Energy/entropy conservation for general hyperbolic systems

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Introduction: the principle of conservation of energy for classical solutions

Let us first focus our attention on the incompressible Euler system

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0,$$
$$\operatorname{div} u = 0,$$

If u is a classical solution, then multiplying the balance equation by u we obtain

$$\frac{1}{2}\partial_t |u|^2 + \frac{1}{2} u \cdot \nabla |u|^2 + u \cdot \nabla p = 0.$$

Integrating the last equality over the space domain Ω yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} |u(x,t)|^2 \, \mathrm{d}x = 0.$$

Consequently, integrating over time in $(0,t)$, gives

$$\int_\Omega \frac{1}{2} |u(x,t)|^2 \, \mathrm{d}x = \int_\Omega \frac{1}{2} |u(x,0)|^2 \, \mathrm{d}x.$$

Weak solutions

However, if u is a weak solution, then

$$\int_\Omega \frac{1}{2}|u(x,t)|^2 \, \mathrm{d}x = \int_\Omega \frac{1}{2}|u(x,0)|^2 \, \mathrm{d}x.$$

might not hold. Technically, the problem is that u might not be regular enough to justify integration by parts in the above derivation.

Motivated by the laws of turbulence Onsager postulated that there is a critical regularity for a weak solution to be a conservative one:

Conjecture, 1949

Let u be a weak solution of incompressible Euler system

- If $u \in \mathrm{C}^\alpha$ with $\alpha > \frac{1}{3}$, then the energy is conserved.
- For any $\alpha < \frac{1}{3}$ there exists a weak solution $u \in C^\alpha$ which does not conserve the energy.

Onsager conjecture for incompressible Euler system

Weak solutions of the incompressible Euler equations which do not conserve energy were constructed:

- Scheffer '93, Shnirelman '97 constructed examples of weak solutions in $L^2(\mathbb{R}^2 \times \mathbb{R})$ compactly supported in space and time
- De Lellis and Székelyhidi 2010 showed how to construct weak solutions for given energy profile

Still incompressible case

- Significant progress has recently been made in constructing energy-dissipating solutions slightly below the Onsager regularity , see e.g.:
  - T. Buckmaster, C. De Lellis, P. Isett, and L. Székelyhidi, Anomalous dissipation for 1/5-Hölder Euler flows. Ann. of Math. (2), 2015
  - T. Buckmaster, C. De Lellis, and L. Székelyhidi, Dissipative Euler flows with Onsager-critical spatial regularity. Comm. Pure and Appl. Math., 2015.

  And the story is closed by the results:
  - **Philip Isett, A Proof of Onsager's Conjecture, arXiv:1608.08301**
  - **Tristan Buckmaster, Camillo De Lellis, László Székelyhidi Jr., Vlad Vicol, Onsager's conjecture for admissible weak solutions, arXiv:1701.08678**

Still incompressible case

Onsager conjecture:

If weak solution v has $C^{0,\alpha}$ (for $\alpha > \frac{1}{3}$) regularity then it conserves energy. In the opposite case it may not conserve energy.

- The first part of this assertion was proved in
  - G. L. Eyink. Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer. Phys. D, 1994
  - P. Constantin, W. E, and E. S. Titi. Onsager's conjecture on the energy conservation for solutions of Euler's equation. Comm. Math. Phys., 1994
  - A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy. Energy conservation and Onsager's conjecture for the Euler equations. Nonlinearity, 2008

Besov spaces

The elements of Besov space $B_p^{\alpha,\infty}(\Omega)$, where $\Omega = (0, T) \times \mathbb{T}^d$ or $\Omega = \mathbb{T}^d$ are functions w for which the norm

$$\|w\|_{B_p^{\alpha,\infty}(\Omega)} := \|w\|_{L^p(\Omega)} + \sup_{\xi \in \Omega} \frac{\|w(\cdot + \xi) - w\|_{L^p(\Omega \cap (\Omega - \xi))}}{|\xi|^\alpha}$$

is finite (here $\Omega - \xi = \{x - \xi : x \in \Omega\}$).
It is then easy to check that the definition of the Besov spaces implies

$$\|w^\epsilon - w\|_{L^p(\Omega)} \leq C\epsilon^\alpha \|w\|_{B_p^{\alpha,\infty}(\Omega)}$$

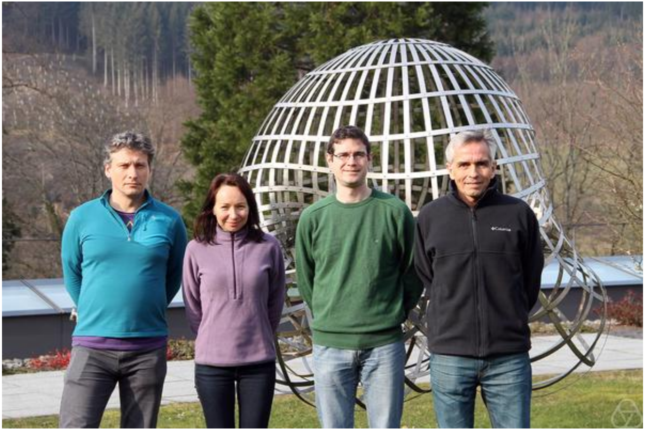
and

$$\|\nabla w^\epsilon\|_{L^p(\Omega)} \leq C\epsilon^{\alpha-1} \|w\|_{B_p^{\alpha,\infty}(\Omega)}.$$

Idea of the proof: P. Constantin, W. E, and E. S. Titi. Onsager's conjecture on the energy conservation for solutions of Euler's equation. Comm. Math. Phys., 1994

- take as the test function doubly mollified solution $(v^\epsilon)^\epsilon$
- problem: estimate term $\int_{\mathbb{T}^d} \mathrm{Tr}(v \otimes v)^\epsilon \cdot \nabla v^\epsilon dx$
- use the identity:
  $(v \otimes v)^\epsilon = v^\epsilon \otimes v^\epsilon + r_\epsilon(v, v) - (v - v^\epsilon) \otimes (v - v^\epsilon)$ where
  $\|r_\epsilon(v, v)\|_{L^{3/2}} \leq C\epsilon^{2\alpha}\|v\|^2_{B_p^{\alpha,\infty}}$

Onsager's conjecture for compressible Euler system



Compressible Euler system

We consider now the isentropic Euler equations,

$$\begin{aligned} \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p(\rho) = 0, \\ \partial_t \rho + \mathrm{div}(\rho u) = 0. \end{aligned} \tag{1}$$

We will use the notation for the so-called pressure potential defined as

$$P(\rho) = \rho \int_1^\rho \frac{p(r)}{r^2} dr.$$

Theorem (Feireisl, Gwiazda, Ś.-G., Wiedemann, ARMA 2017)

Let ϱ, u *be a solution of* (1) *in the sense of distributions. Assume*

$$u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d),\ \varrho, \varrho u \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d), 0 \le \underline{\varrho} \le \varrho \le \overline{\varrho}$$

for some constants $\underline{\varrho}$, $\overline{\varrho}$, *and* $0 \le \alpha, \beta \le 1$ *such that*

$$\beta > \max\left\{1 - 2\alpha; \frac{1-\alpha}{2}\right\}. \tag{2}$$

Assume further that $p \in C^2[\underline{\varrho}, \overline{\varrho}]$, *and, in addition*

$$p'(0) = 0 \text{ as soon as } \underline{\varrho} = 0.$$

Then the energy is locally conserved in the sense of distributions on $(0,T)\times\Omega$, *i.e.*

$$\partial_t\left(\frac{1}{2}\varrho|u|^2 + P(\varrho)\right) + \operatorname{div}\left[\left(\frac{1}{2}\varrho|u|^2 + p(\varrho) + P(\varrho)\right)u\right] = 0.$$

Sharpness of assumptions

Shocks provide examples that show that our assumptions are sharp:

- A shock solution dissipates energy, but ρ and u are in $BV \cap L^\infty$, which embeds into $B_3^{1/3,\infty}$.
- Hence such a solution satisfies (2) with equality but fails to satisfy the conclusion.

Time regularity

The hypothesis on temporal regularity can be relaxed provided

$$\underline{\varrho} > 0$$

Indeed, in this case $\frac{(\varrho u)^\epsilon}{\varrho^\epsilon}$ can be used as a test function in the momentum equation, cf.

T. M. Leslie and R. Shvydkoy. The energy balance relation for weak solutions of the density-dependent Navier- Stokes equations. JDE, 2016.

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General conservation laws

- It is easy to notice similarities in the statements regarding sufficient regularity conditions guaranteeing energy/entropy conservation for various systems of equations of fluid dynamics.
- Especially the differentiability exponent of $\frac{1}{3}$ is a recurring condition.
- One might therefore anticipate that a general statement could be made, which would cover all the above examples and more. Indeed, consider a general conservation law of the form

$$\mathrm{div}_X(G(U(X))) = 0.$$

We consider the conservation law of the form

$$\mathrm{div}_X(G(U(X))) = 0. \tag{3}$$

Here $U : \mathcal{X} \to \mathcal{O}$ is an unknown and $G : \mathcal{O} \to \mathbb{M}^{n\times(d+1)}$ is a given, where $\mathcal{X}$ is an open subset of $\mathbb{R}^{d+1}$ or $\mathbb{T}^3 \times \mathbb{R}$ and the set $\mathcal{O}$ is open in $\mathbb{R}^n$. It is easy to see that any classical solution to (3) satisfies also

$$\mathrm{div}_X(Q(U(X))) = 0, \tag{4}$$

where $Q : \mathcal{O} \to \mathbb{R}^{s\times(d+1)}$ is a smooth function such that

$$D_U Q_j(U) = \mathfrak{B}(U) D_U G_j(U), \quad \text{for all } U \in \mathcal{O},\ j \in 0, \cdots, k, \tag{5}$$

for some smooth function $\mathfrak{B} : \mathcal{O} \to \mathbb{M}^{s\times n}$. The function Q is called a *companion* of G and equation (4) is called a *companion law* of the conservation law (3).

Weak solutions

In many applications some relevant companion laws are *conservation of energy* or *conservation of entropy*. We consider the standard definition of weak solutions to a conservation law.

Definition

We call the function U a weak solution to (3) if $G(U)$ is locally integrable in $\mathcal{X}$ and the equality

$$\int_{\mathcal{X}} G(U(X)) \colon D_X \psi(X) dX = 0$$

holds for all smooth test functions $\psi \colon \mathcal{X} \to \mathbb{R}^n$ with a compact support in $\mathcal{X}$.

How much regularity of a weak solution is required so that it also satisfies the companion law?

Theorem (Gwiazda, Michálek, Ś-G., to appear in ARMA)

Let $U \in B^{\alpha}_{3,\infty}(\mathcal{X};\mathcal{O})$ *be a weak solution of* (3) *with* $\alpha > \frac{1}{3}$. *Assume that* $G \in C^2(\mathcal{O};\mathbb{M}^{n\times(k+1)})$ *is endowed with a companion law with flux* $Q \in C(\mathcal{O};\mathbb{M}^{1\times(k+1)})$ *for which there exists* $\mathcal{B} \in C^1(\mathcal{O};\mathbb{M}^{1\times n})$ *related through identity* (5) *and all the following conditions hold*

$$\mathcal{O} \text{ is convex},$$

$$\mathcal{B} \in W^{1,\infty}(\mathcal{O};\mathbb{M}^{1\times n}),$$

$$|Q(V)| \leq C(1+|V|^3) \text{ for all } V \in \mathcal{O},$$

$$\sup_{i,j\in 1,\dots,d} \|\partial_{U_i}\partial_{U_j} G(U)\|_{C(\mathcal{O};\mathbb{M}^{n\times(k+1)})} < +\infty.$$

Then U *is a weak solution of the companion law* (4) *with the flux* Q.

The essential part of the proof of this Theorem pertains the estimation of the nonlinear commutator

$$[G(U)]_\varepsilon - G([U]_\varepsilon).$$

It is based on the following observation:

Lemma

Let $\mathcal{O}$ be a convex set, $U \in L^2_{loc}(\mathcal{X}, \mathcal{O})$, $G \in C^2(\mathcal{O}; \mathbb{R}^n)$ and let

$$\sup_{i,j \in 1,\dots,d} \|\partial_{U_i}\partial_{U_j} G(U)\|_{L^\infty(\mathcal{O})} < +\infty.$$

Then there exists $C > 0$ depending only on η_1, second derivatives of G and k (dimension of $\mathcal{O}$) such that

$$\|[G(U)]_\varepsilon - G([U]_\varepsilon)\|_{L^q(K)}$$
$$\leq C\left(\|[U]_\varepsilon - U\|^2_{L^{2q}(K)} + \sup_{Y \in \mathrm{supp}\ \eta_\varepsilon} \|U(\cdot) - U(\cdot - Y)\|^2_{L^{2q}(K)}\right)$$

for $q \in [1, \infty)$, where $K \subseteq \mathcal{X}$ satisfies $K^\varepsilon \subseteq \mathcal{X}$.

Remarks

- Due to the assumption on the convexity of $\mathcal{O}$ the previous theorem could be straightforwardly deduced from the result for compressible Euler system (Feireisl, Gwiazda, Ś.-G., Wiedemann ARMA 2017).
- It is worth noting that the convexity of $\mathcal{O}$ might not be natural for all applications (this is e.g. the case of the polyconvex elasticity).

A few words about polyconvex elasticity

Let us consider the evolution equations of nonlinear elasticity

$$\left.\begin{aligned} \partial_t F &= \nabla_x \mathbf{v} \\ \partial_t \mathbf{v} &= \operatorname{div}_x \left(D_F W(F) \right) \end{aligned}\right. \quad \text{in } \mathcal{X},$$

for an unknown matrix field $F\colon \mathcal{X} \to \mathbb{M}^{k\times k}$, and an unknown vector field $\mathbf{v}\colon \mathcal{X} \to \mathbb{R}^k$. Function $W\colon \mathcal{U} \to \mathbb{R}$ is given. For many applications, $\mathcal{U} = \mathbb{M}^{k\times k}_+$ where $\mathbb{M}^{k\times k}_+$ denotes the subset of $\mathbb{M}^{k\times k}$ containing only matrices having positive determinant. Let us point out that $\mathbb{M}^{k\times k}_+$ is a non–convex connected set.

To this purpose, we study the case of non–convex $\mathcal{O}$

- Having $\mathcal{O}$ non–convex, we face the problem that $[U]_\varepsilon$ does not have to belong to $\mathcal{O}$.
- The convexity was crucial to conduct the Taylor expansion argument in error estimates.
- However, a suitable extension of functions G, $\mathcal{B}$ and Q does not alter the previous proof significantly.

How much regularity of a weak solution is required so that it also satisfies the companion law?

Theorem

Let $U \in B_3^{\alpha,\infty}(\mathcal{X};\mathcal{O})$ be a weak solution of (3) *with $\alpha > \frac{1}{3}$.*
Assume that $G \in \mathrm{C}^2(\mathcal{O};\mathbb{M}^{n\times(d+1)})$ is endowed with a companion law with flux $Q \in \mathrm{C}(\mathcal{O};\mathbb{M}^{s\times(d+1)})$ for which there exists $\mathfrak{B} \in \mathrm{C}^1(\mathcal{O};\mathbb{M}^{s\times n})$ related through identity (5) *and the essential image of U is compact in $\mathcal{O}$.*
Then U is a weak solution of the companion law (4) *with the flux Q.*

Remarks

- the generality of the above theorem is achieved at the expense of optimality of the assumptions.
- However given additional information on the structure of the problem at hand one might be able to relax some of these assumptions.
- the theorem provides for instance a conservation of energy result for the system of polyconvex elastodynamics, compressible hydrodynamics et al.
- T. Dębiec, P. Gwiazda, and A. Świerczewska-Gwiazda,
  A tribute to conservation of energy for weak solutions
  arXiv:1707.09794, 2017.

Result of Constantin, E, Titi

Theorem

Let $u \in L^3([0,T], B_3^{\alpha,\infty}(\mathbb{T}^3)) \cap \mathrm{C}([0,T], L^2(\mathbb{T}^3))$ *be a weak solution of the incompressible Euler system. If* $\alpha > \frac{1}{3}$*, then*

$$\int_{\mathbb{T}^3} \frac{1}{2}|u(x,t)|^2 \,\mathrm{d}x = \int_{\mathbb{T}^3} \frac{1}{2}|u(x,0)|^2 \,\mathrm{d}x$$

for each $t \in [0,T]$.

Additional structure of equations

The first lemma gives a sufficient condition to drop the Besov regularity with respect to some variables. It is connected with the columns of G.

Lemma

Let $G = (G_1, \ldots, G_s, G_{s+1}, \ldots G_k)$ where $G_1, \ldots, G_s$ are affine vector–valued functions and $\mathcal{X} = \mathcal{Y} \times \mathcal{Z}$ where $\mathcal{Y} \subseteq \mathbb{R}^s$ and $\mathcal{Z} \subseteq \mathbb{R}^{k+1-s}$. Then it is enough to assume that $U \in L^3(\mathcal{Y}; B^{\alpha}_{3,\infty}(\mathcal{Z}))$ in the main theorem.

We can omit the Besov regularity w.r.t. some components of U.

Lemma

Assume that $U = (V_1, V_2)$ where $V_1 = (U_1, ..., U_s)$ and $V_2 = (U_{s+1}, \ldots, U_n)$. If $\mathcal{B}$ does not depend on V_1 and $G = G(V_1, V_2) = G_1(V_1) + G_2(V_2)$ and G_1 is linear then it is enough to assume $U_1, \ldots, U_s \in L^3(\mathcal{X})$ in the main theorem.

Opposite direction of the Onsager's hypothesis for hyperbolic systems

- It is well known that shock solutions dissipate energy.
- the essence can be already seen even on a simple example of the Burger's equation

$$u_t + (u^2/2)_x = 0.$$

- Classical solutions also satisfy $(u^2/2)_t + (u^3/3)_x = 0$, which can be considered as a companion law.
- The shock solutions to the BE satisfy Rankine-Hugoniot condition $s(u_l - ur) = (u_l^2 - u_r^2)/2$, thus the speed of the shock is $s = (u_l + u_r)/2$, where $u_l = \lim_{y \to x(t)^-} u(y,t)$ and u_r is defined correspondingly.
- Considering the second equation one gets $s = 2(u_l^2 + u_l u_r + u_r^2)/3(u_l + u_r)$.
- If we multiply BE with the function $\mathcal{B}$ then to provide RH conditions to be satisfied for the companion law, we end up with a trivial companion law, namely $\mathcal{B} \equiv const$.

Euler-Korteweg Equations

We now consider the isothermal Euler-Korteweg system in the form

$$\partial_t \rho + \mathrm{div}(\rho u) = 0,$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) = -\rho \nabla_x \left(h'(\rho) + \frac{\kappa'(\rho)}{2} |\nabla_x \rho|^2 - \mathrm{div}(\kappa(\rho) \nabla_x \rho) \right),$$

where $\rho \geq 0$ is the scalar density of a fluid, u is its velocity, $h = h(\rho)$ is the energy density and $\kappa(\rho) > 0$ is the coefficient of capillarity.
In conservative form

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) = \mathrm{div}\, S,$$
$$\partial_t \rho + \mathrm{div}(\rho u) = 0,$$

where S is the Korteweg stress tensor

$$S = [-p(\rho) - \frac{\rho \kappa'(\rho) + \kappa(\rho)}{2} |\nabla_x \rho|^2 + \mathrm{div}(\kappa(\rho) \rho \nabla_x \rho)]\mathbb{I} - \kappa(\rho) \nabla_x \rho \otimes \nabla_x \rho$$

where the local pressure is defined as $p(\rho) = \rho h'(\rho) - h(\rho)$.

Remarks

J.E.Dunn, J.Serrin.
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Energy Equality

It can be shown that smooth solutions to the EK system satisfy the balance of total (kinetic and internal) energy

$$\partial_t \left(\frac{1}{2}\rho|u|^2 + h(\rho) + \frac{\kappa(\rho)}{2}|\nabla_x \rho|^2 \right)$$
$$+ \operatorname{div}\left(\rho u \left(\frac{1}{2}|u|^2 + h'(\rho) + \frac{\kappa'(\rho)}{2}|\nabla_x \rho|^2 - \operatorname{div}(\kappa(\rho)\nabla_x \rho) \right)\right.$$
$$\left. - \kappa(\rho)\partial_t \rho \nabla \rho \right) = 0.$$

Energy Conservation

Theorem (T.Debiec, P.Gwiazda, A.Ś-G., A.Tzavaras, arXiv:1801.00177)

Let (ρ, u) *be a solution to the EK system with constant capillarity in the sense of distributions. Assume*

$$u, \nabla_x u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u, \nabla_x \rho, \Delta\rho \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d),$$

where $0 < \alpha, \beta < 1$ *such that* $\min(2\alpha+\beta, \alpha+2\beta) > 1$.
Then the energy is locally conserved, i.e.

$$\partial_t\left(\frac{1}{2}\rho|u|^2 + h(\rho) + \frac{\kappa}{2}|\nabla_x\rho|^2\right) + \operatorname{div}\left(\frac{1}{2}\rho u|u|^2 + \rho^2 u - \kappa\rho u\Delta\rho - \kappa\partial_t\rho\nabla\rho\right) = 0$$

in the sense of distributions on $(0,T)\times\mathbb{T}^d$.

Thank you for your attention