Propagation of chaos for topological interactions

Mario Pulvirenti

International Research Center on the Mathematics and Mechanics of Complex Systems MeMoCS, University of L'Aquila, Italy and
Dipartimento di Matematica, University of Roma

Background

Propagation of Chaos: basic fundamental property in kinetic theory.

It rules out the transition from the statistical description of a huge number of particles (say N), towards a single PDE (nonlinear).

$$(\partial_t + v \cdot \nabla_x) f(x, v) = C_2(f_2)$$

where f and f_2 are the one-particle and two-particle distributions.
If, in some limiting situations $N \to \infty$, plus something else

$$f_2 \approx f^{\otimes 2}$$

(positions and velocities are becoming i.d.i.r.v.) we get the closed eq.n:

$$(\partial_t + v \cdot \nabla_x) f(x, v) = C_2(f^{\otimes 2})$$

Examples: Boltzmann, Vlasov, Landau .....

Background

The distribution of the particles at time zero, must be assumed chaotic (statistical independence). Chaos propagates, it is not created by the dynamics which does not destroy the correlations. Propagation of Chaos is related to the l.l.n. Actually, defining the empirical distribution (measure valued r.v.)

$$\mu_N(x, v; t) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i(t))\delta(v - v_i(t))$$

$\{x_i(t), v_i(t)\}$ the dynamical flow, P.C is essentially equivalent to show that

$$\mu_N(x, v; t) \to f(x, v; t).$$

weakly in probability or a.e.. Indeed

$$f_2(t) = \mathbb{E}(\mu_N(t)^{\otimes 2}) + O(\frac{1}{N})$$

($\mathbb{E}$ is the expectation w.r.t. the initial distribution of the particles).

The model

We try to apply the ideas of kinetic theory to the following model introduced by Blanchet and Degond to explain the behavior of flocks of birds in Roma, experimentally investigated by a group of physicists Ballerini et al, PNAS (2008). This is an alignment mechanism for which a single agent changes instantaneously its velocity according to that of another agent. The main point is that the interaction is topological, namely the probability of a transition does not depend on the distance of the two agents, but on the rank of the second w.r.t. the first. Topological interaction. This maintains the flock's cohesion under external perturbation as a predation.

The model

N-particle system in $\mathbb{R}^d$, $d = 1, 2, 3 \dots$. Particle i, has a position x_i and velocity v_i. The configuration of the system is

$$Z_N = \{z_i\}_{i=1}^N = \{x_i, v_i\}_{i=1}^N = (X_N, V_N).$$

Given the particle i, we order the remaining particles $j_1, j_2, \cdots j_{N-1}$ according their distance from i, namely by the following relation

$$|x_i - x_{j_s}| \leq |x_i - x_{j_{s+1}}|, \qquad s = 1, 2 \cdots N-1.$$

The rank $R(i,k)$ of particle $k = j_s$ w.r.t. i is s.
The normalized rank is

$$r(i,k) = \frac{R(i,k)}{N-1} \in \{\frac{1}{N-1}, \frac{2}{N-1}, \dots\}.$$

The model

Next we introduce a (smooth) function

$$K : [0,1] \to \mathbb{R}^+ \qquad \text{s.t. } \int_0^1 K(r)dr = 1,$$

and

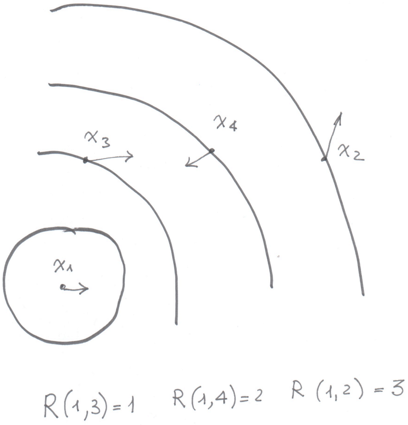
$$\pi_{i,j} = \frac{K(r(i,j))}{\sum_s K(\frac{s}{N-1})} \qquad \sum_j \pi_{i,j} = 1.$$

Let us introduce a stochastic process describing alignment via a topological interaction. The particles go freely $x_i + v_i t$. At some random Poisson time of intensity N, a particle (say i) is chosen with probability $\frac{1}{N}$ and a partner particle, say j, with probability $\pi_{i,j}$. Then the transition

$$(v_i, v_j) \to (v_j, v_j).$$

takes place. After that the system goes freely with the new velocities and so on.

The model



The model

The generator, for any $\Phi \in C_b^1(\mathbb{R}^{2dN})$, is

$$L_N \Phi(x_1, v_1, \cdots x_N, v_N) = \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} \Phi(x_1, v_1, \cdots x_N, v_N) + \sum_{i=1}^{N} \sum_{\substack{1 \le j \le N \\ i \neq j}}$$

$$\pi_{i,j} \big[\Phi(x_1, v_1, \cdots x_i v_j \cdots x_j, v_j \cdots x_N, v_N) - \Phi(x_1, v_1, \cdots x_N, v_N) \big].$$

Note that $\pi_{i,j} = \pi_{i,j}^N$ depends not only on N but also on the whole configuration Z_N.

The law of the process $W^N(Z_N; t)$ solves

$$\partial_t \int W^N(t) \Phi = \int W^N(t) \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} \Phi + \int W^N(t) \sum_{i=1}^{N} \sum_{\substack{1 \le j \le N \\ i \neq j}} \pi_{i,j}$$

$$\big[\Phi(x_1, v_1, \cdots x_i v_j \cdots x_j, v_j \cdots x_N, v_N) - \Phi(x_1, v_1, \cdots x_N, v_N) \big],$$

The model

We assume that the initial measure $W_0^N = W^N(0)$ is chaotic i.e. $W_0^N = f_0^{\otimes N}$ where f_0 is the initial datum for the limiting kinetic equation we are going to establish. Note also that $W^N(Z_N; t)$, for $t \geq 0$, is symmetric in the exchange of particles.
Equivalently

$$(\partial_t + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i}) W^N(t) = -N W^N(t) + \mathcal{L}_N W^N(t),$$

where

$$\mathcal{L}_N W^N(X_N, V_N, t) = \sum_{i=1}^{N} \sum_{\substack{1 \leq j \leq N \\ i \neq j}} \int du \, \pi_{i,j} \, W^N(X_N, V_N^{(i)}(u)) \delta(v_i - v_j).$$

Here $V_N^{(i)}(u) = (v_1 \cdots v_{i-1}, u, v_{i+1} \cdots v_N)$ if $V_N = (v_1 \cdots v_{i-1}, v_i, v_{i+1} \cdots v_N)$.

Kinetic description

$$r(i,j) = \frac{1}{N-1} \sum_{\substack{1 \le k \le N \\ k \neq i}} \chi_{B(x_i, |x_i - x_j|)}(x_k),$$

where $\chi_{B(x_i,|x_i-x_j|)}$ is the characteristic function of the ball $\{y \mid |x_i - y| \le |x_i - x_j|\}$. Moreover, recalling that $\int K = 1$,

$$\sum_s K\left(\frac{s}{N-1}\right) = (N-1)\left(1 + O\left(\frac{1}{N}\right)\right)$$

Therefore

$$\pi_{i,j} = \alpha_N K\left(\frac{1}{N-1} \sum_{k \neq i} \chi_{B(x_i,|x_i-x_j|)}(x_k)\right),$$

where

$$\alpha_N = \frac{1}{(N-1)(1 + (1 + O(\frac{1}{N})))} \approx \frac{1}{(N-1)}.$$

Kinetic description

The heuristic derivation of the kinetic equation (Blanchet and Degond. JSP16). Setting $\Phi(Z_N) = \varphi(z_1)$ in the master equation

$$\partial_t \int f_1^N \varphi = \int f_1^N v \cdot \nabla_x \varphi - \int f_1^N \varphi + \int W^N \sum_{j \neq 1} \pi_{i,j} \varphi(x_1, v_j).$$

Here f_1^N denotes the one-particle marginal of the measure W^N.
We recall that the s-particle marginals are defined by

$$f_s^N(Z_s) = \int W^N(Z_s, z_{s+1} \cdots z_N) dz_{s+1} \cdots dz_N, \qquad s = 1, 2 \cdots N,$$

and are the distribution of the first s particles (or of any group of s tagged particles).
We assume P.C namely

$$f_s^N \approx f_1^{\otimes s}$$

for any fixed integer s.

Kinetic description

In this case the strong law of large numbers does hold, that is for almost all i.i.d. variables $\{z_i(0)\}$ distributed according to $f_1(0) = f_0$, the random measure

$$\frac{1}{N}\sum_j \delta(z - z_j(t))$$

approximates weakly $f_1^N(z,t)$. Then

$$\pi_{i,j} \approx \frac{1}{N-1} K(\frac{1}{N-1}\sum_{k\neq i} \chi_{B(x_i,|x_i-x_j|)}(x_k)) \approx \frac{1}{N-1} K(M_\rho(x_1,|x_1-x_2|))$$

where

$$M_\rho(x,R) = \int_{B(x,R)} \rho(y)dy,$$

where $\rho(x) = \int dv f_1^N(x,v)$ and $B(x,R)$ is the ball of center x and radius R.

Kinetic description

In conclusion if $f_1^N \to f$ and $f_2^N \to f^{\otimes 2}$ in the limit $N \to \infty$, then f solves

$$\partial_t \int f\varphi = \int fv \cdot \nabla_x \varphi - \int f\varphi + \int f(z_1)f(z_2)\varphi(x_1, v_2)K(M_\rho(x_1, |x_1 - x_2|)),$$

or, in strong form,

$$(\partial_t + v \cdot \nabla_x)f(x, v) = -f(x, v) + \rho(x) \int dyK(M_\rho(x, |x - y|))\, f(y, v),$$

which is the equation we want to derive rigorously.

Results

Assume, for $x \in [0,1]$

$$K(x) = \sum_{m=0}^{\infty} a_m x^m, \quad A := \sum_{m=0}^{\infty} |a_m| 8^m < +\infty.$$

Let $f_j^N(t)$ be the j-particle marginals of the N-particle system and $f_j(t) = f(t)^{\otimes j}$, $f(t)$ solution to the kinetic equation, then (Degond and P. 2018)

Theorem

For any $T > 0$ and $\alpha > \log 2$, there exists $N(T,\alpha)$ such that for any $t \in (0, T)$, any $j \in \mathbb{N}$ and for any $N > N(T,\alpha)$, we have

$$\|f_j^N(t) - f_j(t)\|_{L^1} \leq 2^j \Big(\frac{1}{N-1}\Big)^{e^{-\alpha(8At+1)}}.$$

Results

Idea of the proof. Try to use the hierarchy. With this choice of K

$$(\partial_t + \sum_{i=1}^{s} \quad v_i \cdot \nabla_{x_i}) f_s(t) = -s f_s(t) +$$

$$+ \sum_{r=0}^{\infty} a_r \, C_{s,s+r+1} \, f_{s+r+1},$$

where

$$C_{s,s+r+1} f_{s+r+1}(X_s, V_s) = \sum_{i=1}^{s} \int dz_{s+1} \cdots dz_{s+r+1}$$

$$\chi_{i,s+1}(x_{s+2}) \dots \chi_{i,s+1}(x_{k_{s+r+1}}) f_{s+r+1}(X_{s+r+1}, V_{s+r+1}^{(i,s+1)}).$$

Here $\chi_{i,s+1} = \chi_{B(x_i, |x_i - x_{s+1}|)}$ and

$$V_N^{(i,s+1)} = \{v_1 \cdots v_{i-1}, v_{s+1}, v_{i+1} \cdots v_s, v_i, v_{s+2} \cdots v_N\},$$

$v_i \to v_{s+1}$ and $v_{s+1} \to v_i$.

Results

On the other hand

$$(\partial_t + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i}) f_s^N(t) = -s f_s^N(t) + E_s^N(t)$$
$$+ \sum_{r=0}^{\infty} a_r \, C_{s,s+r+1} \, f_{s+r+1}^N,$$

where E^N is an error term depending on the whole measure $W^N(t)$ and contains the correlations. Three steps:

- Estimate E^N in L^1
- Short time estimate of $\|f_s^N(t) - f_s(t)\|_{L^1}$
- Iterate by using the uniform control of the L^1 norm.