Multiple scales
- uncertainty

The Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{1}{\varepsilon} \mathcal{Q}(f, f)(\mathbf{v}), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \ \mathbf{v} \in \mathbb{R}^d$$

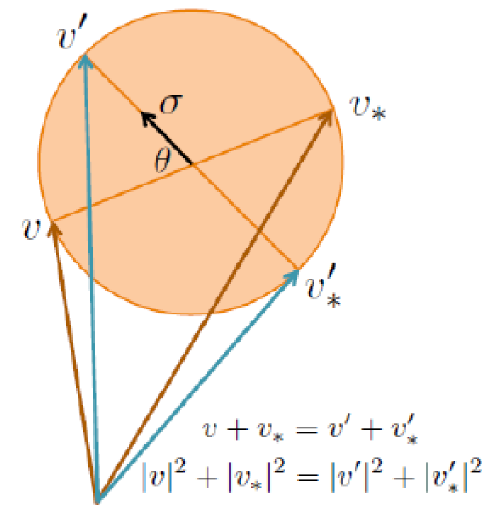
- $f(t, \mathbf{x}, \mathbf{v})$ is the **phase space distribution function** of time t, position $\mathbf{x}$, and velocity $\mathbf{v}$
- ε is the **Knudsen number**, ratio of the mean free path and the characteristic length scale: $\varepsilon \sim O(1)$ kinetic regime; $\varepsilon \ll O(1)$ fluid regime
- $\mathcal{Q}(f, f)$ is the **collision operator**, a quadratic integral operator modeling the interaction of particles

Collision operator

$$\mathcal{Q}(f,f)(\mathbf{v}) = \int_{\mathbb{R}^d}\int_{S^{d-1}} B(\mathbf{v}-\mathbf{v}_*,\sigma)[f(\mathbf{v}')f(\mathbf{v}'_*) - f(\mathbf{v})f(\mathbf{v}_*)]\,\mathrm{d}\sigma\mathrm{d}\mathbf{v}_*$$

$(\mathbf{v},\mathbf{v}_*)$ and $(\mathbf{v}',\mathbf{v}'_*)$ are the velocity pairs before and after collision:

$$\begin{cases} \mathbf{v}' = \dfrac{\mathbf{v}+\mathbf{v}_*}{2} + \dfrac{|\mathbf{v}-\mathbf{v}_*|}{2}\sigma \\ \mathbf{v}'_* = \dfrac{\mathbf{v}+\mathbf{v}_*}{2} - \dfrac{|\mathbf{v}-\mathbf{v}_*|}{2}\sigma \end{cases}$$



$$B(\mathbf{v}-\mathbf{v}_*,\sigma) = B\left(|\mathbf{v}-\mathbf{v}_*|, \tfrac{\sigma\cdot(\mathbf{v}-\mathbf{v}_*)}{|\mathbf{v}-\mathbf{v}_*|}\right)$$

Variable hard sphere (VHS) model

$B = b_\lambda|\mathbf{v}-\mathbf{v}_*|^\lambda,\ -d < \lambda \le 1$

$\lambda = 1$: hard sphere molecule

$\lambda = 0$: Maxwell molecule

Uncertainty in kinetic equations

- Kinetic equations are usually derived from N-body Newton's second law, by mean-field limit, BBGKY hierachy, Grad-Boltzmann limit, etc.
- Collision kernels are often empirical
- Initial and boundary data contain uncertainties due to measurement errors or modelling errors; geometry, forcing
- While UQ has been popular in solid mechanics, CFD, elliptic equations, etc. there has been almost no effort for kinetic equations

Data for scattering cross-section

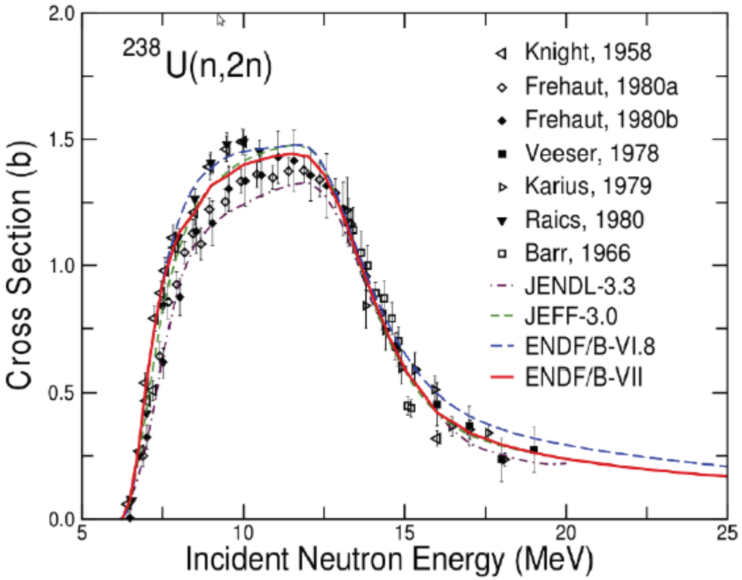


Figure 2: Example of uncertainty associated with a nuclear cross-section (from (Chadwick et al., 2006)). Figure contains values corresponding to several data libraries and measurements.

Uncertainty Quantification (UQ) for kinetic models

For kinetic models, the only thing certain is their uncertainty

- Quantify the propagation of the uncertainty
- efficient numerical methods to study the uncertainty
- understand its statistical moments
- sensitivity analysis (identify sensitive/insensitive parameters), long-time behavior of the uncertainty
- Control of the uncertainty
- dimensional reduction of high dimensional uncertainty
- …

Polynomial Chaos (PC) approximation

- The PC or generalized PC (gPC) approach first introduced by Wiener, followed by Cameron-Martin, and generalized by Ghanem and Spanos, Xiu and Karniadakis etc. has been shown to be very efficient in many UQ applications when the solution has enough regularity in the random variable
- Let z be a random variable with pdf $\rho(z) > 0$
- Let $\Phi_m(z)$ be the orthonormal polynomials of degree m corresponding to the weight $\rho(z) > 0$

$$\int \Phi_i(z)\Phi_j(z)\rho(z)\,dz = \delta_{ij}$$

The Wiener-Askey polynomial chaos for random variables (table from Xiu-Karniadakis SISC 2002)

	Random variables ζ	Wiener-Askey chaos $\{\Phi(\zeta)\}$	Support
Continuous	Gaussian	Hermite-Chaos	$(-\infty, \infty)$
	Gamma	Laguerre-Chaos	$[0, \infty)$
	Beta	Jacobi-Chaos	$[a, b]$
	Uniform	Legendre-Chaos	$[a, b]$
Discrete	Poisson	Charlier-Chaos	$\{0, 1, 2, \dots\}$
	Binomial	Krawtchouk-Chaos	$\{0, 1, \dots, N\}$
	Negative Binomial	Meixner-Chaos	$\{0, 1, 2, \dots\}$
	Hypergeometric	Hahn-Chaos	$\{0, 1, \dots, N\}$

TABLE 4.1

The correspondence of the type of Wiener-Askey polynomial chaos and their underlying random variables ($N \geq 0$ is a finite integer).

Generalized polynomial chaos stochastic Galerkin (gPC-sG) methods

- Take an orthonormal polynomial basis $\{\Phi_j(z)\}$ in the random space

- Expand functions into Fourier series and truncate:

$$f(z) = \sum_{j=0}^{\infty} f_j \phi_j(z) \approx \sum_{j=0}^{K} f_j \phi_j(z) := f^K(z).$$

- Substitute into system, Galerkin projection. Then one gets a deterministic system of the gPC coefficients $(f_0, \ldots, f_K)$

Accuracy and efficiency

- We will consider the gPC-stochastic Galerkin (gPC-SG) method
- Under suitable regularity assumptions this method has a spectral accuracy
- Much more efficient than Monte-Carlo samplings (halfth-order)
- Our regularity/sensitivity analysis is also important for stochastic collocation and other methods

Nonlinear collisional kinetic equations (Liu Liu-J)

- One can extend hypocoercivity theory developed by Herau, Nier， Desvillettes, Villani, Guo, Mouhut, Briant, etc. in velocity space for deterministic problems to study the following properties in random space:

  regularity, sensitivity in random parameter, long-time behavior (exponential decay to global equilibrium, spectral convergence and exponential decay of numerical error for gPC-SG

- for linear kinetic equation with uncertainty: Jin-JG Liu-Ma; Q. Li-L. Wang

The Boltzmann equation with initial uncertainty

$$\begin{cases} \partial_t f + \frac{1}{\epsilon^\alpha} v \cdot \nabla_x f = \frac{1}{\epsilon^{1+\alpha}} \mathcal{Q}(f), \\ f(0,x,v,z) = f_{in}(x,v,z), \qquad x \in \Omega \subset \mathbb{T}^d,\ v \in \mathbb{R}^d, z \in I_z \subset \mathbb{R}. \end{cases}$$

perturbative setting $\quad f = \mathcal{M} + \epsilon M h$

(avoid compressible Euler limit, thus shocks):

Global Maxwellian $\quad \mathcal{M} = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|v|^2}{2}} \qquad M = \sqrt{\mathcal{M}}.$

Euler (acoustic scaling)

$$\partial_t h + v \cdot \nabla_x h = \frac{1}{\epsilon} \mathcal{L}(h) + \mathcal{F}(h,h).$$

(incompressible) Navier-Stokes scaling

$$\partial_t h + \frac{1}{\epsilon} v \cdot \nabla_x h = \frac{1}{\epsilon^2} \mathcal{L}(h) + \frac{1}{\epsilon} \mathcal{F}(h,h).$$

Why it works: hypocoercivity decay of the linear part dominates the bounded (weaker) nonlinear part

Hypocoercivity for linearized Boltzmann operator

$$\langle h, \mathcal{L}(h) \rangle_{L^2_v} \leq -\lambda \, ||h^\perp||_{\Lambda^2_v}$$

$$h^\perp = h - \Pi_{\mathcal{L}}(h)$$

$\Pi_{\mathcal{L}}(h)$ is the orthogonal projection in L^2_v on $N(\mathcal{L})$

$$||h||_{\Lambda_v} = ||h(1+|v|)^{\gamma/2}||_{L^2}$$

$$|| \cdot ||_\Lambda := || \, || \cdot ||_{\Lambda_v} \, ||_{L^2_x}.$$

Boundedness of the nonlinear term

$$\left|\langle \partial^m \partial_l^j \mathcal{F}(h,h),\ f\rangle_{L^2_{x,v}}\right| \le \begin{cases} \mathcal{G}^{s,m}_{x,v,z}(h,h)\,||f||_\Lambda\,, & \text{if } j \neq 0, \\ \mathcal{G}^{s,m}_{x,z}(h,h)\,||f||_\Lambda\,, & \text{if } j = 0. \end{cases}$$

there exists a z-independent $C_{\mathcal{F}} > 0$ such that

$$\sum_{|m|\le r} (\mathcal{G}^{s,m}_{x,v,z}(h,h))^2 \le C_{\mathcal{F}}\,||h||^2_{H^{s,r}_{x,v}}\,||h||^2_{H^{s,r}_\Lambda},$$

$$\sum_{|m|\le r} (\mathcal{G}^{s,m}_{x,z}(h,h))^2 \le C_{\mathcal{F}}\,||h||^2_{H^{s,r}_x L^2_v}\,||h||^2_{H^{s,r}_\Lambda}\,.$$

$$||h||^2_{H^{s,r}_{x,v}} = \sum_{|m|\le r} ||\partial^m h||^2_{H^s_{x,v}}\,, \qquad ||h||^2_{H^{s,r}_\Lambda} = \sum_{|m|\le r} ||\partial^m h||^2_{H^s_\Lambda}$$

$$||h||^2_{H^s_{x,v}} = \sum_{|j|+|l|\le s} ||\partial_l^j h||^2_{L^2_{x,v}} \qquad ||h||^2_{H^{s,r}_x L^2_v} = \sum_{|m|\le r} ||\partial^m h||^2_{H^s_x L^2_v}$$

$$||h||^2_{H^s_\Lambda} = \sum_{|j|+|l|\le s} ||\partial_l^j h||^2_\Lambda \qquad ||h||^2_{H^{s,r}_\Lambda} = \sum_{|m|\le r} ||\partial^m h||^2_{H^s_\Lambda}$$

Convergence to global equilibrium (random initial data)

$$Assume\ ||h_{in}||_{H^{s,r}_{x,v}L^\infty_z} \le C_I$$

(i) Under the incompressible Navier-Stokes scaling,

$$||h||_{H^{s,r}_{x,v}L^\infty_z} \le C_I\, e^{-\tau_s t}\,, \qquad ||h||_{H^{s}_{x,v}H^r_z} \le C_I\, e^{-\tau_s t}\,.$$

(ii) Under the acoustic scaling,

$$||h||_{H^{s,r}_{x,v}L^\infty_z} \le C_I\, e^{-\epsilon\tau_s t}\,, \qquad ||h||_{H^{s}_{x,v}H^r_z} \le C_I\, e^{-\epsilon\tau_s t}\,,$$

where C_I, τ_s *are positive constants independent of* ϵ.

$$||h||^2_{H^{s}_{x,v}H^r_z} = \int_{I_z} ||h||^2_{H^{s,r}_{x,v}}\, \pi(z)\, dz$$

Random collision kernel

$$B(|v-v_*|,\cos\theta,z)=\phi(|v-v_*|)\,b(\cos\theta,z),\qquad \phi(\xi)=C_\phi\,\xi^\gamma,\ \text{with}\ \gamma\in[0,1],$$

$$\forall\eta\in[-1,1],\qquad |b(\eta,z)|\le C_b,\ |\partial_\eta b(\eta,z)|\le C_b,\ \text{ and }\ |\partial_z^k b(\eta,z)|\le C_b^*,\ \forall\,0\le k\le r\,.$$

- Need to use a weighted Sobolev norm in random space as in Jin-Ma-J.G. Liu

$$||g||_{L^{2,r*}_{x,v}}:=\sum_{m=0}^{r}\widetilde{C}_{m,r+1}\,||\partial^m g||_{L^2_{x,v}},$$

- Similar decay rates can be obtained

gPC-SG approximation

$$f(t,x,v,z) \approx \sum_{|\mathbf{k}|=1}^{K} f_{\mathbf{k}}(t,x,v)\psi_{\mathbf{k}}(z) := f^K(t,x,v,z),$$

$$h(t,x,v,z) \approx \sum_{|\mathbf{k}|=1}^{K} h_{\mathbf{k}}(t,x,v)\psi_{\mathbf{k}}(z) := h^K(t,x,v,z).$$

- Perturbative setting

$$f_{\mathbf{k}} = \mathcal{M} + \epsilon M h_{\mathbf{k}}$$

$$\begin{cases} \partial_t h_{\mathbf{k}} + \frac{1}{\epsilon} v \cdot \nabla_x h_{\mathbf{k}} = \frac{1}{\epsilon^2}\mathcal{L}_{\mathbf{k}}(h^K) + \frac{1}{\epsilon}\mathcal{F}_{\mathbf{k}}(h^K, h^K), \\ h_{\mathbf{k}}(0,x,v) = h_{\mathbf{k}}^0(x,v), \qquad x \in \Omega \subset \mathbb{T}^d,\ v \in \mathbb{R}^d, \end{cases}$$

- Assumptions: z bounded

$$|\partial_z b| \le O(\epsilon)$$

(following R. Shu-Jin) $||\psi_k||_{L^\infty} \le C k^p, \qquad \forall\, k,$

Let $q > p + 2$, *define the energy* E^K *by*

$$E^K(t) = E^K_{s,q}(t) = \sum_{k=1}^{K} ||k^q h_k||^2_{H^s_{x,v}},$$

Regularity and exponential decay

(i) Under the incompressible Navier-Stokes scaling,

$$E^K(t) \leq \eta\, e^{-\tau t} \qquad ||h^K||_{H^s_{x,v} L^\infty_z} \leq \eta\, e^{-\tau t}$$

(ii) Under the acoustic scaling,

$$E^K(t) \leq \eta\, e^{-\epsilon \tau t}\,, \qquad ||h^K||_{H^s_{x,v} L^\infty_z} \leq \eta\, e^{-\epsilon\, \tau t}$$

gPC-SG error

(i) Under the incompressible Navier-Stokes scaling,

$$||h^e||_{H^s_{x,v}L^2_z} \leq C_e \frac{e^{-\lambda t}}{K^r},$$

(ii) Under the acoustic scaling,

$$||h^e||_{H^s_{x,v}L^2_z} \leq C_e \frac{e^{-\epsilon\lambda t}}{K^r},$$

with the constants C_e, $\lambda > 0$ *independent of* K *and* ϵ.

A general framework

- This framework works for general linear and nonlinear collisional kinetic equations

- Linear and nonlinear Boltzmann, Landau, relaxation-type quantum Boltzmann, etc.

Vlasov-Poisson-Fokker-Planck system
(J., & Y. Zhu)

$$
\begin{cases}
\partial_t f + \frac{1}{\delta}\mathbf{v}\cdot\nabla_{\mathbf{x}} f - \frac{1}{\epsilon}\nabla_{\mathbf{x}}\phi\cdot\nabla_{\mathbf{v}} f = \frac{1}{\delta\epsilon}\mathcal{F}f, \\
-\Delta_{\mathbf{x}}\phi = \rho - 1, \quad t>0, \quad \mathbf{x}\in\Omega\subset\mathbb{R}^N, \ \mathbf{v}\in\mathbb{R}^N
\end{cases}
$$

$$
\mathcal{F}f = \nabla\cdot\left(M\nabla\left(\frac{f}{M}\right)\right) \qquad M = \frac{1}{(2\pi)^{\frac{N}{2}}}e^{-\frac{|\mathbf{v}|^2}{2}}
$$

$$
f(0,\mathbf{x},\mathbf{v},\mathbf{z}) = f^0(\mathbf{x},\mathbf{v},\mathbf{z}), \quad \mathbf{x}\in\Omega, \ \mathbf{v}\in\mathbb{R}^N, \ \mathbf{z}\in I_{\mathbf{z}}.
$$

Asymptotic regimes

- High field regime: $\delta = 1$

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \nabla_{\mathbf{x}} \phi) = 0, \\ -\triangle_{\mathbf{x}} \phi = \rho - 1, \end{cases}$$

- Parabolic regime: $\delta = \epsilon$

$$\begin{cases} \partial_t \rho - \nabla \cdot (\nabla_{\mathbf{x}} \rho - \rho \nabla_{\mathbf{x}} \phi) = 0, \\ -\triangle_{\mathbf{x}} \phi = \rho - 1. \end{cases}$$

Norms and energies

$$d\mu = d\mu(\mathbf{x}, \mathbf{v}, \mathbf{z}) = \pi(\mathbf{z}) d\mathbf{x} d\mathbf{v} d\mathbf{z}.$$

$$\langle f, g \rangle = \int_\Omega \int_{\mathbb{R}^N} \int_{I_\mathbf{z}} f g \, d\mu(\mathbf{x}, \mathbf{v}, \mathbf{z}), \quad \text{or,} \quad \langle \rho, j \rangle = \int_\Omega \int_{I_\mathbf{z}} \rho j \, d\mu(\mathbf{x}, \mathbf{z}), \quad \text{with norm } \|f\|^2 = \langle f, f \rangle.$$

$$h = \frac{f - M}{\sqrt{M}}, \quad \sigma = \int_\mathbb{R} h \sqrt{M} \, dv, \quad u = \int_\mathbb{R} h \, v \sqrt{M} \, dv,$$

$$\Pi_1 h = \sigma \sqrt{M}, \quad \Pi_2 h = u v \sqrt{M}, \quad \Pi h = \Pi_1 h + \Pi_2 h.$$

$$\|f\|_{H^m}^2 = \sum_{l=0}^m \left\| \partial_z^l f \right\|^2$$

hypocoercivity

- Linearized Fokker-Planck operator

$$\mathcal{L}h = \frac{1}{\sqrt{M}}\mathcal{F}\left(M + \sqrt{M}h\right) = \frac{1}{\sqrt{M}}\partial_v\left(M\partial_v\left(\frac{h}{\sqrt{M}}\right)\right)$$

- Duan-Fornaiser-Toscani '10

(a) $-\langle \mathcal{L}h, h\rangle = -\langle L(1-\Pi)h, (1-\Pi)h\rangle + \|u\|^2$;

(b) $-\langle \mathcal{L}(1-\Pi)h, (1-\Pi)h\rangle = \|\partial_v(1-\Pi)h\|^2 + \frac{1}{4}\|v(1-\Pi)h\|^2 - \frac{1}{2}\|(1-\Pi)h\|^2$;

(c) $-\langle \mathcal{L}(1-\Pi)h, (1-\Pi)h\rangle \geq \|(1-\Pi)h\|^2$;

(d) *There exists a constant* $\lambda_0 > 0$, *such that the following hypocoercivity holds,*

$$-\langle \mathcal{L}h, h\rangle \geq \lambda_0 \|(1-\Pi)h\|_v^2 + \|u\|^2,$$

and the largest $\lambda_0 = \frac{1}{7}$ *in one dimension.*

- Energy terms:
  - $E_h^m = \|h\|_{H^m}^2 + \|\partial_x h\|_{H^{m-1}}^2 \,, \quad E_\phi^m = \|\partial_x \phi\|_{H^m}^2 + \left\|\partial_x^2 \phi\right\|_{H^{m-1}}^2;$
- Dissipation terms:
  - $D_h^m = \|(1-\Pi)h\|_{H^m}^2 + \|(1-\Pi)\partial_x h\|_{H^{m-1}}^2, \quad D_\phi^m = E_\phi^m,$
  - $D_u^m = \|u\|_{H^m}^2 + \|\partial_x u\|_{H^{m-1}}^2 \,, \quad D_\sigma^m = \|\sigma\|_{H^m}^2 + \|\partial_x \sigma\|_{H^{m-1}}^2 \,.$

Previous energy estimates

Without the multiple scales and randomness, (Hwang, Jang, 2013) gives the energy estimates,

$$\frac{1}{2}\partial_t \hat{E}^m + \left(D_h^m + D_{\mathbf{u}}^m\right) + D_\sigma^m + D_\phi^m \lesssim \sqrt{\hat{E}^m}(D_h^m + D_{\mathbf{u}}^m + D_\sigma^m + D_\phi^m), \qquad (3.9$$

where $\hat{E}^m \sim \|h\|_{H_{\mathbf{x}}^m} + \|\nabla_{\mathbf{x}}\phi\|_{H_{\mathbf{x}}^m}$, $D_h^m = \|(1-\Pi)h\|_v^2$, $D_a^m = \|a\|^2$ for $a = \sigma, \mathbf{u}, \phi$.
However, when small parameters are involved,

$$\frac{1}{2}\partial_t \hat{E}_{\epsilon^2}^m + \frac{1}{\epsilon\delta}(D_h^m + D_{\mathbf{u}}^m) + \frac{1}{\epsilon\delta}D_\sigma^m + \frac{1}{\epsilon^2}D_\phi^m \le \frac{1}{\epsilon}\sqrt{\hat{E}_{\epsilon^2}^m}(D_h^m + D_{\mathbf{u}}^m) + \frac{1}{\epsilon^2}\sqrt{\hat{E}_{\epsilon^2}^m}D_\sigma^m + \frac{1}{\epsilon^2}\sqrt{\hat{E}_{\epsilon^2}^m}D_\phi^m.$$

So the requirement for initial data to obtain the uniform regularity of $\hat{E}^m$ is

- High-Field regime,

$$\hat{E}_{\epsilon^2}^m(0) \sim E_h^m(0) + \frac{1}{\epsilon^2}E_\phi^m(0) \le O(\epsilon)$$

- Parabolic regime,

$$\hat{E}_{\epsilon^2}^m(0) \sim E_h^m(0) + \frac{1}{\epsilon^2}E_\phi^m(0) \le O(1)$$

Our new estimates

- High-Field regime,

$$\frac{1}{2}\partial_t \hat{E}^m_{\epsilon^2} + \frac{1}{\epsilon}(D^m_h + D^m_{\mathbf{u}}) + \frac{1}{\epsilon}D^m_\sigma + \frac{1}{\epsilon^2}D^m_\phi$$
$$\lesssim \frac{1}{\sqrt{\epsilon}}\sqrt{\hat{E}^m_{\epsilon^2}}(D^m_h + D^m_{\mathbf{u}}) + \frac{1}{\sqrt{\epsilon}}\sqrt{E_\phi D^m_\sigma} + \frac{1}{\epsilon^2}\sqrt{E_\phi}(D^m_\sigma + D^m_\phi) + \frac{1}{\epsilon\sqrt{\epsilon}}\sqrt{E^m_h}D^m_\phi$$
$$\lesssim \frac{1}{\sqrt{\epsilon}}\sqrt{\hat{E}^m_{\epsilon^2}}(D^m_h + D^m_{\mathbf{u}}) + \frac{1}{\epsilon}\sqrt{\hat{E}^m_{\epsilon^2}}D^m_\sigma + \frac{1}{\epsilon\sqrt{\epsilon}}\sqrt{\hat{E}^m_{\epsilon^2}}D^m_\phi,$$
$$\Rightarrow \quad \hat{E}^m_{\epsilon^2}(0) \sim E^m_h(0) + \frac{1}{\epsilon^2}E^m_\phi(0) \le O(1)$$

- Parabolic regime,

$$\frac{1}{2}\partial_t \hat{E}^m_{\epsilon} + \frac{1}{\epsilon^2}(D^m_h + D^m_{\mathbf{u}}) + \frac{1}{\epsilon}D^m_\sigma + \frac{1}{\epsilon}D^m_\phi$$
$$\lesssim \frac{1}{\epsilon\sqrt{\epsilon}}\sqrt{\hat{E}^m_{\epsilon}}(D^m_h + D^m_{\mathbf{u}}) + \frac{1}{\sqrt{\epsilon}}\sqrt{E^m_\phi}D^m_\sigma + \frac{1}{\epsilon}\sqrt{E_\phi}(D^m_\sigma + D^m_\phi) + \frac{1}{\sqrt{\epsilon}}\sqrt{E^m_h}D^m_\phi$$
$$\lesssim \frac{1}{\epsilon\sqrt{\epsilon}}\sqrt{\hat{E}^m_{\epsilon}}(D^m_h + D^m_{\mathbf{u}}) + \frac{1}{\sqrt{\epsilon}}\sqrt{\hat{E}^m_{\epsilon}}D_\sigma + \frac{1}{\sqrt{\epsilon}}\sqrt{\hat{E}^m_{\epsilon}}D^m_\phi$$
$$\Rightarrow \quad \hat{E}^m_{\epsilon}(0) \sim E^m_h(0) + \frac{1}{\epsilon}E^m_\phi(0) \le O(\frac{1}{\epsilon})$$

Long time behavior (sensitivity/regularity)

Theorem 4.1. *Consider the VPFP system with multiple scales and uncertainty, for the high field regime ($\delta = 1$), if*

$$E_h^m(0) + \frac{1}{\epsilon^2} E_\phi^m(0) \leq C_1, \tag{4.1}$$

then,

$$E_h^m(t) + \frac{1}{\epsilon^2} E_\phi^m(t) \leq C_3 e^{-\frac{\lambda_0}{3}t} \left(E_h^m(0) + \frac{1}{\epsilon^2} E_\phi^m(0) \right). \tag{4.2}$$

For the parabolic regime ($\delta = \epsilon$), if

$$E_h^m(0) + \frac{1}{\epsilon} E_\phi^m(0) \leq \frac{C_2}{\epsilon}, \tag{4.3}$$

then,

$$E_h^m(t) + \frac{1}{\epsilon} E_\phi^m(t) \leq C_4 e^{-\frac{\lambda_0}{2}t} \left(E_h^m(0) + \frac{1}{\epsilon} E_\phi^m(0) \right). \tag{4.4}$$

Here C_1, C_2, C_3, C_4 are both constants independent of ϵ, and, $E_f^m = \|f\|^2_{L^2_{v,x}(H_z^m)} + \|\partial_x f\|^2_{L^2_{v,x}(H_z^{m-1})}$, for $f = h$ or $\partial_x \phi$.

UQ for many different kinetic equations

- Stochastic Asymptotic-Preserving: （Jin-Xiu-Zhu ‘16)
- Boltzmann: a fast algorithm for collision operator (J. Hu-Jin, JCP ‘16), sparse grid for high dimensional random space (J. Hu-Jin-R. Shu ‘16):
- Landau equation (J. Hu-Jin-R. Shu, ‘16)
- Landau damping (regularity of Landau damping solution, R. Shu-Jin)
- Best N+approximation & greedy algorithm for high dimensional random space (Jin-Zhu-Zuazua, on-going)

conclusion

- Hypocoercivity based regularity and sensitivity analysis can be done for general linear and nonlinear collision kinetic equations and VPFP system, which imply (uniform) spectral convergence and exponential time decay of error of gPC methods
- Kinetic equations have the good regularity in the random space, even for the nonlinear kinetic equation: good problem for UQ!
- Many kinetic ideas useful for UQ problems: mean-field approximations; moment closure; etc. （stochastic Asymptotic-Preserving is one example）
- Many open questions , very few existing works
- Kinetic equations are good problems for UQ; ** UQ + Multiscale **