ABOUT THE SPLITTING ALGORITHM FOR BOLTZMANN AND B.G.K. EQUATIONS

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Abstract

We prove the convergence of splitting algorithms for Boltzmann and B.G.K. equations. The proof in the case of the Boltzmann equation is made in the framework of renormalized solutions.

1 Introduction

A rarefied gas is usually described by the Boltzmann equation (Cf. [Ce], [Ch, Co], [Tr, Mu]). In this model, the dynamics of the gas is given by the nonnegative density $f(t,x,v)$ of particles which at time $t \in [0,T]$ and point $x \in I\!R^3$, move with velocity $v \in I\!R^3$, where T is a stricly positive number.

Such a density satisfies the Boltzmann equation,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f), \tag{1.1}$$

$$f(0,x,v) = f_0(x,v), \tag{1.2}$$

where Q is a quadratic collision kernel acting only on velocities and defined (with the notations of [DP, L]) by

$$Q(f) = Q^+(f) - Q^-(f), \tag{1.3}$$

$$Q^+(f)(v) = \int_{v_* \in I\!R^3} \int_{\omega \in S^2} f(v')\, f(v'_*)\, B(v - v_*, \omega)\, d\omega dv_*, \tag{1.4}$$

$$A(z) = \int_{\omega \in S^2} B(z,\omega)\, d\omega, \tag{1.5}$$

$$L(f) = A *_v f, \tag{1.6}$$

$$Q^-(f)(v) = f(v)\, L(f)(v). \tag{1.7}$$

In formula (1.4), the post–collisional velocities v' and v'_* are parametrized by

$$v' = v + ((v_* - v) \cdot \omega)\, \omega, \tag{1.8}$$

$$v'_* = v_* - ((v_* - v) \cdot \omega)\, \omega, \tag{1.9}$$

where ω is a unit vector varying in the sphere S^2.

Finally, the nonnegative cross section B is assumed to satisfy the following properties, first introduced in [DP, L]:

Assumption 1: *The function $B(z,\omega)$ belongs to $L^1_{loc}(I\!R^3 \times S^2)$ and depends only on $|z|$ and $|z \cdot \omega|$.*

Moreover, the function A satisfies for all $R > 0$,

$$(1 + |z|^2)^{-1} \int_{v \in B_R} A(z+v)\, dv \underset{|z| \to \infty}{\longrightarrow} 0, \tag{1.10}$$

where B_R (or B_v^R) is the set $\{v \in I\!R^3; |v| < R\}$.

Finally, we assume that the nonnegative initial datum f_0 satisfies the following physically relevant assumption:

Assumption 2: *The function f_0 is such that*

$$\int_{x\in\mathbb{R}^3}\int_{v\in\mathbb{R}^3} f_0(x,v)\,\{1+|x|^2+|v|^2+|\log f_0(x,v)|\,\}\,dvdx < +\infty. \quad (1.11)$$

R.J. DiPerna and P-L. Lions proved in [DP, L] that under assumptions 1 and 2, the Boltzmann equation (1.1) – (1.9) admits a nonnegative renormalized solution in $C([0,T],L^1(\mathbb{R}^3\times\mathbb{R}^3))$.

The proof uses the averaging lemmas introduced by F. Golse, B. Perthame and R. Sentis in [G, P, S], and developed by F. Golse, P-L. Lions, B. Perthame and R. Sentis in [G, L, P, S] and by R.J. DiPerna, P-L. Lions and Y. Meyer in [DP, L, M].

Note that a new and simpler proof was given by P-L. Lions in [L 1].

We shall also consider in the sequel a simpler model of rarefied gases, namely the B.G.K. model, first introduced in [Bh, Gr, Kr].

The gas is still described by a nonnegative density $f(t,x,v)$, but the equation satisfied by f now becomes

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = M[f] - f, \quad (1.12)$$

$$f(0,x,v) = f_0(x,v), \quad (1.13)$$

where $M[f](t,x,v)$ is a Maxwellian function of v:

$$M[f](t,x,v) = \frac{\rho(t,x)}{(2\pi\,\mathcal{T}(t,x))^{3/2}}\exp\Big\{-\frac{|v-u(t,x)|^2}{2\mathcal{T}(t,x)}\Big\}, \quad (1.14)$$

and ρ, u, $\mathcal{T}$ are the respective density, global velocity and temperature of the gas. More precisely,

$$\rho(t,x) = \int_{v\in\mathbb{R}^3} f(t,x,v)\,dv, \quad (1.15)$$

$$\rho(t,x)\,u(t,x) = \int_{v\in\mathbb{R}^3} v\,f(t,x,v)\,dv, \quad (1.16)$$

$$\rho(t,x)\,\{|u(t,x)|^2 + 3\,\mathcal{T}(t,x)\} = \int_{v\in\mathbb{R}^3} |v|^2\,f(t,x,v)\,dv. \quad (1.17)$$

Note that the previous quantities are not well–defined when $\rho = 0$, therefore we define $M[0] = 0$.

The existence of a global nonnegative solution for the B.G.K. system (1.12) – (1.17) under assumption 2 on the initial datum was proved by B. Perthame in [Pe]. The proof was based on a dispersion lemma. Another proof was given by E. Ringeissen in [Ri], allowing to take into account a gas in a bounded domain with boundary conditions.

Equation (1.1) – (1.9) as well as (1.12) – (1.17) can be written in the form

$$\frac{\partial f}{\partial t} = \mathcal{A}f + \mathcal{B}f, \tag{1.18}$$

$$f(t=0) = f_0, \tag{1.19}$$

where

$$\mathcal{A} = -v \cdot \nabla_x, \tag{1.20}$$

and $\mathcal{B}$ is a nonlinear operator acting only on the variable v.

Therefore, in order to compute numerically their solution, it is usual to solve equations

$$\frac{\partial f}{\partial t} = \mathcal{A}f \tag{1.21}$$

and

$$\frac{\partial f}{\partial t} = \mathcal{B}f \tag{1.22}$$

one after another and to apply Trotter's formula

$$e^{t(\mathcal{A}+\mathcal{B})} = \lim_{n\to+\infty} (e^{\frac{t}{n}\mathcal{A}} e^{\frac{t}{n}\mathcal{B}})^n. \tag{1.23}$$

This procedure is known as a splitting method for system (1.18), (1.19) and it is said to converge if Trotter's formula (1.23) holds when $\mathcal{A}$ and $\mathcal{B}$ are the operators introduced in (1.18). A large amount of splitting algorithms involving discretization in time can be found in [L, M].

We intend to prove that the splitting method converges for the Boltzmann and B.G.K. equations in the cases described earlier.

Note that this method is actually used in the numerical computation of both equations (Cf. [De, Pr]).

Note also that we proved in an earlier work the convergence of the splitting algorithm in the simpler cases of the "grey" radiative transfer equation and of Vlasov–Maxwell system (Cf. [De 1] and [De 2]). The proofs of existence of global solutions for the Boltzmann equation (Cf. [DP, L]) and for the B.G.K. model (Cf. [Pe]) were already known at that time, but it seemed difficult to prove the convergence of the splitting algorithm in the context of those works. Namely, the analysis of sub- and supersolutions in [DP, L] did not seem well–adapted to the splitting algorithm, and the dispersion lemma of [Pe] seemed also inoperant in this context.

However, the new proof of existence for the Boltzmann equation of [L 1], and the proof of existence for the B.G.K. model of [Ri] are better–adapted to the method of splitting and can therefore be followed, as will be seen in the sequel.

Therefore, in section 2, we prove the convergence of Trotter's formula for the Boltzmann equation, and the corresponding result for B.G.K. model in section 3.

2 Splitting for Boltzmann equation

2.1 Introduction and main result

In this section, we introduce the splitting algorithm for equation (1.1) – (1.9). We define

$$\mathcal{A}f = -v \cdot \nabla_x f, \tag{2.1.1}$$

$$\mathcal{B}f = Q(f), \tag{2.1.2}$$

and we intend to prove Trotter's formula (1.23) in this context.

Therefore, we define for every n in $I\!N_*$ and k in $[0, n-1]$ two sequences f_n^k and g_n^k by the following procedure:

we note

$$\Delta T = \frac{T}{n}, \qquad t_k = k\Delta T, \tag{2.1.3}$$

and the functions f_n^k and g_n^k are defined on $[t_k, t_{k+1}]$ by induction on k:

$$f_n^0(0) = f_0, \tag{2.1.4}$$

$$\frac{\partial f_n^k}{\partial t} = \mathcal{A}f_n^k, \tag{2.1.5}$$

$$f_n^k(t_k) = g_n^{k-1}(t_k) \text{ when } k > 0, \tag{2.1.6}$$

$$\frac{\partial g_n^k}{\partial t} = \mathcal{B}g_n^k, \tag{2.1.7}$$

$$g_n^k(t_k) = f_n^k(t_{k+1}). \tag{2.1.8}$$

This definition is meaningful because the solutions of equations (2.1.5), (2.1.6) and (2.1.7), (2.1.8) belong to $C([t_k, t_{k+1}], L^1(I\!R^3 \times I\!R^3))$.

Then, we define

$$f_n(t) = f_n^k(t), \tag{2.1.9}$$

$$g_n(t) = g_n^k(t), \tag{2.1.10}$$

for every t lying in $[t_k, t_{k+1}[$.

The functions f_n and g_n are therefore piecewise continuous with respect to the time variable on $[0, T]$ with values in $L^1(I\!R^3 \times I\!R^3)$, and their discontinuities appear at points t_k for each k in $[1, n]$.

The main result of this section is the following:

Theorem 1: *Under assumptions 1 and 2 on the cross section and initial datum, the sequences f_n and g_n defined in (2.1.4) – (2.1.10) converge up to extraction to the same nonnegative limit f in $L^\infty([0,T]; L^1(I\!R^3 \times I\!R^3))$ weak *, and this limit satisfies equation (1.1) – (1.9) in the sense of renormalized solutions. More precisely,*

$$\frac{Q^\pm(f)}{1+f} \in L^1_{loc}([0,T] \times I\!R^3 \times I\!R^3), \tag{2.1.11}$$

and

$$\{\frac{\partial}{\partial t} + v \cdot \nabla_x\} \log(1+f) = \frac{Q(f)}{1+f} \tag{2.1.12}$$

in the sense of distributions.

Remark: This property exactly means that Trotter's formula (1.23) holds for $\mathcal{A}$ and $\mathcal{B}$ defined in (2.1.1), (2.1.2).

The proof of this theorem is given in subsections 2.2 to 2.7.

2.2 Equation satisfied by f_n and g_n

For all nonnegative and smooth function β such that

$$\beta(0) = 0, \qquad |\beta'(s)| \le \frac{1}{1+s}, \tag{2.2.1}$$

we compute:

$$\frac{\partial \beta(f_n)}{\partial t} = \sum_{i=1}^{n} \frac{\partial \beta(f_n)}{\partial t} 1_{](i-1)\Delta T, i\Delta T[} + \sum_{i=1}^{n} \Big\{ \beta(f_n^i)(i\Delta T) - \beta(f_n^{i-1})(i\Delta T) \Big\} \delta_{i\Delta T}$$

$$= \sum_{i=1}^{n} -v \cdot \nabla_x \beta(f_n)\ 1_{](i-1)\Delta T, i\Delta T[}$$

$$+ \sum_{i=1}^{n} \left\{ \beta(g_n^{i-1})(i\Delta T) - \beta(g_n^{i-1})((i-1)\Delta T) \right\} \delta_{i\Delta T}$$

$$= -v \cdot \nabla_x \beta(f_n) + \sum_{i=1}^{n} \left(\int_{(i-1)\Delta T}^{i\Delta T} \frac{\partial \beta(g_n)}{\partial t}(s) ds \right) \delta_{i\Delta T}$$

$$= -v \cdot \nabla_x \beta(f_n) + \sum_{i=1}^{n} \left(\int_{(i-1)\Delta T}^{i\Delta T} \beta'(g_n)(s)\, Q(g_n)(s) ds \right) \delta_{i\Delta T}.$$

Therefore, we obtain the following equation for f_n and g_n:

$$\frac{\partial \beta(f_n)}{\partial t} + v \cdot \nabla_x \beta(f_n) = \sum_{i=1}^{n} \left(\int_{(i-1)\Delta T}^{i\Delta T} \beta'(g_n)(s)\, Q(g_n)(s) ds \right) \delta_{i\Delta T}. \quad (2.2.2)$$

In order to pass to the limit in equation (2.2.2), we need estimates for the sequences f_n and g_n.

2.3 Estimates on f_n and g_n

Lemma 1: *The sequences f_n and g_n defined in Theorem 1 are nonnegative and satisfy for some nonnegative constant C_T:*

$$\sup_{t\in[0,T]} \int_{x\in\mathbb{R}^3} \int_{v\in\mathbb{R}^3} f_n(t,x,v) \{1 + |x|^2 + |v|^2 + |\log f_n(t,x,v)|\}\, dxdv \le C_T, \quad (2.3.1)$$

$$\sup_{t\in[0,T]} \int_{x\in\mathbb{R}^3} \int_{v\in\mathbb{R}^3} g_n(t,x,v) \{1 + |x|^2 + |v|^2 + |\log g_n(t,x,v)|\}\, dxdv \le C_T. \quad (2.3.2)$$

Moreover, the quantity $e(g_n)$ defined by

$$e(g_n)(t,x,v) = \int_{v_*\in\mathbb{R}^3} \int_{\omega\in S^2} \{g_n(t,x,v'_*) g_n(t,x,v') - g_n(t,x,v_*) g_n(t,x,v)\}$$

$$\times \log \left\{ \frac{g_n(t,x,v'_*) g_n(t,x,v')}{g_n(t,x,v_*) g_n(t,x,v)} \right\} B(v-v_*,\omega)\, d\omega dv_* \quad (2.3.3)$$

satisfies the following estimate:

$$\int_0^T \int_{x\in\mathbb{R}^3} \int_{v\in\mathbb{R}^3} e(g_n)(t,x,v)\, dvdxdt \le C_T. \quad (2.3.4)$$

Proof: The total density and total energy are conserved in the two steps of the splitting algorithm, therefore

$$\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f_n(t)\,(1+|v|^2)\,dxdv = \int\int_{\mathbb{R}^3\times\mathbb{R}^3} g_n(t)\,(1+|v|^2)\,dxdv$$

$$= \int\int_{\mathbb{R}^3\times\mathbb{R}^3} f_0\,(1+|v|^2)\,dxdv. \tag{2.3.5}$$

During the first step we also get

$$\int\int_{\mathbb{R}^3\times\mathbb{R}^3} |x-vt|^2\, f_n^k(t)\,dxdv = \int\int_{\mathbb{R}^3\times\mathbb{R}^3} |x-vt_k|^2\, f_n^k(t_k)\,dxdv \tag{2.3.6}$$

for all $t\in[t_k,t_{k+1}[$, whereas during the second step we have

$$\int\int_{\mathbb{R}^3\times\mathbb{R}^3} |x-vt_k|^2\, g_n^k(t)\,dxdv = \int\int_{\mathbb{R}^3\times\mathbb{R}^3} |x-vt_k|^2\, g_n^k(t_k)\,dxdv \tag{2.3.7}$$

for all $t\in[t_k,t_{k+1}[$. Therefore,

$$\int\int_{\mathbb{R}^3\times\mathbb{R}^3} |x-vt|^2\, f_n(t)\,dxdv = \int\int_{\mathbb{R}^3\times\mathbb{R}^3} |x|^2\, f_0\,dxdv \tag{2.3.8}$$

for all $t\in[0,T[$, and

$$\sup_{t\in[0,T]}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} g_n(t)\,|x|^2\,dxdv \le \sup_{t\in[0,T]}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f_n(t)\,|x|^2\,dxdv$$

$$\le \int\int_{\mathbb{R}^3\times\mathbb{R}^3} 2\,f_0\,(|x|^2+T^2|v|^2)\,dxdv. \tag{2.3.9}$$

Finally, we prove the estimate on the entropy production:

$$\frac{d}{dt}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f_n^k\,\log f_n^k\,dxdv = 0,$$

and

$$\frac{d}{dt}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} g_n^k\,\log g_n^k\,dxdv = -\frac{1}{4}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} e(g_n^k)\,dxdv \le 0$$

for all $t\in[t_k,t_{k+1}[$. Therefore,

$$\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f_n\,\log f_n\,dxdv(t) \le \int\int_{\mathbb{R}^3\times\mathbb{R}^3} f_0\,\log f_0\,dxdv, \tag{2.3.10}$$

and

$$\int\int_{\mathbb{R}^3\times\mathbb{R}^3} g_n \log g_n \, dxdv(t) - \int\int_{\mathbb{R}^3\times\mathbb{R}^3} f_0 \log f_0 \, dxdv$$
$$\leq -\frac{1}{4}\int_0^T \int_{\mathbb{R}^3\times\mathbb{R}^3} e(g_n)\, dxdvdt \qquad (2.3.11)$$

for all $t \in [0, T[$. Finally, it is now classical (Cf. [DP, L]) that estimates (2.3.5) and (2.3.9) – (2.3.11) ensure the existence of a constant C_T such that

$$\sup_{t\in[0,T]} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} f_n \, |\log f_n| \, dxdv \leq C_T, \qquad (2.3.12)$$

$$\sup_{t\in[0,T]} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} g_n \, |\log g_n| \, dxdv \leq C_T, \qquad (2.3.13)$$

and

$$\int_0^T \int_{\mathbb{R}^3\times\mathbb{R}^3} e(g_n)\, dxdvdt \leq C_T, \qquad (2.3.14)$$

which ends the proof of lemma 1.

According to lemma 1, we can extract from the sequences f_n and g_n subsequences still denoted by f_n and g_n, which converge respectively to f and g in $L^\infty([0,T]; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ weak *.

2.4 Weak compactness of the renormalized collision terms

We present here the main estimate on the collision term:

Lemma 2: *The sequences* $\frac{Q^\pm(g_n)}{1+g_n}$ *and* $\frac{Q^+(g_n)}{1+L(g_n)}$ *belong to a weakly compact set of* $L^1([0,T] \times \mathbb{R}^3_x \times B^R_v)$*, for all* $R > 0$.

Proof: We only prove here that the sequences are bounded in L^1. The reader will find in [DP, L] the proof of weak compactness.

For all $R > 0$, we compute

$$\int_{x\in\mathbb{R}^3}\int_{v\in B_R} \frac{Q^-(g_n)}{1+g_n}\, dvdx \leq \int_{x\in\mathbb{R}^3}\int_{v\in B_R} L(g_n)\, dvdx$$
$$\leq \int_{x\in\mathbb{R}^3}\int_{v_*\in\mathbb{R}^3} g_n(v_*) \left\{ \int_{v\in B_R} A(v-v_*) dv \right\} dv_* dx$$
$$\leq \sup_{z\in\mathbb{R}^3} \left\{ (1+|z|^2)^{-1} \int_{v\in B_R} A(v-z)\, dv \right\}$$

$$\times \int_{x\in I\!R^3}\int_{v_*\in I\!R^3} g_n(v_*)\,(1+|v_*|^2)\,dv_*dx, \qquad (2.4.1)$$

which is bounded because of assumption 1 and estimate (2.3.2).

Then, the boundedness of $\frac{Q^+(g_n)}{1+g_n}$ and $\frac{Q^+(g_n)}{1+L(g_n)}$ comes out of the boundedness of $\frac{Q^-(g_n)}{1+g_n}$ and of estimate (2.3.4).

More precisely, we recall that for all $K > 0$,

$$Q^+(g_n) \le KQ^-(g_n) + \frac{e(g_n)}{\log K}. \qquad (2.4.2)$$

2.5 The sequences f_n and g_n converge to the same limit

In order to pass to the limit in equation (2.2.2) we need to know that $\beta(g_n)$ and $\beta(f_n)$ converge to the same limit, and that the same holds for f_n and g_n.

lemma 3: *Up to extraction, the sequences f_n and g_n satisfy the following properties:*

i) For all nonnegative and smooth function β such that (2.2.1) holds, $\beta(f_n)$ and $\beta(g_n)$ converge to the same limit in $L^1([0,T]\times I\!R^3\times I\!R^3)$ weak.

ii) the sequences f_n and g_n have the same limit f in $L^1([0,T]\times I\!R^3\times I\!R^3)$ weak.

Proof:

Step 1 : We prove i). Because of lemma 1, we just have to show that $\beta(f_n)$ and $\beta(g_n)$ converge to the same limit in the sense of distributions. Let φ belong to $D(]0,T[\times I\!R^3\times I\!R^3)$ and K be its compact support.

We compute

$$|\int_0^T \int_{x\in I\!R^3}\int_{v\in I\!R^3} (\beta(f^n)-\beta(g^n))(t,x,v)\,\varphi(t,x,v)\,dv\,dx\,dt|$$

$$\le \sum_{j=0}^{n-1} |\int_{j\Delta T}^{(j+1)\Delta T}\int_{x\in I\!R^3}\int_{v\in I\!R^3} (\beta(f^n)(t,x,v)$$

$$-\beta(f_n^j)((j+1)\Delta T,\ x,v))\,\varphi(t,x,v)\,dv\,dx\,dt|$$

$$+ \sum_{j=0}^{n-1} | \int_{j\Delta T}^{(j+1)\Delta T} \int_{x\in\mathbb{R}^3} \int_{v\in\mathbb{R}^3} (\beta(g^n)(t,x,v)$$

$$-\beta(g_n^j)(j\Delta T,\ x,v))\,\varphi(t,x,v)\,dx\,dv|\,dt$$

$$\leq \sum_{j=0}^{n-1} \int_{j\Delta T}^{(j+1)\Delta T} \int_{x\in\mathbb{R}^3} \int_{v\in\mathbb{R}^3} \int_t^{(j+1)\Delta T} -v\cdot\nabla_x\beta(f^n)(s,x,v)\,ds$$

$$\varphi(t,x,v)\,dx\,dv|\,dt$$

$$+ \sum_{j=0}^{n-1} \int_{j\Delta T}^{(j+1)\Delta T} \int_{x\in\mathbb{R}^3} \int_{v\in\mathbb{R}^3} | \int_{j\Delta T}^{t} \beta'(g^n)(s)\,Q(g_n)(s)\,dsdxdv|\,dt$$

$$\leq \ \Delta T\,\|v\cdot\nabla_x\,\varphi\|_{L^\infty(K)}\,\|\beta(f^n)\|_{L^1([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}$$

$$+ \Delta T\,\|\varphi\|_{L^\infty(K)}\|\beta'(g_n)\,Q(g_n)\|_{L1(K)}, \tag{2.5.1}$$

which clearly tends to 0 when $\Delta T = \frac{T}{n}$ tends to 0.

Step 2 : We prove ii). Taking $\beta_\delta(s) = \frac{s}{1+\delta s}$, we note that

$$0 \leq s - \beta_\delta(s) \leq \delta R\,s + s\,1_{s>R}. \tag{2.5.2}$$

Therefore,

$$0 \leq g_n - \beta_\delta(g_n) \leq \delta R\,g_n + g_n\,\frac{|\log g_n|}{\log R}, \tag{2.5.3}$$

and the same estimate holds for f_n. Using then estimates (2.3.12) and (2.3.13), we get

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|f_n - \beta_\delta(f_n)\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \underset{\delta\to 0}{\longrightarrow} 0, \tag{2.5.4}$$

and

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|g_n - \beta_\delta(g_n)\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \underset{\delta\to 0}{\longrightarrow} 0. \tag{2.5.5}$$

According to step 1 and estimates (2.5.4), (2.5.5), we get ii).

2.6 Strong compactness for velocity averages

In this section we get some informations on the limits of the sequence $Q^{\pm}(g_n)$ which follow from the strong compactness of the velocity averages of g_n.

lemma 4: *For all* φ *in* $L^{\infty}([0,T]\times I\!R^3\times I\!R^3)$*, the sequence*

$$j_n(t,x)=\int_{v\in I\!R^3} g_n(t,x,v)\psi(t,x,v)\,dv \tag{2.6.1}$$

lies in a strongly compact set of $L^1([0,T]\times I\!R^3)$*.*

Proof: The proof is divided in six steps. During the five first steps, we fix a function ψ in $L_c^{\infty}(I\!R^3)$, a nonnegative and smooth function β satisfying (2.2.1). Denoting for every function h in $L^1_{loc}([0,T]\times I\!R^3\times I\!R^3)$,

$$\tilde{h}(t,x)=\int_{I\!R^3} h(t,x,v)\;\psi(v)\;dv \tag{2.6.2}$$

we prove that $\widetilde{\beta(g_n)}$ lies in a strongly compact set of $L^1([0,T]\times I\!R^3)$.

Step 1 : We compute

$$\frac{\partial}{\partial t}\beta(g_n)=\beta'(g_n)Q(g_n)+\sum_{j=1}^{n-1}\left\{\beta(g_n^j)(j\Delta T)-\beta(g_n^{j-1})(j\Delta T)\right\}\,\delta_{j\Delta T}(t)$$

$$=\beta'(g_n)Q(g_n)+\sum_{j=1}^{n-1}\left\{\beta(f_n^j)((j+1)\Delta T)-\beta(f_n^j)(j\Delta T)\right\}\,\delta_{j\Delta T}(t)$$

$$=\beta'(g_n)Q(g_n)+\sum_{j=1}^{n-1}\left\{\int_{j\Delta T}^{(j+1)\Delta T}-v\cdot\nabla_x\beta(f_n)(s,x,v)\,ds\right\}\delta_{j\Delta T}(t). \tag{2.6.3}$$

Therefore, for all function ϕ in $C_c^1(I\!R_x^3)$, we have

$$\frac{d}{dt}\int_{x\in I\!R^3}\widetilde{\beta(g_n)}\,\varphi(x)\;dx=\int\int_{I\!R^3\times I\!R^3}\beta'(g_n)\,Q(g_n)\,\psi(v)\,\phi(x)\,dv\,dx$$

$$+\sum_{j=1}^{n-1}\int_{j\Delta T}^{(j+1)\Delta T}\left\{\int_{x\in I\!R^3}\int_{v\in I\!R^3}\beta(f_n)\;\psi(v)\;v\cdot\nabla_x\phi(x)\;dxdv\right\}ds\,\delta_{j\Delta T}(t), \tag{2.6.4}$$

which is a bounded sequence of measures in $[0,T]$, thanks to (2.3.1) and lemma 2.

Thus, the quantity

$$a_n(t) = \int_{x \in I\!R^3} \widetilde{\beta(g_n)}(t,x)\, \varphi(x)\, dx \tag{2.6.5}$$

is bounded in $BV([0,T])$.

Considering now a sequence of (compactly supported) mollifiers $\rho_\varepsilon(x)$, the previous statement implies that for every fixed $\varepsilon > 0$, the sequence $\widetilde{\beta(g_n)} *_x \rho_\varepsilon$ is strongly compact in $L^1([0,T] \times I\!R^3)$.

Thanks to the identity

$$\widetilde{\beta(g_n)} = \widetilde{\beta(g_n)} *_x \rho_\varepsilon + \left\{ \widetilde{\beta(g_n)} - \widetilde{\beta(g_n)} *_x \rho_\varepsilon \right\}, \tag{2.6.6}$$

we only need to prove that the second term is uniformly (in n) small in $L^1([0,T] \times I\!R^3)$ when ε tends to 0 to get the strong compactness in $L^1([0,T] \times I\!R^3)$ of $\widetilde{\beta(g_n)}$.

This will in turn be true if we prove that

$$I_h = \sup_{n \in I\!N} \int_0^T \int_{I\!R^3} |\widetilde{\beta(g_n)}(t, x+h) - \widetilde{\beta(g_n)}(t,x)|\, dx\, dt \tag{2.6.7}$$

tends to 0 when h tends to 0. Steps 2 to 5 are devoted to the proof of this estimate.

Step 2 : We compute

$$I_h \le \sup_{n \in I\!N} \sum_{j=0}^{n-1} \int_{I\!R^3} \int_{j\Delta T}^{(j+1)\Delta T} |\widetilde{\beta(g_n)}(t, x+h) - \widetilde{\beta(g_n^j)}(j\Delta T, x+h)|\, dx\, dt$$

$$+ \sup_{n \in I\!N} \sum_{j=0}^{n-1} \int_{I\!R^3} \int_{j\Delta T}^{(j+1)\Delta T} |\widetilde{\beta(g_n^j)}(j\Delta T, x+h) - \widetilde{\beta(g_n^j)}(j\Delta T, x)|\, dx\, dt$$

$$+ \sup_{n \in I\!N} \sum_{j=0}^{n-1} \int_{I\!R^3} \int_{j\Delta T}^{(j+1)\Delta T} |\widetilde{\beta(g_n^j)}(j\Delta T, x) - \widetilde{\beta(g_n)}(t,x)|\, dx\, dt$$

$$\le 2\, \Delta T\, ||\psi||_{L^\infty(I\!R^3_v)} \sup_{n \in I\!N} ||\beta'(g_n)\, Q(g_n)||_{L^1([0,T] \times I\!R^3 \times Supp\psi)} + J_h, \tag{2.6.8}$$

where

$$J_h = \sup_{n\in \mathbb{N}} \Delta T \int_{\mathbb{R}^3} \sum_{j=0}^{n-1} |\widetilde{\beta(f_n^j)}((j+1)\Delta T, x+h) - \widetilde{\beta(f_n^j)}((j+1)\Delta T, x)|\, dx. \tag{2.6.9}$$

Step 3: In order to use the Fourier transform, we recast in this step the problem in an L^2 setting.

Fixing $\varepsilon > 0$ and using lemma 2, we decompose $\beta'(g_n)Q(g_n)$ in such a way that

$$\beta'(g_n)Q(g_n) = q_n^{1,\epsilon} + q_n^{2,\epsilon}, \tag{2.6.10}$$

and

$$\|q_n^{1,\epsilon}\|_{L^1([0,T]\times \mathbb{R}^3 \times Supp\psi)} \le \varepsilon, \qquad \|q_n^{2,\epsilon}\|^2_{L^2([0,T]\times \mathbb{R}^3 \times Supp\psi)} \le C_\varepsilon. \tag{2.6.11}$$

We also decompose $\beta(f_0)$ in such a way that

$$\beta(f_0) = h_0^{1,\epsilon} + h_0^{2,\epsilon}, \tag{2.6.12}$$

and

$$\|h_0^{1,\epsilon}\|_{L^1(\mathbb{R}^3 \times \ Supp\psi)} \le \varepsilon, \qquad \|h_0^{2,\epsilon}\|^2_{L^2(\mathbb{R}^3\times \ Supp\psi)} \le C_\varepsilon. \tag{2.6.13}$$

Then, we define $h_n^{1,\epsilon}, h_n^{2,\epsilon}$ the solutions of the (linear) problems

$$\frac{\partial h_n^{p,\epsilon}}{\partial t} + v\cdot \nabla_x h_n^{p,\epsilon} = \sum_{j=0}^{n-1} (\int_{j\Delta T}^{(j+1)\Delta T} q_n^{p,\epsilon}(s)\ ds)\ \delta_{(j+1)\Delta T}(t), \tag{2.6.14}$$

$$h_n^{p,\epsilon}(0,.) = h_0^{p,\epsilon}(.). \tag{2.6.15}$$

Note that h_n^p is not continuous with respect to the time variable at points $j\,\Delta T$ for $j \in \mathbb{Z}$. Therefore, we denote by $h_n^p(j\,\Delta T^+)$ (respt. $h_n^p(j\,\Delta T^-)$)) the right–hand (respt. left–hand) limit of h_n^p at points $j\,\Delta T$.

Classical estimates yield then the following result:

lemma 5: *The sequences $h_n^{1,\epsilon}$ and $h_n^{2,\epsilon}$ are such that*

$$\Delta T \sum_{j=0}^{n-1} \int_{x\in \mathbb{R}^3} \int_{v\in\mathbb{R}^3} |h_n^{1,\epsilon}((j+1)\Delta T^-, x, v)\ \psi(v)| dx\ dv$$

$$\le C_T \|\psi\|_{L^\infty} (\|h_0^{1,\epsilon}\|_{L^1(\mathbb{R}^3\times \ Supp\psi)} + \|q_n^{1,\epsilon}\|_{L^1([0,T]\times\mathbb{R}^3\times \ Supp\ \psi)}), \tag{2.6.16}$$

$$\Delta T \sum_{j=0}^{n-1} \int_{x\in I\!R^3} \int_{v\in I\!R^3} |h_n^{2,\epsilon}((j+1)\Delta T^-, x, v)\,\psi(v)|^2 dx\, dv$$

$$\leq C_T \|\psi\|_{L^\infty}^2 (\|h_0^{2,\epsilon}\|^2_{L^2(I\!R^3\times\ Supp\ \psi)} + \|q_n^{2,\epsilon}\|^2_{L^2([0,T]\times I\!R^3\times\ Supp\ \psi)}). \quad (2.6.17)$$

The definitions of the sequences $h_n^{p,\epsilon}$ imply that

$$\beta(f_n) = h_n^{1,\epsilon} + h_n^{2,\epsilon}. \quad (2.6.18)$$

Using then the Cauchy-Schwarz inequality and lemma 5, we get

$$J_h \leq \sup_{n\in I\!N} \Delta T \sum_{j=0}^{n-1} \int_{I\!R^3} (|\widetilde{h_n^{1,\epsilon}}((j+1)\Delta T^-, x+h)| + |\widetilde{h_n^{1,\epsilon}}((j+1)\Delta T^-, x)|\, dx$$

$$+ \sup_{n\in I\!N} \quad \Delta T \sum_{j=0}^{n-1} \int_{I\!R^3} |\widetilde{h_n^{2,\epsilon}}((j+1)\Delta T^-, x+h) - \widetilde{h_n^{2,\epsilon}}(j\Delta T^-, x)|\, dx$$

$$\leq 2\, C_T\, \|\psi\|_{L^\infty} \epsilon + (T\, R^3)^{1/2} \sup_{n\in I\!N} (K_{h,\epsilon}(n))^{1/2}, \quad (2.6.19)$$

where

$$K_{h,\epsilon}(n) = \Delta T \sum_{j=0}^{n-1} \int_{I\!R^3} |\widetilde{h_n^{2,\epsilon}}((j+1)\Delta T^-, x+h) - \widetilde{h_n^{2,\epsilon}}((j+1)\Delta T^-, x)|^2 dx. \quad (2.6.20)$$

We now need to prove that for any fixed $\varepsilon > 0$, the quantity $K_{h,\epsilon}(n)$ tends uniformly (in n) to 0 when h tends to 0.

This property is proved in steps 4 and 5. Note that in the sequel, we shall not write down explicitly the dependance of K_h or h_n^2 with respect to the parameter ϵ.

Step 4: We now denote by $\hat{f} = \hat{f}(t,\xi,v)$ and $\hat{\tilde{f}} = \hat{\tilde{f}}(t,\xi)$ the Fourier transform with respect to the space variable of the functions $f(t,x,v)$ and $\tilde{f}(t,x)$.
We shall assume in this step that there exists a bounded sequence k_n in $L^2(I\!R^3_\xi)$ such that

$$\Delta T \sum_{j=0}^{n-1} |\widehat{\widetilde{h_n^2}}((j+1)\Delta T^-, \xi)|^2 \leq \left(\frac{1}{n} + \frac{1}{|\xi|}\right) |k_n(\xi)|^2. \quad (2.6.21)$$

This fact will be proved in the fifth step.

Thanks to the Fourier-Plancherel identity, one gets

$$K_h(n) = \Delta T \sum_{j=0}^{n-1} \int_{\xi \in I\!R^3} |e^{-ih\xi} - 1|^2 |\widehat{\widetilde{h_n^2}}((j+1)\Delta T^-, \xi)|^2 d\xi$$

$$\leq \Delta T \sum_{j=0}^{n-1} \int_{|\xi| \leq R} (hR)^2 |\widehat{\widetilde{h_n^2}}((j+1)\Delta T^-, \xi)|^2 d\xi$$

$$+\Delta T \sum_{j=0}^{n-1} \int_{\xi \geq R} 4 |\widehat{\widetilde{h_n^2}}(j\Delta T^-, \xi)|^2 d\xi. \tag{2.6.22}$$

Using lemma 5 and estimates (2.6.11), (2.6.13), we get

$$K_h(n) \leq (hR)^2 \, \|\psi\|^2_{L^\infty} 2C_\varepsilon + 4 \int_{|\xi| \geq R} \left(\frac{1}{n} + \frac{1}{|\xi|} \right) |k_n(\xi)|^2 d\xi. \tag{2.6.23}$$

Therefore, $K_h(n)$ tends to 0 uniformly (in n) when h tends to 0.

This ends the proof of the compactness of $\widetilde{\beta(g_n)}$ in $L^1([0,T] \times I\!R^3)$ under assumption (2.6.21).

Step 5 is dedicated to the proof of this assumption.

Step 5 : In this step, we prove in fact the averaging lemma replacing in the context of the splitting method the averaging lemma used in [DP, L].

Taking the Fourier transform with respect to the variable x of eq. (2.6.14), (2.6.15) for $p = 2$, we get

$$\frac{\partial}{\partial t} \hat{h_n^2} + iv \cdot \xi \, \hat{h_n^2} = \sum_{j=0}^{n-1} \left(\int_j^{(j+1)\Delta T} \hat{q_n^2}(s) \, ds \right) \delta_{(j+1)\Delta T}(t). \tag{2.6.24}$$

The Duhamel representation of the solution of this equation is

$$\hat{h_n^2}(t, \xi, v) = \hat{h_0^2}(\xi, v) \, e^{-i\xi \cdot vt}$$

$$+ \int_0^t e^{-iv\xi(t-s)} \sum_{j=0}^{n-1} \left(\int_{j\Delta T}^{(j+1)\Delta T} \hat{q_n^2}(\sigma) d\sigma \right) \delta_{(j+1)\Delta T}(s) \, ds. \tag{2.6.25}$$

The velocity average $\widehat{\widetilde{h_n^2}}$ satisfies

$$\widehat{\widetilde{h_n^2}}(t, \xi) = \int_{I\!R^3} \psi(v) \, \widehat{h_0^2}(\xi, v) \, e^{-itv \cdot \xi} \, dv$$

$$+\sum_{0\le(j+1)\Delta T\le t}\int_{I\!R^3}\psi(v)e^{-iv.\xi(t-(j+1)\Delta T)}\left(\int_{j\Delta T}^{(j+1)\Delta T}\widehat{q_n^2}(\sigma)d\sigma\right)dv. \quad (2.6.26)$$

Then, we compute

$$\Delta T\sum_{k=0}^{n-1}|\widehat{\widehat{h_n^2}}((k+1)\Delta T^-,\xi)|^2\le 2\Delta T\sum_{k=0}^{n-1}|\int_{I\!R^3}\psi(v)\,\widehat{h_0^2}(\xi,v)\,e^{-i(k+1)v\cdot\xi\Delta T}dv|^2$$

$$+2\Delta T\sum_{k=0}^{n-1}T\sum_{j=0}^{k}\int_{j\Delta T}^{(j+1)\Delta T}|\int_{I\!R^3}\psi(v)\widehat{q_n^2}(\sigma,\xi,v)e^{-iv\cdot\xi\Delta T(k-j)}dv|^2d\sigma. \quad (2.6.27)$$

We prove the bound (2.6.21) only for the second term of the right–hand side of (2.6.27) (which will be denoted by $L_n(\xi)$). Note that the first term could be bounded in the same way.

We make the change of variables

$$v=\frac{v_1}{\Delta T|\xi|^2}\xi+v^{\perp}\quad\text{with}\quad v^{\perp}\cdot\xi=0. \quad (2.6.28)$$

Then,

$$L_n(\xi)\le 2\Delta T\,T\sum_{l=0}^{n-1}\int_0^T|\int_{I\!R^3}\psi(v)\widehat{q_n^2}(\sigma,\xi,v)e^{-iv\cdot\xi\Delta Tl}dv|^2d\sigma$$

$$\le 2T\Delta T\sum_{l\in Z\!\!Z}\int_0^T|\int_{v_1\in I\!R}e^{-iv_1l}\int_{v^{\perp}\in I\!R^2}(\psi\widehat{q_n^2})(\sigma,\xi,\frac{v_1}{\Delta T|\xi|^2}\xi+v^{\perp})\frac{dv_1dv^{\perp}}{\Delta T|\xi|}|^2d\sigma. \quad (2.6.29)$$

We use then he Poisson identity with respect to the variable v_1.

$$\sum_{l\in Z\!\!Z}|\mathcal{F}_{v_1}(\varphi)(l)|^2=\sum_{l\in Z\!\!Z}\mathcal{F}_{v_1}(\varphi*\varphi)(l)=2\pi\sum_{l\in Z\!\!Z}(\varphi*_{v_1}\varphi)(2\pi l). \quad (2.6.30)$$

Therefore, using the change of variables $v_1\to w=\Delta T|\xi|v_1$, we get

$$L_n(\xi)\le 4\pi\frac{T}{\Delta T|\xi|^2}\sum_{l\in Z\!\!Z}\int_0^T|\int_{v_1\in I\!R}\int_{y\in I\!R^2}(\psi\widehat{q_n^2})(\sigma,\xi,\frac{v_1}{\Delta T|\xi|^2}\xi+y)dy$$

$$\times\int_{z\in I\!R^2}(\psi\widehat{q_n^2})(\sigma,\xi,\frac{2\pi l-v_1}{\Delta T|\xi|^2}\xi+z)\,dzdv_1|\,d\sigma$$

$$\le 4\pi\frac{T}{|\xi|}\sum_{l\in Z\!\!Z}\int_0^T\int_{w\in I\!R}\int_{y\in I\!R^2}|(\psi\widehat{q_n^2})(\sigma,\xi,w\frac{\xi}{|\xi|}+y)|dy$$

$$\times \int_{z\in \mathbb{R}^2} |(\widehat{\psi q_n^2})(\sigma,\xi,(\frac{2\pi l}{\Delta T|\xi|}-w)\frac{\xi}{|\xi|}+z)|dzdwd\sigma. \qquad (2.6.31)$$

Because of the compact support of ψ, the integrations with respect to y, z and w are made over a compact set. Therefore, the sum over l is different from 0 only if $|l| \le \Delta T\,|\xi|\,C_\psi$. Then,

$$L_n(\xi) \le 4\pi\frac{T}{|\xi|}\sum_{|l|\le \Delta T|\xi|C_\psi}\int_0^T\left\{\int_{w\in\mathbb{R}} |\int_{|y|\le C_\psi}(\widehat{\psi q_n^2})\left(\sigma,\xi,w\frac{\xi}{|\xi|}+y\right)dy|^2\right.$$

$$\left. +|\int_{|z|\le C_\psi}(\widehat{\psi q_n^2})\left(\sigma,\xi,\left(\frac{2\pi l}{\Delta T|\xi|}-w\right)\frac{\xi}{|\xi|}+z\right)dz|^2dw\right\}d\sigma$$

$$\le 4\pi\frac{T}{|\xi|}(1+\Delta T|\xi|C_\psi)\,2\int_0^T \|\psi\|^2_{L_\infty}(2C_\psi)^3\int_{v\in\mathbb{R}^3}|\widehat{q_n^2}(\sigma,\xi,v)|^2dv\;d\sigma$$

$$\le C_{T,\psi}\left(\frac{1}{|\xi|}+\Delta T\right)\int_0^T\int_{v\in\mathbb{R}^3}|\widehat{q_n^2}(\sigma,\xi,v)|^2dv\;d\sigma. \qquad (2.6.32)$$

This estimate clearly yields (2.6.21).

step 6 We wish to prove that for all $\psi \in L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$, the sequence

$$s_n(t,x) = \int_{v\in\mathbb{R}^3}\beta(g_n)(t,x,v)\,\psi(t,x,v)\,dv \qquad (2.6.33)$$

is strongly compact in $L^1([0,T]\times\mathbb{R}^3)$.

This result is an immediate consequence of steps 1 to 5 in the case of separated variables, namely when

$$\psi(t,x,v) = \sum_{i=1}^m \psi_i^1(t,x)\,\psi_i^2(v), \quad \psi_i^1 \in L_c^\infty([0,T]\times\mathbb{R}^3), \quad \psi_i^2\in L_c^\infty(\mathbb{R}^3). \qquad (2.6.34)$$

Therefore, in order to establish the general case, we fix an arbitrary ψ and consider a sequence ψ^k of separated functions which are uniformy bounded in $L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$ by $\|\psi\|_\infty$ and converge to ψ in $L^1_{\rm loc}([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$. We only have to prove that:

$$\sup_{n\in\mathbb{N}}\|\beta(g_n)\,(\psi^k-\psi)\|_{L^1([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}\underset{k\to\infty}{\longrightarrow}0. \qquad (2.6.35)$$

In order to prove this assertion, note that

$$\|\beta(g_n)\,(\psi^k-\psi)\,\chi\|_{L^1([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \le 2\,\|\psi\|_{L^\infty}\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\beta(g_n)\,\chi\,dvdxdt, \qquad (2.6.36)$$

where

$$\chi = 1_{|x|\geq R} + 1_{|v|\geq R} + 1_{g_n \geq R}. \tag{2.6.37}$$

Using the weak compactness of g_n, we can see that the last term of (2.6.36) is arbitrary small for large R. Finally, we note that

$$\int_0^T \int_{B_R} \int_{B_R} \beta(g_n)\, 1_{g_n \leq R}\, |\psi^k - \psi|\, dv dx dt \leq C_R \int_0^T \int_{B_R} \int_{B_R} |\psi^k - \psi|\, dv dx dt, \tag{2.6.38}$$

where

$$C_R = \sup_{s \in [0,R]} |\beta(s)|, \tag{2.6.39}$$

and we conclude by letting k and R go to $+\infty$.

Finally, we choose $\beta_\delta(s) = \frac{s}{1+\delta s}$ and we let δ go to 0. Then, the arguments used in the proof of the step 2 of lemma 3 yield lemma 4.

We now state some facts that come out of lemmas 1 to 5. The proof exactly follows that of [DP, L].

Lemma 6: *The following properties hold for g_n:*
i) The sequence $L(g_n)$ converges towards $L(f)$ strongly in $L^1([0,T] \times I\!R^3 \times B_v^R)$.
ii) For all $\nu > 0$, the sequence $\frac{Q^+(g_n)}{1+\nu L(g_n)}$ converges towards $\frac{Q^+(f)}{1+\nu L(f)}$ weakly in $L^1([0,T] \times I\!R^3 \times I\!R^3)$.

2.7 Passing to the limit

We now wish to prove that f is a renormalised solution of (1.1). Following [L 1], we consider $\beta_\delta(s) = \frac{s}{1+\delta s}$ and pass to the limit as n goes to $+\infty$ weakly in equation (2.2.2) with $\beta = \beta_\delta$. Then we renormalize the resulting limit equation and let δ go to 0 in order to recover (2.1.12).

We shall use in the sequel some notations. Without loss of generality, extracting subsequences if necessary, we may assume that for all $\delta > 0$,

$$\beta_\delta(f^n), \beta_\delta(g^n) \underset{n \to +\infty}{\rightharpoonup} \beta_\delta \text{ weakly in } L^1([0,T] \times I\!R_x^3 \times I\!R_v^3), \tag{2.7.1}$$

$$\gamma_\delta^n = g_n\,(1+\delta g_n)^{-2} \underset{n \to +\infty}{\rightharpoonup} \gamma_\delta \text{ weakly in } L^1([0,T] \times I\!R_x^3 \times I\!R_v^3), \tag{2.7.2}$$

$$\frac{Q^\pm(g_n)}{(1+\delta g_n)^2} \underset{n\to+\infty}{\rightharpoonup} Q_\delta^\pm \text{ weakly in } L^1([0,T]\times I\!R^3_x \times B^R_v). \tag{2.7.3}$$

Then, we use the following lemma (Cf. [De 1]):

Lemma 7: *Assume that a sequence h_n converges weakly in $L^1_{loc}([0,T]\times I\!R^3\times I\!R^3)$ to h.*

Then,

$$i_n = \sum_{i=1}^{n-1} \delta_{i\Delta T} \int_{(i-1)\Delta T}^{i\Delta T} h_n(s)\, ds \tag{2.7.4}$$

converges to h weakly in the sense of measures.

Passing to the limit in (2.2.2) as n tends to $+\infty$, we get:

$$\frac{\partial \beta_\delta}{\partial t} + v\cdot\nabla_x \beta_\delta = Q_\delta^+ - Q_\delta^- \text{ in } \mathcal{D}'. \tag{2.7.5}$$

Since $Q_\delta^+, Q_\delta^- \in L^1_{\text{loc}}([0,T]\times I\!R^3\times I\!R^3)$, we can see that $\beta_\delta \in C([0,T], L^1(I\!R^3\times I\!R^3))$. Thus, estimate (2.5.17) implies that β_δ converges to f as δ tends to 0 in the space $C([0,T], L^1(I\!R^3\times I\!R^3))$. Moreover, using the convexity of $-\beta_\delta$, we get

$$\beta_\delta \le \beta_\delta(f), \tag{2.7.6}$$

and (because $-\frac{t}{(1+\delta t)^2} = -\beta_\delta(t)(1-\delta\beta_\delta(t))$ is convex with respect to β_δ),

$$\gamma_\delta \le \beta_\delta\,(1-\delta\beta_\delta) \le \beta_\delta(f). \tag{2.7.7}$$

Finally, because of lemma 5,

$$Q_\delta^- = \gamma_\delta\, L(f). \tag{2.7.8}$$

We now renormalize (2.7.5) by $\beta(s) = \log(1+s)$ and get

$$(\frac{\partial}{\partial t} + v\cdot\nabla_x)\beta(\beta_\delta) = \frac{Q_\delta^+}{1+\beta_\delta} - \frac{Q_\delta^-}{1+\beta_\delta} \text{ in } \mathcal{D}'. \tag{2.7.9}$$

To prove that f is a renormalized solution of Boltzmann equation we let δ go to 0^+ and thanks to the strong convergence of β_δ towards f, we only have to prove the weak compactness of $\frac{Q_\delta^\pm}{1+\beta_\delta}$ in $L^1([0,T]\times I\!R^3_x\times B^R_v)$ and the following convergence:

$$\frac{Q_\delta^\pm}{1+\beta_\delta} \underset{\delta\to 0^+}{\longrightarrow} \frac{Q^\pm(f)}{1+f} \quad a.e. \tag{2.7.10}$$

The weak compactness comes out of estimate (2.3.1) and the following inequality (based on estimates (2.7.7) and (2.7.8)):

$$\frac{Q_\delta^-}{1+\beta_\delta} = \frac{\gamma_\delta L(f)}{1+\beta_\delta} \leq L(f). \tag{2.7.11}$$

For more details, we refer to [L 1].

Then, we prove that the convergence (2.7.10) holds. The proof exactly follows that of [L 1] and therefore we only give the main steps.

Note that for all $R > 1$

$$0 \leq g_n - g_n\,(1+\delta g_n)^{-2} \leq R\,\delta\, g_n + g_n\, 1_{(g_n > R)}, \tag{2.7.12}$$

therefore γ_δ converges to f in $C([0,T]; L^1(I\!R^3 \times I\!R^3))$ when δ go to 0_+ . This remark ensures that

$$\frac{Q_\delta^-}{1+\beta_\delta} = \frac{\gamma_\delta L(f)}{1+\beta_\delta} \underset{\delta\to 0}{\longrightarrow} \frac{f L(f)}{1+f} \text{ a.e. .} \tag{2.7.13}$$

It remains to prove eq. (2.7.10) for Q_δ^+. Using lemma 6 and the inequality

$$\frac{Q^+(g_n)}{1+L(g_n)} \geq \frac{Q^+(g_n)}{(1+\delta g_n)^2}\,\frac{1}{1+L(g_n)}, \tag{2.7.14}$$

we get the following estimate:

$$\frac{Q^+(f)}{1+L(f)} \geq \frac{Q_\delta^+}{1+L(f)} \quad \text{for all } \delta > 0. \tag{2.7.15}$$

Then, for any $\nu, K > 0$,

$$(1+\delta R)^{-2}(1+\nu L(g_n))^{-1} Q^+(g_n)$$

$$\leq \frac{Q^+(g_n)}{(1+\delta g_n)^2} + \frac{Q^+(g_n)}{1+\nu L(g_n)}\, 1_{g_n \geq R}. \tag{2.7.16}$$

Using then inequality (2.4.2), one gets

$$(1+\delta R)^{-2}(1+\nu L(g_n))^{-1} Q^+(g_n)$$

$$\leq (1+\delta g_n)^{-2} Q^+(g_n) + \frac{e(g_n)}{\log K} + \frac{K}{\nu}\, g_n\, 1_{g_n \geq R}. \tag{2.7.17}$$

Passing to the limit in (2.7.17), we get

$$(1+\delta R)^{-2}(1+\nu L(f))^{-1}Q^{+}(f) \leq Q_{\delta}^{+} + \frac{e_0}{\log K} + \frac{K}{\nu} g_R, \tag{2.7.18}$$

where g_R is the weak limit of $g_n 1_{(g_n>R)}$. Letting δ go to 0, then R and K go to $+\infty$ and finally ν go to 0_+, we can see that

$$Q^{+}(f) \leq \liminf_{\delta\to 0_+} Q_{\delta}^{+} \text{ a.e. }. \tag{2.7.19}$$

Therefore, theorem 1 holds.

Remark: It is also possible to prove that the convergence of f_n and g_n towards f is in fact strong. The proof is exactly the same as that of [L 1].

3 Splitting for the B.G.K. model

We prove in this section the result corresponding to theorem 1 in a slightly different context, namely that of the B.G.K. model.

3.1 Definition and main result

As in section 2, we introduce a splitting algorithm for equations (1.12) – (1.17), where

$$\mathcal{A}f = -v \cdot \nabla_x f, \tag{3.1.1}$$

and

$$\mathcal{B}f = M[f] - f. \tag{3.1.2}$$

We define therefore for every n in $\mathbb{N}_*$ and every k in $[0, n-1]$ the functions f_n^k, g_n^k. We denote

$$\Delta T = \frac{T}{n}, \quad t_k = k\Delta T, \tag{3.1.3}$$

and f_n^k, g_n^k, are defined on $[t_k, t_{k+1}]$ by induction on k and according to formulas (2.1.4) – (2.1.8), where $\mathcal{A}$ and $\mathcal{B}$ are defined by (3.1.1) – (3.1.2). This definition is meaningful because the solutions of (2.1.5), (2.1.6) and of (2.1.7), (2.1.8) are such that f_n^k, g_n^k belong to $C([t_k, t_{k+1}]; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$. Then, we define

$$f_n(t) = f_n^k(t), \tag{3.1.4}$$

$$g_n(t) = g_n^k(t), \tag{3.1.5}$$

for every t lying in $[t_k, t_{k+1}[$. The functions f_n and g_n defined by (3.1.4) and (3.1.5) are piecewise continuous with respect to the time variable on $[0,T]$ with values in $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$, and their discontinuities appear at each point t_k for k in $[1,n]$.

The main result of this section is the following:

Theorem 2: *We suppose that assumption 2 on the initial datum holds. Then, the sequences f_n and g_n defined in (3.1.4) – (3.1.5) converge up to extraction to the same limit f in $L^\infty([0,T]; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ weak * . This limit f satisfies equation (1.12) – (1.17) in the sense of distributions.*

Remark: This property exactly means that formula (1.23) holds for $\mathcal{A}$ and $\mathcal{B}$ defined in (3.1.1) – (3.1.2).

The proof of this theorem is given in subsections 3.2 and 3.3 .

3.2 Estimates on f_n and g_n

We proceed as in subsection 2.2, and give for f_n and g_n the following equation:

$$\frac{\partial f_n}{\partial t} + v \cdot \nabla_x f_n = \sum_{i=1}^{n} \int_{(i-1)\Delta T}^{i\Delta T} \Big\{ M[g_n](s) - g_n(s) \Big\} \, ds \; \delta_{i\Delta T}. \tag{3.2.1}$$

We now wish to pass to the limit in equation (3.2.1), we give therefore some estimates on the different terms of the equation.

Lemma 8: *The sequences f_n and g_n defined in theorem 2 satisfy the following bound for some nonnegative constant C_T:*

$$\sup_{t\in[0,T]} \int_{x\in\mathbb{R}^3} \int_{v\in\mathbb{R}^3} f_n(t,x,v)\{1+|v|^2+|x|^2+|\log f_n(t,x,v)|\} \, dvdx \le C_T, \tag{3.2.2}$$

$$\sup_{t\in[0,T]} \int_{x\in\mathbb{R}^3} \int_{v\in\mathbb{R}^3} g_n(t,x,v)\{1+|v|^2+|x|^2+|\log g_n(t,x,v)|\} \, dvdx \le C_T, \tag{3.2.3}$$

$$\sup_{t\in[0,T]} \int_{x\in\mathbb{R}^3} \int_{v\in\mathbb{R}^3} M[g_n](t,x,v)$$
$$\times \{1+|v|^2+|x|^2+|\log M[g_n](t,x,v)|\} \, dvdx \le C_T. \tag{3.2.4}$$

Proof: The proof is quite similar to that of lemma 1, we only present here the estimate relative to the entropy. Denoting $H(s) = s \log s$, we get during the transport step:

$$\frac{d}{dt} \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} H(f_n)\, dxdv = 0 \qquad (3.2.5)$$

for all t in $[t_k, t_{k+1}[$, whereas during the collision step we have

$$\frac{d}{dt} \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} H(g_n)\, dxdv = \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} (M[g_n] - g_n)\, H'(g_n)\, dxdv. \qquad (3.2.6)$$

Therefore,

$$\begin{aligned} \frac{d}{dt} \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} H(g_n)\, dxdv = \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} & (M[g_n] - g_n) \\ & \times (H'(g_n) - H'(M[g_n])\, dxdv \\ + \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} & (M[g_n] - g_n)\, (1 + \log M[g_n]) dxdv. \end{aligned} \qquad (3.2.7)$$

But because of the definition of $M[g_n]$,

$$\int_{v \in \mathbb{R}^3} \Big(M[g_n](t,x,v) - g_n(t,x,v) \Big)\, \psi(v)\, dv = 0 \qquad (3.2.8)$$

when $\psi \in \mathrm{Vect}\Big\{ 1, v_i, |v|^2 \Big\}$. Therefore, for $t \in [t_k, t_{k+1}]$,

$$\frac{d}{dt} \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} H(g_n)(t,x,v)\, dvdx \leq 0. \qquad (3.2.9)$$

Using now the convexity of H, we get

$$H(g_n) \geq H(M[g_n]) + (g_n - M[g_n])\, (1 + \log M[g_n]), \qquad (3.2.10)$$

and thus

$$\int\int_{\mathbb{R}^3 \times \mathbb{R}^3} H(M(g_n))\, dvdx \leq \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} H(g_n)\, dvdx. \qquad (3.2.11)$$

The passage from $g_n \log g_n$ to $g_n |\log g_n|$ in estimate (3.2.3) – (3.2.4) is then classical.

The previous estimates can be improved in the following way:

Lemma 9: *There exists a nonnegative convex and nondecreasing function Φ defined on $[0,+\infty[$ such that*

$$\frac{\Phi(\tau)}{\tau} \underset{\tau\to+\infty}{\longrightarrow} +\infty, \tag{3.2.12}$$

$$\Phi(\lambda\tau) \le (1+\lambda^2)\,\Phi(\tau) \tag{3.2.13}$$

for all $\tau,\lambda > 0$ and

$$\sup_{t\in[0,T]} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \Phi(|v|^2)\, g_n\, dvdx \le C_T. \tag{3.2.14}$$

This lemma will be a consequence of the following results, first given by E. Ringeisen (Cf. [Ri]):

Lemma 10 (E. Ringeisen): *Let f_0 be a nonnegative function in $L^1((1+|v|^2)dvdx)$. Then there exists a function Φ satisfying the assumptions of lemma 9 and $C_1 > 0$ such that*

$$\int_{\mathbb{R}^3} \Phi(|v|^2)\, f_0(x,v)\, dxdv \le C_1. \tag{3.2.15}$$

Lemma 11 (E. Ringeisen): *Let ϕ be a nonnegative convex function defined on $[0,+\infty[$ and satisfying $\phi(\lambda\tau) \le P(\lambda)\,\phi(\tau)$ for all $\lambda,\tau \ge 0$, where P is a polynomial.*

Then we can find a constant $C_0 > 1$ such that for every nonnegative function h in $L^1(\mathbb{R}^3;(1+|v|^2)dv)$, the inequality

$$\int_{\mathbb{R}^3} \phi(|v|^2)\, M[h](v)\, dv \le C_0 \int_{\mathbb{R}^3} \phi(|v|^2) h(v) dv \tag{3.2.16}$$

holds.

Proof of Lemma 9: During the transport step, we can write

$$\frac{d}{dt}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \Phi(|v|^2)\, f_n^k\, dxdv = 0, \tag{3.2.17}$$

whereas during the collisions step we get

$$\frac{d}{dt}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\Phi(|v|^2)\,g_n^k(t)\,e^t\,dxdv=\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\Phi(|v|^2)\,M[g_n^k(t)]\,e^t\,dxdv$$

$$\leq C_0\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\Phi(|v|^2)\,g_n^k(t)\,e^t\,dxdv. \qquad (3.2.18)$$

Thanks to Gronwall lemma, we get for all $t\in[t_k,t_{k+1}]$ (and according to lemma 11),

$$\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\Phi(|v|^2)\,g_n^k(t)\,dxdv\leq e^{(C_0-1)(t-t_k)}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\Phi(|v|^2)\,g_n^k(t_k)\,dxdv. \qquad (3.2.19)$$

Then by induction and using (3.2.17) (note that the first step of the induction is given by lemma 10), we get

$$\sup_{t\in[0,T]}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\Phi(|v|^2)\,g_n(t)\,dxdv\leq e^{(C_0-1)\,T}\,C_1. \qquad (3.2.20)$$

According to lemmas 8 and 9, we can extract from the sequences f_n, g_n and $M[g_n]$ subsequences still denoted by f_n, g_n and $M[g_n]$, which converge weakly * in $L^\infty([0,T];L^1(\mathbb{R}^3\times\mathbb{R}^3))$ respectively to f, g and M.

3.3 Averaging and passing to the limit

We now give the averaging lemma allowing us to pass to the limit in eq. (3.2.1).

Lemma 12: *The weak limits f and g are equal. Moreover, the macroscopic quantities $\rho_n=\int_{v\in\mathbb{R}^3}g_n dv$, $\rho_n u_n=\int_{v\in\mathbb{R}^3}g_n v dv$ and $\rho_n|u_n|^2+3\rho_n\mathcal{T}_n=\int_{v\in\mathbb{R}^3}g_n|v|^2dv$ are strongly compact in $L^1([0,T]\times\mathbb{R}^3)$.*

Proof: We get the result of lemma 4 with exactly the same proof. (In fact the proof can be simplified a little since it is not necessary here to renormalize the equation. Note also that step 6 is not used here).

Then, lemma 9 ensures that the same result also holds when ψ does not increase more rapidly than a quadratic function.

Note then that (up to extraction) the quantities ρ_n, u_n and $\mathcal{T}_n$ pass to the limit a.e. and converge (when $\rho\neq 0$) towards ρ, u and $\mathcal{T}$. Finally, $M[g_n]$ converges to $M[f]$ a.e. and therefore in L^1 strong (since it is weakly compact in L^1). Using then lemma 7, one can pass to the limit in (3.2.1) and prove theorem 2.

Remark: The convergence in theorem 2 is in fact strong in L^1. Namely, one can prove that $\log(1+f_n)$ tends weakly towards $\log(1+f)$.

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