Some Mathematical Theories of MHD Boundary Layers

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In 1904, Prandtl developed the boundary layer theory by resolving the difference between viscous and inviscid flow near a boundary with no-slip boundary condition:

• outside a layer of thickness of $\sqrt{\varepsilon}$, convection dominates so that the flow is governed by Euler equations;

• inside a layer (boundary layer) of thickness of $\sqrt{\varepsilon}$, convection and viscosity balanced so that the flow is governed by the Prandtl layer equation.

Well-posedness? Justification?

(Incompressible Navier-Stokes equations)

$$\begin{cases} \partial_t \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon + \nabla p^\varepsilon - \varepsilon \Delta \mathbf{u}^\varepsilon = 0 \\ \nabla \cdot \mathbf{u}^\varepsilon = 0 \\ \mathbf{u}^\varepsilon|_{z=0} = 0 \quad \textit{noslip boundary condition} \end{cases}$$

with flat boundary $\{(x,y) \in D, z = 0\}$, set $\mathbf{u}^\varepsilon = (u^\varepsilon, v^\varepsilon, w^\varepsilon)^T$:

$$\begin{cases} u^\varepsilon(t,x,y,z) = u(t,x,y,\frac{z}{\sqrt{\varepsilon}}) + o(1) \\ v^\varepsilon(t,x,y,z) = v(t,x,y,\frac{z}{\sqrt{\varepsilon}}) + o(1) \\ w^\varepsilon(t,x,y,z) = \sqrt{\varepsilon} w(t,x,y,\frac{z}{\sqrt{\varepsilon}}) + o(\sqrt{\varepsilon}) \end{cases}$$

(Prandtl boundary layer equations)

$$\begin{cases} \partial_t u + (u\partial_x + v\partial_y + w\partial_Z)u + \partial_x p^E(t,x,y,0) = \partial_z^2 u \\ \partial_t v + (u\partial_x + v\partial_y + w\partial_Z)v + \partial_y p^E(t,x,y,0) = \partial_z^2 v \\ \partial_x u + \partial_y v + \partial_z w = 0 \\ (u,v,w)|_{z=0} = 0, \qquad \lim_{z\to+\infty}(u,v) = (u^E,v^E)(t,x,y,0), \end{cases}$$

with p^E and $\mathbf{u}^E = (u^E, v^E, 0)(t,x,y,0)$ satisfy

(Bernoulli's law)

$$\partial_t \mathbf{u}^E + (\mathbf{u}^E \cdot \nabla)\mathbf{u}^E + \nabla p^E = 0.$$

(Oleinik's monotonicity condition (2D))

- *Coordinate transformations*
  - *von Mise transformation for steady layer (Oleinik):*

$$(x,y) \to (x,\phi), \quad w = u^2,$$
$$w_x = \sqrt{w} w_{yy} - 2P^E_x.$$

  - *Crocco transformation for unsteady layer (Oleinik):*

$$(t,x,y) \to (t,x,u), \quad w = u_y,$$
$$w_t + u w_x = w^2 w_{uu}.$$

- Cancellations

$$\partial_t u + u\partial_x u + v\partial_y u + \partial_x P^E = \partial_y^2 u$$

$$\cdots + (\frac{u\partial_x u}{\partial_y u} + v)_y + \cdots = \cdots$$

$$\cdots + u(\frac{\partial_x u}{\partial_y u})_y + \partial_x u + \partial_y v + \cdots = \cdots.$$

Denote $w = \partial_y u$, then

$$f_0 = (\frac{u}{\partial_y u})_y = \frac{w^2 - w_y u}{w^2}. \quad \text{(Alexandre-Wang-Xu-Y.)}$$

Another way to look at the cancellation:

$$w_t + u\partial_x w + v\partial_y w = \partial_y^2 w,$$

$$u_t + u\partial_x u_1 + v\partial_y u = \partial_y^2 u.$$

Set

$$f_1 = w - \frac{w_y}{w} u = w f_0. \quad \text{(Masmoudi-Wong)}$$

A similar function $f_2 = w w_x - w_y u_{1x}$ satisfying an equation without loss of derivative is used to study solutions in Gevrey function space without monotonicity condition, cf. Li-Y.

(Well-posedness with finite order of regularity)

- *Oleinik ('60): local existence of classical solutions;*
- *Xin-Zhang ('04): existence of global weak solution with additional favorable pressure $\partial_x p^E(t,x,0) \leq 0$;*
- *Alexandre-Wang-Xu-Y.('12), Masmoudi-Wong ('12): local existence in Sobolev spaces.*
- $\cdots$

How about 3D?

(Ill-posedness without the monotonicity condition)

- *E & Engquist ('97): construction of blowup solutions;*
- *Grenier ('00): unstable Euler shear flow yields instability of Prandtl;*
- *Gérard-Varet & Dormy ('10), Guo & Nguyen ('11): shear flow with non-degenerate critical point in 2D;*
- *Grenier-Nguyen ('17): unstable even for Rayleigh's stable shear flow;*
- *Liu-Wang-Y. ('15): ill-posedness in 3D if $U(z) \not\equiv kV(z)$. Not know nonlinear stability even $U(z) = kV(z)$ with $U_z(z) > 0$ and finite regularity.*

(Well-posedness with infinite order of regularity)

- *Sammartino & Caflisch ('98): Well-posedness of Prandtl system, and justification of the Prandtl ansatz when the data is analytic;*
- *Gérard-Varet & Masmoud: 2D with Gevrey index =* $\frac{7}{4}$*;*
- *Li-Y.('17): 2D optimal Gevrey index* $(1,2]$*;*
- *Li-Y.('18): 3D with index* $(1,2]$ *and monotonicity in one direction.*

Optimal Gevrey index 2 implied by the ill-posedness theory of Gérard-Varet & Dormy.

(High Reynolds number limit)

- *Kato ('84): a necessary and sufficient condition;*
- *Bardos-Titi ('18): relation to Onsager conjecture;*
- *Maekawa ('14): initial vorticity is supported away from the boundary for 2D flow;*
- *Gérard-Varard, Maekawa & Masmoudi('15): Gevrey stability of Prandtl expansion in 2D;*
- *Guo-Nguyen('17): steady fow over a moving plate;*
- ...

Well-posedness with Gevrey regularity in 2D

Defintion Let $\ell > 3/2$. For (ρ,σ), $\rho > 0, \sigma \geq 1$, $X_{\rho,\sigma}$ is a Gevrey function space with the norm

$$
\begin{aligned}
\|f\|_{\rho,\sigma} &= \sup_{m\geq 6} \frac{\rho^{m-5}}{\left[(m-6)!\right]^{\sigma}} \left\| < y >^{\ell-1} \partial_x^m f \right\|_{L^2} \\
&+ \sup_{m\geq 6} \frac{\rho^{m-5}}{\left[(m-6)!\right]^{\sigma}} \left\| < y >^{\ell} \partial_x^m (\partial_y f) \right\|_{L^2} \\
&+ \sup_{\substack{1\leq j\leq 4 \\ i+j\geq 6}} \frac{\rho^{i+j-5}}{\left[(i+j-6)!\right]^{\sigma}} \left\| < y >^{\ell+1} \partial_x^i \partial_y^j (\partial_y f) \right\|_{L^2} \\
&+ \ldots
\end{aligned}
$$

(Theorem Li-Y., JEMS, '17)

Let $\sigma \in (1,2]$, $u_0 \in X_{2\rho_0,\sigma}$ *with*

$$\|u_0\|_{\rho_0,\sigma} \leq \eta_0.$$

Suppose u_0 *satisfies the compatibility conditions, then there is a unique solution* $u \in L^\infty\big([0,T];\, X_{\rho,\sigma}\big)$ *for some* $T > 0$ *and some* $0 < \rho < \rho_0$, *provided* η_0 *is sufficiently small.*

Three types of cancellations

The first type cancellation: Consider

$$\left(\partial_t + (u^s + u)\partial_x + v\partial_y - \partial_y^2\right)\partial_x^m u + (\partial_x^m v)(\omega^s + \omega) = \cdots$$

and vorticity equation

$$\left(\partial_t + (u^s + u)\partial_x + v\partial_y - \partial_y^2\right)\partial_x^m \omega + (\partial_x^m v)(\partial_y \omega^s + \partial_y \omega) = \cdots$$

In the region of monotonicity, introduce

$$f_m = \partial_x^m \omega - \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega}\partial_x^m u;$$

cf. Masmoudi-Wong, Alexandre-Wang-Xu-Y.

The second type cancellation: For estimation on $\partial_x^m \omega$ in the neighborhood of the critical point, use the vorticity equation

$$\left(\partial_t + (u^s + u)\partial_x + v\partial_y - \partial_y^2\right)\partial_x^m \omega + (\partial_x^m v)(\partial_y \omega^s + \partial_y \omega) = \cdots$$

with inner product with

$$\frac{\partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega};$$

and use

$$\left((\partial_x^m v)(\partial_y \omega^s + \partial_y \omega),\ \frac{\partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega}\right)_{L^2} = (\partial_x^{m+1} u, \partial_x^m u)_{L^2} = 0,$$

cf. Gérard-Varet and Masmoudi.

The third type cancellation: Note that the equation for $\partial_y\omega$:

$$\partial_t(\partial_y\omega)+(u^s+u)\partial_x(\partial_y\omega)+v(\partial_y^2\omega^s+\partial_y^2\omega)-\partial_y^2(\partial_y\omega)=-g_1,$$

with $g_1=(\omega^s+\omega)\partial_x\omega-(\partial_y\omega^s+\partial_y\omega)\partial_x u,$ we have

$$\left(\partial_t+(u^s+u)\partial_x+v\partial_y-\partial_y^2\right)\partial_x^m\partial_y\omega+(\partial_x^m v)(\partial_y^2\omega^s+\partial_y^2\omega)$$
$$=-\partial_x^m g_1+\cdots$$

In the neighborhood of the critical point, consider

$$h_m=\partial_x^m\partial_y\omega-\frac{\partial_y^2\omega^s+\partial_y^2\omega}{\partial_y\omega^s+\partial_y\omega}\partial_x^m\omega.$$

(Property of g_1)

Note that $g_1 = (\omega^s + \omega) f_1$ *satisfies:*

$$\left(\partial_t + (u^s + u)\partial_x + v\partial_y - \partial_y^2\right) g_1 =$$

$$2(\partial_y^2 \omega^s + \partial_y^2 \omega)\partial_x \omega - 2(\partial_y \omega^s + \partial_y \omega)\partial_x \partial_y \omega.$$

- *The order of derivative in* x *on the RHS is the same as* g_1*, an extra* m *in front of* g_m *can be added to its Gevrey norm.*

Sketch of Proof

Define three auxilliary functions f_m, h_m and g_m:

$$f_m = \chi_1 \partial_x^m \omega - \chi_1 \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega_\varepsilon} \partial_x^m u = \chi_1 (\omega^s + \omega) \partial_y \left(\frac{\partial_x^m u}{\omega^s + \omega} \right),$$

$$h_m = \chi_2 \partial_x^m \partial_y \omega - \chi_2 \frac{\partial_y^2 \omega^s + \partial_y^2 \omega}{\partial_y \omega^s + \partial_y \omega} \partial_x^m \omega,$$

and

$$g_m = \partial_x^{m-1} \Big((\omega^s + \omega_\varepsilon) \partial_x \omega - (\partial_y \omega^s + \partial_y \omega) \partial_x u \Big).$$

(Define an equivalent norm)

$$
\begin{aligned}
|u|_{\rho,\sigma} = \|u\|_{\rho,\sigma} + \sup_{1\leq m\leq 5} \Big(& m\,\|g_m\|_{L^2} + \big\|< y >^{\ell} f_m\big\|_{L^2} + \|h_m\|_{L^2} \\
& + \sup_{1\leq m\leq 5} \|\chi_2 \partial_y \partial_x^m \omega\|_{L^2} \Big) \\
& + \sup_{m\geq 6} \frac{\rho^{m-5}}{\big[(m-6)!\big]^{\sigma}} \Big(m\,\|g_m\|_{L^2} + \big\|< y >^{\ell} f_m\big\|_{L^2} + \|h_m\|_{L^2} \Big) \\
& + \sup_{m\geq 6} \frac{\rho^{m-5}}{\big[(m-6)!\big]^{\sigma}} \|\chi_2 \partial_y \partial_x^m \omega\|_{L^2} .
\end{aligned}
$$

(Uniform estimate in Gevrey space)

$$|u(t)|^2_{\rho,\sigma} \lesssim |u_0|^2_{\rho,\sigma} + \int_0^t (|u|^2_{\rho,\sigma} + |u|^4_{\rho,\sigma})\, ds + \int_0^t \frac{|u|^2_{\tilde{\rho}(s),\sigma}}{\tilde{\rho}(s) - \rho}\, ds.$$

Define

$$\|u\|_{(\lambda,T)} \overset{def}{=} \sup_{\rho,t} \left(\frac{\rho_0 - \rho - \lambda t}{\rho_0 - \rho}\right)^{1/2} |u(t)|_{\rho,\sigma}\,,\ \tilde{\rho}(s) = \frac{\rho_0 + \rho - \lambda s}{2},$$

where the supremum is over $\rho > 0$, $0 \le t \le T$ and $\rho + \lambda t < \rho_0$.
For small $\|u_0\|_{2\rho_0,\sigma}$, there exists R and λ such that

$$|||u|||_{(\lambda,\frac{\rho_0}{4\lambda})} \le R.$$

(2D incompressible MHD)

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \frac{1}{\nu_0}(\mathbf{H}\cdot\nabla)\mathbf{H} - \nabla\left(\frac{|\mathbf{H}|^2}{2\nu_0}\right) + \frac{1}{Re}\triangle \mathbf{u}, \\ \partial_t \mathbf{H} - \nabla\times(\mathbf{u}\times\mathbf{H}) = \frac{1}{Re_m}\triangle \mathbf{H}, \qquad in \quad \Omega, \\ \nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{H} = 0, \\ \mathbf{u}|_\Gamma = 0, \quad \mathbf{H}|_\Gamma = \text{ Perfect conducting.} \end{cases}$$

$\mathbf{u} = (u_1, u_2) : velocity,\ \mathbf{H} = (h_1, h_2) : magnetic\ filed,\ p : pressure,$

$\nu_0 : permeability,\ R_e = \nu^{-1} : Reynolds\ number,\ \nu : viscosity,$

$Re_m = \nu_0\sigma : magnetic\ Reynolds\ number,\ \sigma : electrical\ conductivity,$

$magnetic\ Prandtl\ number :\ Pr_m := \frac{Re_m}{Re} = \nu\nu_0\sigma.$

Case 1. $Pr_m \ll 1$ or $\sigma \ll Re$

The velocity has characteristic boundary layer with thickness $\frac{1}{\sqrt{Re}}$ with leading order of the magnetic field unchanged.

Denote $\mathbf{H} \cdot \vec{n}|_\Gamma = B$. There is an extra term $\sigma B^2(U - u_1)$ in the equation for u_1 and it is similar to the classical Prandtl equations.

Case 2. $Pr_m = O(1)$ or $\sigma = O(Re)$

The boundary layer for both the velocity and magnetic fields.

- $\mathbf{H}\cdot\vec{n}|_\Gamma = B \neq 0$: The non-characteristic boundary layer $(u_1, \frac{1}{Re}u_2, h_1, B+\frac{1}{Re}h_2)(x, Re\ y)$ is called Hartmann layer:

$$\begin{cases} \partial_Y^2 u_1 + \frac{B}{\mu}\partial_Y h_1 = 0, \qquad -B\partial_Y u_1 = \frac{1}{Pr_m}\partial_Y^2 h_1, \\ \partial_x u_1 + \partial_Y u_2 = 0, \quad \partial_x h_1 + \partial_Y h_2 = 0, \quad \lim\limits_{Y\to+\infty} u_1 = U. \end{cases}$$

$$\Longrightarrow \quad \partial_Y^2 u_1 - \mu^{-1} Pr_m B^2 (u_1 - U) = 0, \quad \partial_Y h_1 = Pr_m B(U - u_1).$$

$$\Longrightarrow \quad u_1 = U\left[1 - exp\{-\sqrt{\mu^{-1}Pr_m B^2}\ Y\}\right]. \quad \textit{Hartmann Layer}$$

- $\mathbf{H}\cdot\vec{n}|_{\Gamma}=0$: Inside the boundary layer,
  friction force $\sim$ inertia force $\sim$ Lorentz force.
  The velocity and magnetic field have characteristic boundary layer profile $(u_1, \frac{1}{\sqrt{Re}}u_2, h_1, \frac{1}{\sqrt{Re}}h_2)(t,x,\sqrt{Re}\,y)$.

  Note that this MHD boundary layer system behaves very differently from the classical Prandtl system.

Case 3. $Pr_m \gg 1$ or $\sigma \gg Re$

The magnetic field has characteristic boundary layer with the boundary layer thickness $\frac{1}{\sqrt{Re_m}}$, while the leading order of the velocity field in the layer remains unchanged. The magnetic boundary layer profile $(h_1, B + \frac{1}{\sqrt{Re_m}} h_2)(t, x, \sqrt{Re_m}\, y)$ satisfies

$$\partial_t h_1 = \partial_Y^2 h_1 + \mu^{-1} B^2 Re\, (H - h_1) + H_t, \qquad \partial_x h_1 + \partial_Y h_2 = 0,$$

where $H := \lim_{Y \to +\infty} h_1$.

(Derivation of the boundary layer system in Case 2)

$$\begin{cases} \partial_t \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon - (\mathbf{H}^\varepsilon \cdot \nabla)\mathbf{H}^\varepsilon + \nabla\big(p^\varepsilon + \frac{|\boldsymbol{H}^\varepsilon|^2}{2}\big) = \varepsilon \triangle \mathbf{u}^\varepsilon, \\ \partial_t \mathbf{H}^\varepsilon - \nabla \times (\mathbf{u}^\varepsilon \times \mathbf{H}^\varepsilon) = \kappa\varepsilon \triangle \mathbf{H}^\varepsilon, \qquad in\ \{t, y > 0, x \in \mathbb{T}\}, \\ \nabla \cdot \mathbf{u}^\varepsilon = 0, \quad \nabla \cdot \mathbf{H}^\varepsilon = 0, \\ (\mathbf{u}^\varepsilon, \mathbf{H}^\varepsilon \cdot n, (\nabla \times \mathbf{H}^\varepsilon) \times n)|_{y=0} = 0. \quad \textit{(perfect conducting)} \end{cases}$$

Near the boundary as $\varepsilon \to 0$:

$$(\boldsymbol{u}^\varepsilon, \boldsymbol{H}^\varepsilon, p^\varepsilon)(t,x,y) \sim \big(u_1, \sqrt{\varepsilon}u_2, h_1, \sqrt{\varepsilon}h_2, p\big)(t, x, \frac{y}{\sqrt{\varepsilon}}).$$

MHD boundary layer system

$$
(\textbf{\textit{Main}})\begin{cases}
\partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 - h_1 \partial_x h_1 - h_2 \partial_y h_1 = \partial_y^2 u_1 - P_x, \\
\partial_t h_1 + \partial_y (u_2 h_1 - u_1 h_2) = \kappa \partial_y^2 h_1, \\
\partial_x u_1 + \partial_y u_2 = 0, \quad \partial_x h_1 + \partial_y h_2 = 0, \\
u_1|_{t=0} = u_{10}(x,y), \quad h_1|_{t=0} = h_{10}(x,y), \\
(u_1, u_2, \partial_y h_1, h_2)|_{y=0} = 0, \\
\lim_{y \to +\infty} (u_1, h_1) = (u_1^e, h_1^e)(t,x,0) := (U,H)(t,x).
\end{cases}
$$

Theories in analytic and Gevrey function spaces hold because of the same singularity and degeneracy as Prandtl equations.

(Observation)

Motivatd by von Mise and Crocco transforms

The stream function of the magnetic field

$$\psi(t,x,y): \; h_1 = \partial_y \psi, \; h_2 = -\partial_x \psi, \; \psi|_{y=0} = 0,$$

satisfies

$$\partial_t \psi + u_1 \partial_x \psi + u_2 \partial_y \psi = \kappa \partial_y^2 \psi.$$

If $h_1 \neq 0$, then ψ is monotone in y.

(Coordinate transformation)

$$\tau = t,\ \xi = x,\ \eta = \psi(t,x,y),$$

(Symmetric quasilinear system)

$$\begin{cases} \partial_\tau u_1 + u_1 \partial_\xi u_1 - h_1 \partial_\xi h_1 + (\kappa - 1) h_1 \partial_\eta h_1 \partial_\eta u_1 = h_1^2 \partial_\eta^2 u_1, \\ \partial_\tau h_1 - h_1 \partial_\xi u_1 + u_1 \partial_\xi h_1 = \kappa h_1^2 \partial_\eta^2 h_1, \\ (u_1, h_1 \partial_\eta h_1)|_{y=0} = 0, \quad \lim_{\eta \to +\infty} (u_1, h_1) = (U, H). \end{cases}$$

(Cancellation)

$$u_1^m := \partial_x^m u_1 - \frac{\partial_y u_1}{h_1}\partial_x^m \psi,\; h_1^m := \partial_x^m h_1 - \frac{\partial_y h_1}{h_1}\partial_x^m \psi.$$

(Symmetry system for (u_1^m, h_1^m))

$$\begin{cases} \partial_t u_1^m + (u_1\partial_x + u_2\partial_y)u_1^m - (h_1\partial_x + h_2\partial_y)h_1^m = \partial_y^2 u_1^m + \ldots, \\ \partial_t h_1^m + (u_1\partial_x + u_2\partial_y)h_1^m - (h_1\partial_x + h_2\partial_y)u_1^m = k\partial_y^2 h_1^m + \ldots \end{cases}$$

Theorem (Liu-Xie-Y., CPAM, 2017)

Let $m \geq 5, l \geq 0$*, and assume*

$$\Big(u_{10}(x,y) - U(0,x), h_{10}(x,y) - H(0,x)\Big) \in H_l^{3m+2}(\Omega),\ h_{10}(x,y) \geq 2\delta_0,$$

and the compatibility conditions up to m*-th order. Here,* $\|f\|_{H_l^m(\Omega)} = \big(\sum_{m_1+m_2\leq m} \|\langle y\rangle^{l+m_2}\partial_x^{m_1}\partial_y^{m_2} f\|_{L^2(\Omega)}^2\big)^{\frac{1}{2}}$*. Then, there exist a time* $0 < T \leq T_0$*, and a unique solution* (u_1, u_2, h_1, h_2) *to* ***(Main)****, such that* $h_1 \geq \delta_0$*,*

$$(u_1 - U, h_1 - H) \in \bigcap_{i=0}^{m} W^{i,\infty}\Big(0, T; H_l^{m-i}(\Omega)\Big).$$

Justification of the high Reynolds numbers limit

Consider

$$\begin{cases} \partial_t \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon - (\mathbf{H}^\varepsilon \cdot \nabla)\mathbf{H}^\varepsilon + \nabla p^\varepsilon = \mu\varepsilon \triangle \mathbf{u}^\varepsilon, \quad (x,y) \in \mathbb{R}^2_+, \\ \partial_t \mathbf{H}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{H}^\varepsilon - (\mathbf{H}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon = \kappa\varepsilon \triangle \mathbf{H}^\varepsilon, \\ \nabla \cdot \mathbf{u}^\varepsilon = 0, \quad \nabla \cdot \mathbf{H}^\varepsilon = 0, \\ \mathbf{u}^\varepsilon|_{y=0} = \mathbf{0}, \qquad \partial_y h_1^\varepsilon|_{y=0} = 0, \quad h_2^\varepsilon|_{y=0} = 0, \\ (\mathbf{u}^\varepsilon, \mathbf{H}^\varepsilon)|_{t=0} = (\mathbf{u}_0, \mathbf{h}_0)(x,y), \end{cases}$$

with no-slip boundary and perfect conducting boundary conditions.

Assume $\mathbf{u}^e = (u_1^e, u_2^e), \mathbf{H}^e = (h_1^e, h_2^e)$ and p^e solve ideal MHD equations with the same initial data and corresponding BC:

$$
\begin{cases}
\partial_t \mathbf{u}^e + (\mathbf{u}^e \cdot \nabla)\mathbf{u}^e - (\mathbf{H}^e \cdot \nabla)\mathbf{H}^e + \nabla p^e = 0, \\
\partial_t \mathbf{H}^e + (\mathbf{u}^e \cdot \nabla)\mathbf{H}^e - (\mathbf{H}^e \cdot \nabla)\mathbf{u}^e = 0, \\
\nabla \cdot \mathbf{u}^e = 0, \quad \nabla \cdot \mathbf{H}^e = 0, \\
(u_2^e, h_2^e)|_{y=0} = \mathbf{0}, \quad (\mathbf{u}^e, \mathbf{H}^e)|_{t=0} = (\mathbf{u}_0, \mathbf{h}_0)(x,y).
\end{cases}
$$

The boundary layer profile $(u_1^b, u_2^b, h_1^b, h_2^b)(t,x,Y)$ is given by

$$\begin{cases} (u_1^b, h_1^b)(t,x,Y) := (u_1^p, h_1^p)(t,x,Y) - (u_1^e, h_1^e)(t,x,0), \\ u_2^b(t,x,Y) := \int_Y^\infty \partial_x u_1^b(t,x,z)dz, \quad h_2^b(t,x,Y) := \int_Y^\infty \partial_x h_1^b(t,x,z)dz, \end{cases}$$

where $(u_1^p, h_1^p)(t,x,Y)$ solves

$$\begin{cases} \partial_t u_1^p + (u_1^p \partial_x + u_2^p \partial_Y) u_1^p - (h_1^p \partial_x + h_2^p \partial_Y) h_1^p = \mu \partial_Y^2 u_1^p - \partial_x p^e(t,x,0), \\ \partial_t h_1^p + (u_1^p \partial_x + u_2^p \partial_Y) h_1^p - (h_1^p \partial_x + h_2^p \partial_Y) u_1^p = \kappa \partial_Y^2 h_1^p, \\ \partial_x u_1^p + \partial_Y u_2^p = 0, \quad \partial_x h_1^p + \partial_Y h_2^p = 0, \\ (u_1^p, u_2^p, \partial_Y h_1^p, h_2^p)|_{Y=0} = \mathbf{0}, \quad \lim_{Y\to+\infty} (u_1^p, h_1^p)(t,x,Y) = (u_1^e, h_1^e)(t,x,0), \\ (u_1^p, h_1^p)|_{t=0} = (u_1^e, h_1^e)(0,x,0); \end{cases}$$

(Theorem Liu-Xie-Y., '17)

Let the initial data $(\mathbf{u}_0, \mathbf{h}_0)(x,y)$ *be compatible and satisfy*

$$h_{10}(x,0) \geq \delta_0 \quad \text{for some constant} \quad \delta_0 > 0.$$

There is a time $T_* > 0$ *independent of* ε*, such that*

$$\|(\mathbf{u}^\varepsilon, \mathbf{H}^\varepsilon)(t,x,y) - (\mathbf{u}^e, \mathbf{H}^e)(t,x,y) - \big(u_1^b, \sqrt{\varepsilon}u_2^b, h_1^b, \sqrt{\varepsilon}h_2^b\big)(t,x,\frac{y}{\sqrt{\varepsilon}})\|_{L^\infty_{xy}} \sim \varepsilon^{\frac{1}{2}} \to 0, \quad \varepsilon \to 0.$$

$$\begin{cases} (\mathbf{u}^\varepsilon, \mathbf{H}^\varepsilon)(t,x,y) & = (\mathbf{u}^a, \mathbf{H}^a)(t,x,y) + \varepsilon(\mathbf{u}, \mathbf{h})(t,x,y), \\ p^\varepsilon(t,x,y) & = p^a(t,x,y) + \varepsilon p(t,x,y), \end{cases}$$

Then, the remainder $(\mathbf{u}, \mathbf{h})(t,x,y)$ satisfies

$$\begin{cases} \partial_t u_1 + (\mathbf{u}^\varepsilon \cdot \nabla) u_1 - (\mathbf{H}^\varepsilon \cdot \nabla) h_1 + (\mathbf{u} \cdot \nabla) u_1^a - (\mathbf{h} \cdot \nabla) h_1^a + \partial_x p - \mu\varepsilon \triangle u_1 = r_1^\varepsilon, \\ \partial_t u_2 + (\mathbf{u}^\varepsilon \cdot \nabla) u_2 - (\mathbf{H}^\varepsilon \cdot \nabla) h_2 + (\mathbf{u} \cdot \nabla) u_2^a - (\mathbf{h} \cdot \nabla) h_2^a + \partial_y p - \mu\varepsilon \triangle u_2 = r_2^\varepsilon, \\ \partial_t h_1 + (\mathbf{u}^\varepsilon \cdot \nabla) h_1 - (\mathbf{H}^\varepsilon \cdot \nabla) u_1 + (\mathbf{u} \cdot \nabla) h_1^a - (\mathbf{h} \cdot \nabla) u_1^a - \kappa\varepsilon \triangle h_1 = r_3^\varepsilon, \\ \partial_t h_2 + (\mathbf{u}^\varepsilon \cdot \nabla) h_2 - (\mathbf{H}^\varepsilon \cdot \nabla) u_2 + (\mathbf{u} \cdot \nabla) h_2^a - (\mathbf{h} \cdot \nabla) u_2^a - \kappa\varepsilon \triangle h_2 = r_4^\varepsilon, \\ \nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{h} = 0, \\ (u_1, u_2, \partial_y h_1, h_2)|_{y=0} = \mathbf{0}, \quad (\mathbf{u}, \mathbf{h})|_{t=0} = \mathbf{0}. \end{cases}$$

Here, r_i^ε, $i = 1 \sim 4$ are the error terms determined by the approximate solution. By a careful construction of the approximate solution $(\mathbf{u}^a, \mathbf{H}^a)$, we can have

$$\|\partial_{tx}^\alpha r_i^\varepsilon(t,\cdot)\|_{L^2} \leq C, \quad |\alpha| \leq 3,\ i = 1, \cdots, 4$$

for some positive constant C independent of ε.

(Difficulty)

The following four terms in the first and the third equations can not be estimated directly:

$$\begin{cases} u_2\partial_y u_1^a - h_2\partial_y h_1^a = \varepsilon^{-\frac{1}{2}}\left(u_2\partial_Y u_1^b - h_2\partial_Y h_1^b\right) + O(1), \\ u_2\partial_y h_1^a - h_2\partial_y u_1^a = \varepsilon^{-\frac{1}{2}}\left(u_2\partial_Y h_1^b - h_2\partial_Y u_1^b\right) + O(1), \end{cases}$$

Denote the stream function ψ of the magnetic field:

$$h_1 = \partial_y \psi, \quad h_2 = -\partial_x \psi, \quad \psi|_{y=0}, \quad \psi|_{t=0} = 0.$$

Note that ψ satisfies

$$\partial_t \psi + (\mathbf{u}^\varepsilon \cdot \nabla)\psi - h_2^a u_1 + h_1^a u_2 - \kappa \triangle \psi = \partial_y^{-1} r_3^\varepsilon \triangleq r_5^\varepsilon.$$

Set

$$\eta_0^p(t,x,y) \ := \ \frac{u_1^p(t,x,\frac{y}{\sqrt{\varepsilon}})}{h_1^p(t,x,\frac{y}{\sqrt{\varepsilon}})}, \quad \eta_1^p(t,x,y) := \frac{\partial_y u_1^p(t,x,\frac{y}{\sqrt{\varepsilon}})}{h_1^p(t,x,\frac{y}{\sqrt{\varepsilon}})},$$

$$\eta_2^p(t,x,y) \ := \ \frac{\partial_y h_1^p(t,x,\frac{y}{\sqrt{\varepsilon}})}{h_1^p(t,x,\frac{y}{\sqrt{\varepsilon}})}.$$

Note $\eta_0^p(t,x,Y), \sqrt{\varepsilon}\eta_i^p(t,x,Y), \ \sqrt{\varepsilon}Y\eta_i^p(t,x,Y) = O(1), \ i=1,2$ uniformly in ε so that

$$\eta_0^p(t,x,y), \ \sqrt{\varepsilon}\partial_y\eta_0^p(t,x,y), \ y\partial_y\eta_0^p(t,x,y),$$
$$\sqrt{\varepsilon}\eta_i^p(t,x,y), \ y\eta_i^p(t,x,y) \ = \ O(1), \quad i=1,2.$$

(Cancellation)

$$\begin{cases} u(t,x,y) := u_1(t,x,y) - \partial_y(\eta_0^p \cdot \psi)(t,x,y), \\ v(t,x,y) := u_2(t,x,y) + \partial_x(\eta_0^p \cdot \psi)(t,x,y), \\ h(t,x,y) := h_1(t,x,y) - (\eta_2^p \cdot \psi)(t,x,y), \quad g(t,x,y) := h_2(t,x,y). \end{cases}$$

Then $U(t,x,y) := (u,v,h,g)^T(t,x,y)$ satisfies

$$\begin{cases} \partial_t U + A_1(U)\partial_x U + A_2(U)\partial_y U + C(U)U + \psi D + \\ \qquad (p_x, p_y, 0, 0)^T - \varepsilon B \triangle U = E^\varepsilon, \\ \partial_x u + \partial_y v = 0, \\ (u, v, \partial_y h, g)|_{y=0} = \mathbf{0}, \quad U|_{t=0} = \mathbf{0}. \end{cases}$$

Estimates on the coefficients:

$$A_i(U) \;=\; A_i^a + \sqrt{\varepsilon}A_i^p + \varepsilon\tilde{A}_i(U), \quad i = 1,2,$$

$$C(U) \;=\; C^a + \varepsilon\tilde{C}(U), \quad D \;=\; D^a + \varepsilon\psi D^p,$$

For $|\alpha| \leq 2,\ i = 1,2,$

$$\|\partial_{tx}^\alpha A_i^a(t,\cdot)\|_{L^\infty}, \|\partial_{tx}^\alpha A_i^p(t,\cdot)\|_{L^\infty}, \|y^2\partial_{tx}^\alpha D^p(t,\cdot)\|_{L^\infty}, \|\partial_{tx}^\alpha B(t,\cdot)\|_{L^\infty} \leq C,$$

$$\|\partial_{tx}^\alpha C^a(t,\cdot)\|_{L^\infty} + \|y\partial_{tx}^\alpha D^a(t,\cdot)\|_{L^\infty} \;\leq\; C,$$

$$\|\partial_{tx}^\alpha E^\varepsilon(t,\cdot)\|_{L^2} \leq \; C.$$

Global existence of smooth solutions?

(Classical Prandtl equations)

- *Zhang-Zhang ('14): lower bound of life span* $\varepsilon^{-\frac{4}{3}}$ *for outflow velocity of the order* $\varepsilon^{\frac{5}{3}}$ *and perturbation* ε*;*
- *Ignatova-Vicol ('16): almost global with lower bound of the life span* $e^{(\varepsilon \log(\varepsilon^{-1}))^{-1}}$ *for* ε *order perturbation of shear flow in the form Guassian error function.*

Observation: For shear flow as Guassian error function, a damping $\frac{g}{<t>}$ can be obtained by using cancellation.

MHD boundary layer

(Notations)

$$\theta_\alpha(t,y) = \exp\left(\frac{\alpha z^2}{4}\right), \quad z = \frac{y}{\sqrt{<t>}}, \quad M_m = \frac{\sqrt{m+1}}{m!}.$$

$$X_m(f,\tau) = \|\theta_\alpha \partial_x^m f\|_{L^2} \tau^m M_m, \quad Y_m(f,\tau) = \|\theta_\alpha \partial_x^m f\|_{L^2} \tau^{m-1} m M_m.$$

$$\|f\|_{X_{\tau,\alpha}} = \sum_{m\geq 0} X_m(f,\tau), \quad \|f\|_{D_{\tau,\alpha}} = \|\partial_y f\|_{X_{\tau,\alpha}}, \quad \|f\|_{Y_{\tau,\alpha}} = \sum_{m\geq 1} Y_m(f,\tau).$$

A family of solutions

$$(\bar{u}\phi(t,y),0,\bar{b},0), \quad \phi(t,y)=\frac{1}{\sqrt{\pi}}\int_0^{y/\sqrt{\langle t\rangle}}\exp(-\frac{z^2}{4})dz.$$

Theorem (Xie-Y., '18) Suppose

$$\|u_0-\bar{u}\phi(0,y)\|_{X_{\tau_0,1/2}}, \|b_0-\bar{b}\|_{X_{\tau_0,1/2}} \le \varepsilon, \quad \varepsilon << 1.$$

Then there exists a unique solution to the MHD prandtl equations satisfying

$$(u_1-\bar{u}\phi(t,y), b_1-\bar{b}) \in X_{\tau,1/2}, \quad 0\le t\le T_\varepsilon, \quad T_\varepsilon \ge C\varepsilon^{-2+}.$$

(Idea of the proof)

Denote the perturbation by (u, v, b, g). *Motivated by the cancellation and Ignatova-Vicol's work, set*

$$\tilde{u} = u - \frac{\bar{u}}{\bar{b}} \partial_y \phi \psi, \qquad \tilde{b} = b.$$

$$\begin{cases} \partial_t \tilde{u} - \partial_y^2 \tilde{u} + (\bar{u}\phi + u)\partial_x \tilde{u} + v\partial_y \tilde{u} \\ \qquad -(\bar{b} + b)\partial_x \tilde{b} - g\partial_y \tilde{b} - 2\partial_y^2 \phi \tilde{b} + v\partial_y^2 \phi \psi = 0, \\ \partial_t \tilde{b} - \partial_y^2 \tilde{b} - (\bar{b} + b)\partial_x \tilde{u} - g\partial_y \tilde{u} + (\bar{u}\phi + u)\partial_x \tilde{b} + v\partial_y \tilde{b} - g\partial_y^2 \phi \psi = 0. \end{cases}$$

(Key estimates)

$$\frac{d}{dt}(\|\tilde{u}\|_{X_{\tau,\alpha}}+K\|\tilde{b}\|_{X_{\tau,\alpha}})+\sum_{m\geq 0}\tau^m M_m\Big(\frac{\|\theta_\alpha\partial_x^m\partial_y\tilde{u}\|_{L^2}^2}{\|\theta_\alpha\partial_x^m\tilde{u}\|_{L^2}}+K\frac{\|\theta_\alpha\partial_x^m\partial_y\tilde{b}\|_{L^2}^2}{\|\theta_\alpha\partial_x^m\tilde{b}\|_{L^2}}\Big)$$

$$-\frac{\alpha}{2\langle t\rangle}\|\tilde{u}\|_{X_{\tau,\alpha}}-(2+\frac{K\alpha}{2})\frac{1}{\langle t\rangle}\|\tilde{b}\|_{X_{\tau,\alpha}}$$

$$\leq\left(\dot{\tau}(t)+\frac{C_0(K+1)}{(\tau(t))^{1/2}}\left(\langle t\rangle^{-1/4}(\|\tilde{u}\|_{X_{\tau,\alpha}}+\|\tilde{b}\|_{X_{\tau,\alpha}})+\right.\right.$$

$$\left.\left.\langle t\rangle^{1/4}(\|\tilde{u}\|_{D_{\tau,\alpha}}+\|\tilde{b}\|_{D_{\tau,\alpha}})\right)\right)\times(\|\tilde{u}\|_{Y_{\tau,\alpha}}+K\|\tilde{b}\|_{Y_{\tau,\alpha}}).$$

(A simplied system of equations)

$$\begin{cases} \frac{d}{dt}X(t)+\frac{1}{4<t>}X(t)+\frac{\delta}{\sqrt{<t>}}D(t)\leq 0, \\ \tau(t)^{\frac{3}{2}}=\tau^{\frac{3}{2}}(0)-C\int_0^t(<s>^{-\frac{1}{4}}X(s)+<s>^{\frac{1}{4}}D(s))ds. \end{cases}$$

$$X(t)\lesssim <t>^{-\frac{1}{4}}\varepsilon,\quad \int_0^t <s>^{-\frac{1}{4}}D(s)ds\leq C\varepsilon.$$

$$\tau^{\frac{3}{2}}(t)\geq \tau^{\frac{3}{2}}(0)-C<t>^{\frac{1}{2}}\varepsilon.$$

- Stabilizing effect of the magnetic field on life span?

(Boundary layer in the Prandtl-Hartmann regime)

Derived by Gerard-Varet & Prestipino

$$\begin{cases} \partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 = \partial_y b_1 + \partial_y^2 u_1, \\ \partial_y u_1 + \partial_y^2 b_1 = 0, \\ \partial_x u_1 + \partial_y u_2 = 0, \end{cases}$$

admits the classical Hartmann layer solution

$$u_1 = (1 - e^{-y})\bar{u}, \qquad u_2 = 0.$$

Denote the perturbation by (u,v), note that $\omega = u_y$ satisfies

$$\partial_t \omega + (\bar{u}(1-e^{-y}) + u)\partial_x \omega - v e^{-y} + v\partial_y \omega = -\omega + \partial_y^2 \omega.$$

Corresponding to the cancellation f_1 for Prandtl equations by noting $\frac{u_{1yy}}{u_{1y}} = -1$, set

$$g = u + \omega$$

to have

$$\partial_t g + (\bar{u}(1-e^{-y}) + u)\partial_x g + v\partial_y g = -g + \partial_y^2 g.$$

(Theorem Xie-Y., '18)

Assume $\|\partial_y u_{10} + u_{10} - \bar{u}\|_{X^{r,\beta}_{\tau_0,\alpha}} \leq \delta_0 << 1$, *with* $0 < \alpha < \sqrt{2}/2$ *and* $1 \leq \beta < \min\{(2r+1)/3, 2r-1\}, r > 1$. *Then there exists a unique global solution* (u_1, u_2) *satisfying*

$$\|g\|^2_{X^{r,\beta}_{\tau(t),\alpha}} < c(t), \quad \tau(t) > \frac{\tau_0}{2}.$$

Here, $M_m = \frac{(m+1)^r}{(m!)^\beta}$ *with* $r > 1$ *and* $\beta \geq 1$,

$$X_m = \|e^{\alpha y}\partial_x^m g\|_{L^2}\tau^m M_m, \quad \|g\|^2_{X^{r,\beta}_{\tau,\alpha}} = \sum_{m\geq 0} X_m^2.$$

THANK YOU!