On controllability of linear viscoelastic flows

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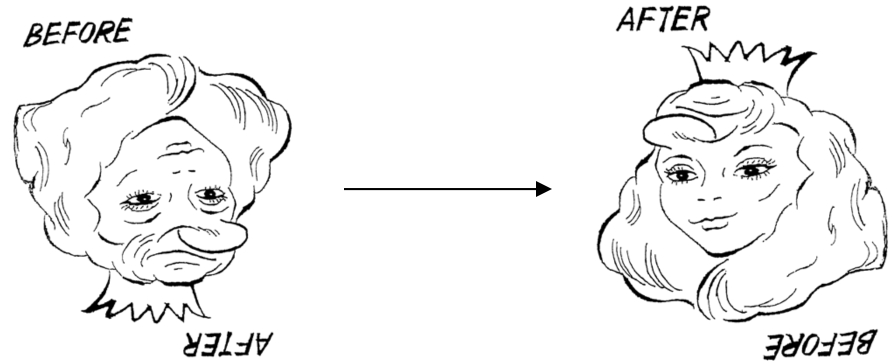
Controllability:

$$\dot{x} = Ax + f(t), \;\; x(0) = x_0.$$



Can f(t) be chosen such that x(T) assumes a given value?

Restrictions of f:

f may be present only in some equations, but not others.
f may be restricted to parts of the spatial domain or the boundary (for PDEs).

General setup: A generates a C_0 semigroup in a Hilbert space H. f takes values in some subspace U of H.

The usual way to prove controllability is by considering "observability" for an adjoint system.

$$\dot{y} = -A^* y.$$

A simple calculation leads to

$$(y(T), x(T)) - (y(0), x(0)) = \int_0^T (y(t), f(t))\, dt.$$

Exact controllability:

Assume that solutions of the adjoint system satisfy

$$\|y(T)\| \leq C\|y\|_{L^2(0,T;U^*)}.$$

Then the system is exactly controllable, i.e. every state x(T) can be reached with some control.

Proof: Let L be the bounded mapping from $L^2(0,T,U^*)$ to H^*. Set $x(0)=0$ and define $f=L^*x(T)$.

Null controllability:

Assume that solutions of the adjoint system satisfy

$$\|y(0)\| \leq C\|y\|_{L^2(0,T;U^*)}.$$

Then the system is null controllable, i.e. x(T)=0 can be reached from any given initial state.

Proof: Let L be the bounded mapping from $L^2(0,T,U^*)$ to H^*. Set $f=-L^*x(0)$.

Approximate controllability:

Assume that solutions of the adjoint are uniquely determined by the restriction to U^*. Then the system is approximately controllable, i.e. a dense set of final states $x(T)$ can be reached.

Proof: Suppose not. Set $x(0)=0$, and pick $y(T)$ to be orthogonal to every final state which can be reached. It follows that the restriction of y to U^* must vanish. By assumption this implies $y=0$.

Case of ODEs:

For simplicity, assume the eigenvalues of A^* are simple. The solution of the adjoint equation has the form

$$y = \sum_{i=1}^{n} \exp(\lambda_i t) v_i.$$

For any u in U, we have

$$(y, u) = \sum_{i=1}^{n} (v_i, u) \exp(\lambda_i t).$$

If there is a u which is not orthogonal to any of the v_i, the system is controllable.

But for PDEs:

1. Can we represent solutions in terms of eigenfunctions?
2. Even if we can, can we extract coefficients from an infinite sum of exponentials?

Fluid behavior

Newtonian	non-Newtonian
water, air, gasoline, liquid metals (runny stuff)	molten plastics, eggwhite, ketchup, concrete (icky, sticky, gooey stuff)
simple molecules	complex structure
internal forces depend linearly on velocity gradient	internal forces depend on the interaction of the flow with the microstructure

We see how quickly through the colander
The wines will flow; on the other hand,
The sluggish olive-oil delays; no doubt,
Because ‘tis wrought of elements more large
Or else more crooked and entangled.

Lucretius ~ 60 BC

Newtonian fluid

$$\mathbf{T} = \eta(\nabla \mathbf{u} + (\nabla \mathbf{u})^T).$$

Linear viscoelasticity

$$\mathbf{T}(\mathbf{x}, t) = \int_0^\infty G(s)(\nabla \mathbf{u}(\mathbf{x}, t-s) + (\nabla \mathbf{u}(\mathbf{x}, t-s))^T)\, ds$$

$$(+\eta(\nabla \mathbf{u}(\mathbf{x}, t) + (\nabla \mathbf{u}(\mathbf{x}, t))^T)).$$

Equation of motion

$$\rho \frac{\partial \mathbf{u}}{\partial t} = \operatorname{div} \mathbf{T} - \nabla p + \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0.$$

The function G is called the stress relaxation modulus. It is commonly assumed to be of the form

$$G(s) = \sum_{i=1}^{\infty} \kappa_i e^{-\lambda_i s}.$$

We can then write

$$\mathbf{T} = \sum_{i=1}^{\infty} \mathbf{T}_i,$$

and each $\mathbf{T}_i$ satisfies

$$\frac{\partial \mathbf{T}_i}{\partial t} + \lambda_i \mathbf{T}_i = \kappa_i(\nabla \mathbf{u} + (\nabla \mathbf{u})^T).$$

We note that the right hand side of this equation is always the symmetric part of the gradient of a divergence free vector field. Hence we shall set

$$\mathbf{T}_i = \nabla \mathbf{v}_i + (\nabla \mathbf{v}_i)^T,$$
$$\frac{\partial \mathbf{v}_i}{\partial t} = -\lambda_i \mathbf{v}_i + \kappa_i \mathbf{u}.$$

We thus end up with the following system of equations:

$$\rho \mathbf{u}_t = \eta \Delta \mathbf{u} + \sum_{i=1}^{\infty} \Delta \mathbf{v}_i - \nabla p + \mathbf{f} \chi_{\mathcal{O}},$$
$$(\mathbf{v}_i)_t = -\lambda_i \mathbf{v}_i + \kappa_i \mathbf{u},$$
$$\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{v}_i = 0,$$
$$\mathbf{u} = \mathbf{v}_i = \mathbf{0} \ \text{ on } \ \partial\Omega.$$

The control is localized on a subdomain $\mathcal{O}$.

Results:

Approximate controllability holds in the following cases. In all of these cases, we assume

$$\sum_{i=1}^{\infty} \frac{\kappa_i}{\lambda_i} < \infty,$$

$$\lambda_{i+1} - \lambda_i \geq \delta > 0.$$

Case 1: η>0.
Case 2: η=0 and

$$\sup_{\mathrm{Re}\,\lambda \geq 0} \left| \arg\Big(\sum_{i=1}^{\infty} \frac{\kappa_i}{\lambda + \lambda_i} \Big) \right| < \pi/2.$$

Note: This condition implies that

$$\sum_{i=1}^{\infty} \kappa_i = \infty.$$

Case 3:

$$\sum_{i=1}^{\infty} \kappa_i \lambda_i < \infty,$$

and the following geometric condition holds: For some point $\mathbf{x}_0$ outside Ω, define

$$\Gamma = \{\mathbf{x} \in \partial\Omega \,|\, (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n} \geq 0\}.$$

Then the controlled region must contain a neighborhood of Γ. Moreover, let

$$R(\mathbf{x}_0) = \max_{\mathbf{x} \in \overline{\Omega}} |\mathbf{x} - \mathbf{x}_0|, \qquad T(\mathbf{x}_0) = 2R(\mathbf{x}_0)\sqrt{\rho / \sum_{i=1}^{\infty} \kappa_i}.$$

We require that $T > T(\mathbf{x}_0)$.

Earlier results

Renardy (2005): One space dimension, finite number of exponentials
Doubova, Fernandez-Cara (2012), Boldrini, Doubova, Fernandez-Cara, Gonzalez-Burgos (2012): single exponential
Chowdhury, Mitra, Ramaswamy, Renardy (2016); finite number of exponentials

Outline of proof

Case 1

1. Define the following function spaces:

$$V_n^0(\Omega) = \Big\{ \mathbf{u} \in (L^2(\Omega))^d \mid \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u} \cdot n = 0 \quad \text{in} \partial\Omega \Big\},$$

$$\ell_w^2 = \{(a_i) \mid a_i \in \mathbb{R}, \sum_{i=1}^{\infty} \frac{\lambda_i}{\kappa_i} |a_i|^2 < \infty\},$$

$$\ell_{\frac{1}{w}}^2 = \{(a_i) \mid a_i \in \mathbb{R}, \sum_{i=1}^{\infty} \frac{\kappa_i}{\lambda_i} |a_i|^2 < \infty\},$$

$$H = V_n^0(\Omega) \times V_n^0(\Omega; \ell_w^2), \quad H^* = V_n^0(\Omega) \times V_n^0(\Omega; \ell_{\frac{1}{w}}^2).$$

2. We can formulate the governing equations as an abstract evolution problem of the form $dx/dt=Ax$ in H.
3. Prove that A generates an analytic semigroup in H (and hence A^* generates an analytic semigroup in H^*). It follows that solutions of the adjoint problem are analytic in time.
4. Prove that the resolvent of A^* is bounded in a domain which has holes near the eigenvalues.
5. Prove that the linear span of the generalized eigenfunctions of A is dense.
6. Assume a solution of the adjoint problem has velocity component vanishing on $\mathcal{O}$. Take its Laplace transform. Since the Laplace transform of the semigroup is the resolvent, we expect poles at the eigenvalues, with the coefficient of the singularity given by the eigenfunctions. But since none of the eigenfunctions have velocity component vanishing on $\mathcal{O}$, all the poles must have strength zero. This together with 5, implies the solution is zero.

Case 2

1. Let A_0 be the Stokes operator. Define

$$\mathbf{w}_i = \sqrt{\frac{1}{\kappa_i \rho}} A_0^{1/2} \mathbf{v}_i,$$

and write the equations in the form

$$\mathbf{u}_t = -\rho^{-1/2} A_0^{1/2} \sum_{i=1}^{\infty} \kappa_i^{1/2} \mathbf{w}_i + \mathbf{f} \chi_{\mathcal{O}},$$

$$(\mathbf{w}_i)_t = -\lambda_i \mathbf{w}_i + (\frac{\kappa_i}{\rho})^{1/2} A_0^{1/2} \mathbf{u}.$$

2. Formulate as an abstract evolution problem in

$$H = V_n^0(\Omega) \times V_n^0(\Omega; \ell^2),$$

and show that A generates an analytic semigroup.

3. The rest goes as for Case 1.

Case 3

1. Choose function spaces as for Case 2. However, the operator A does not generate an analytic semigroup.
2. Instead we can decompose into a part that does generate an analytic semigroup, and another part that is a small perturbation of a damped wave equation.

3. Now consider a solution of

$$\dot{y} = A^* y,$$

   which has velocity component vanishing on $\mathcal{O} \times (0, T)$.
   Decompose $y=y_1+y_2$, where y_1 is the analytic semigroup part and y_2 is the damped wave equation part. We know y_1 is analytic in time , hence the velocity component of y_2 is analytic in time on $\mathcal{O} \times (0, T)$.
4. Use observability estimates for the damped wave equation to show that actually y_2 is analytic in time and can be continued analytically to $(0,\infty)$.
5. Now use Laplace transforms as before.

Exaxt null controllability?

Does not hold for Jeffreys models or Maxwell models with more than one relaxation mode. Reason: Propagation of singularities along vertical characteristics.

Jeffreys models: Effect of control is C^∞ outside the controlled region.

Maxwell models: Microlocal analysis.

Variable coefficients

Consider a single mode Maxwell model

$$\frac{\partial \mathbf{T}}{\partial t} + \lambda(\mathbf{x})\mathbf{T} = \kappa(\mathbf{x})(\nabla \mathbf{u} + (\nabla \mathbf{u})^T).$$

If the coefficients are variable, we cannot set

$$\mathbf{T} = \nabla \mathbf{v} + (\nabla \mathbf{v})^T.$$

What are the constraints on controllability?

At the level of the adjoint problem, this leads to the question of existence of pure stress modes:

$$\frac{\partial \mathbf{T}}{\partial t} + \lambda(\mathbf{x})\mathbf{T} = \mathbf{0},$$

$$\operatorname{div} \mathbf{T} - \nabla p = \mathbf{0}.$$

It can be shown that pure stress modes are given by

$$\mathbf{T} = \mathbf{T_0} \exp(-\lambda(x)t),$$

$$\mathbf{T_0} = p_0\mathbf{I} + \mathbf{Q},$$

$$\operatorname{div} \mathbf{Q} = \mathbf{0},$$

$$(\mathbf{Q} + \phi'(\lambda)\mathbf{I})\nabla\lambda = \mathbf{0}.$$

Two dimensions: Under generic hypothesis, only trivial solutions exist.
Three dimensions: We can prescribe the function ϕ'(λ) arbitrarily. On each level surface λ=const. we are left with a PDE system that is elliptic if the level surface has positive Gauss curvature and hyperbolic if it has negative Gauss curvature.

Nonlinear systems

A simple example:

$$\dot{x} = f(t),$$

$$\dot{y} = x^2.$$

No matter how we choose f, we can control x, but y will always increase. The system is not controllable.

Upper convected Maxwell model, nonlinear case

We specialize to parallel shear flow, i.e. the velocity is (u(y,t),0), and the viscoelastic stress is

$$\begin{pmatrix} \sigma(y,t) & \tau(y,t) \\ \tau(y,t) & 0 \end{pmatrix}.$$

The full constitutive law is

$$\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{T} - (\nabla \mathbf{u})\mathbf{T} - \mathbf{T}(\nabla \mathbf{u})^T + \lambda \mathbf{T} = \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T).$$

In shear flow this leads to

$$\sigma_t - 2\tau u_y + \lambda\sigma = 0,$$

$$\tau_t + \lambda\tau = \mu u_y,$$

$$\rho u_t = \tau_y + f(y,t).$$

Homogeneous Dirichlet conditions for u: u(0,t)=u(1,t)=0.
Initial and final conditions:

$$u(y,0) = u_0(y),\ \tau(y,0) = \tau_0(y),\ \sigma(y,0) = \sigma_0(y).$$

$$u(y,T) = u_f(y),\ \tau(y,T) = \tau_f(y),\ \sigma(y,T) = \sigma_f(y).$$

We shall require that

$$\int_0^1 \tau(y,t)\,dy = 0.$$

Positive definiteness

$$\frac{d}{dt}(\mu\sigma - \tau^2) = -\lambda(\mu\sigma - \tau^2) + \lambda\tau^2.$$

Hence

$$\mu\sigma_f(y) - \tau_f(y)^2 > e^{-\lambda T}(\mu\sigma_0(y) - \tau_0(y)^2).$$


"That's all Folks!"
ACME ACRES
USA
U.S.
MAIL
USA
33
2001

Questions?