SOME APPLICATIONS OF THE METHOD OF MOMENTS FOR THE HOMOGENEOUS BOLTZMANN AND KAC EQUATIONS

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Abstract

Using the method of moments, we prove that any polynomial moment of the solution of the homogeneous Boltzmann equation with hard potentials or hard spheres is bounded as soon as a moment of order strictly higher than 2 exists initially. We also give partial results of convergence towards the Maxwellian equilibrium in the case of soft potentials. Finally, exponential as well as Maxwellian estimates are introduced for the Kac equation.

1 Introduction

The spatially homogeneous Boltzmann equation of rarefied gas dynamics writes

$$\frac{\partial f}{\partial t}(t,v) = Q(f)(t,v), \tag{1.1}$$

where f is a nonnegative function of the time t and the velocity v, and Q is a quadratic collision kernel taking in account any collisions preserving momentum and kinetic energy:

$$Q(f)(t,v) = \int_{v_1 \in I\!R^3} \int_{\omega \in S^2} \{ f(t,v')\, f(t,v_1') - f(t,v)\, f(t,v_1) \}$$

$$B(|v - v_1|, |\omega \cdot \frac{v - v_1}{|v - v_1|}|)\, d\omega dv_1, \tag{1.2}$$

with

$$v' = v - (\omega \cdot (v - v_1))\,\omega, \tag{1.3}$$

$$v_1' = v_1 + (\omega \cdot (v - v_1))\,\omega, \tag{1.4}$$

and the nonnegative cross section B depends on the type of interaction between molecules (Cf. [Ce], [Ch, Co], [Tr, Mu]).

In a gas of hard spheres, the cross section is

$$B(x, y) = x\,y. \tag{1.5}$$

However, for inverse s^{th} power forces with angular cut–off (Cf. [Ce], [Gr]),

$$B(x, y) = x^{\alpha}\,\beta(y), \tag{1.6}$$

where $\alpha = \frac{s-5}{s-1}$, and there exists $\beta_1 > 0$ such that for a.e. $y \in [0, 1]$,

$$0 < \beta(y) \leq \beta_1. \tag{1.7}$$

When $s > 5$, the potentials are said to be hard and $0 < \alpha < 1$. But when $3 < s < 5$, the potentials are said to be soft and $-1 < \alpha < 0$. The intermediate case when $s = 5$ is called "Maxwellian molecules" and makes exact computations possible (Cf. [Tr], [Tr, Mu] and [Bo]).

Since hard and soft potentials are fairly involved, (the function β is defined implicitly), engineers often use in numerical computations the simpler variable hard spheres (VHS) model, in which

$$B(x, y) = x^{\alpha}\,y, \tag{1.8}$$

and $0 < \alpha \leq 1$. Note that, at least formally, for every function $\psi(v)$,

$$\int_{v \in I\!R^3} Q(f)(t, v)\psi(v)dv = \int_{v \in I\!R^3} \int_{v_1 \in I\!R^3} \int_{\omega \in S^2} \{\psi(v') - \psi(v)\} f(t, v) f(t, v_1)$$
$$B(|v - v_1|, |\omega \cdot \frac{v - v_1}{|v - v_1|}|)\, d\omega dv_1 dv, \tag{1.9}$$

and also

$$\int_{v \in I\!R^3} Q(f)(t, v)\psi(v)dv = -\frac{1}{4} \int_{v \in I\!R^3} \int_{v_1 \in I\!R^3} \int_{\omega \in S^2} \{\psi(v') + \psi(v_1')$$

$$- \psi(v) - \psi(v_1)\} \{f(t,v')\, f(t,v_1') - f(t,v)\, f(t,v_1)\}$$
$$B(|v - v_1|, |\omega \cdot \frac{v - v_1}{|v - v_1|}|)\, d\omega dv_1 dv. \tag{1.10}$$

When $\psi(v) = 1, v, \frac{|v|^2}{2}$ in (1.10), one obtains the conservation of mass, momentum and energy for the Boltzmann kernel:

$$\int_{v \in I\!R^3} Q(f)(t,v)\,(1, v, \frac{|v|^2}{2})\, dv = 0. \tag{1.11}$$

Moreover, using (1.10) with $\psi = \log f$, one obtains the entropy estimate:

$$\int_{v \in I\!R^3} Q(f)(t,v)\, \log f(t,v)\, dv \le 0. \tag{1.12}$$

According to [A 1], [A 2], for any of the cross sections previously presented, there exists a nonnegative solution $f(t,v)$ of eq. (1.1) satisfying $f(0,v) = f_0(v)$ as soon as f_0 is nonnegative and

$$\int_{v \in I\!R^3} f_0(v)(1 + \frac{|v|^r}{2} + |\log f_0(v)|)\, dv < +\infty \tag{1.13}$$

for some $r > 2$. Moreover, estimates (1.11) and (1.12) hold for this solution, and therefore f satisfies

$$\int_{v \in I\!R^3} f(t,v)\,(1, v, \frac{|v|^2}{2})\, dv = \int_{v \in I\!R^3} f_0(v)\,(1, v, \frac{|v|^2}{2})\, dv, \tag{1.14}$$

$$\int_{v \in I\!R^3} f(t,v)\, \log f(t,v)\, dv \le \int_{v \in I\!R^3} f_0(v)\, \log f_0(v)\, dv \tag{1.15}$$

when $t \ge 0$.

Note that condition (1.13) can be relaxed by taking $r = 2$ for the proof of existence, but in that case (1.14) may not hold (at least for soft potentials). Note also the results in [DP, L 1] of existence and weak stability for the inhomogeneous equation.

In this work, when we consider solutions of the Boltzmann equation (1.1), it will always be the nonnegative solutions of [A 1] or [A 2].

It is now well–known that in the case of VHS models (including hard spheres) and hard potentials (including Maxwellian molecules), the moments of the solution of the Boltzmann equation

$$l_r(t) = \int_{v \in I\!R^3} f(t,v)\, |v|^r\, dv \tag{1.16}$$

for $r > 2$, are bounded on $[0, +\infty[$ as soon as they exist at time $t = 0$ (Cf. [El 1]).

The same estimate holds for soft potentials, except that $l_r(t)$ is bounded only on $[0, T]$ for $T > 0$ and may blow up when t goes to infinity (Cf. [A 2]).

Note finally that the case of Maxwellian molecules is treated extensively in [Tr, Mu] and [Bo].

We shall prove in section 2 that in fact, for VHS models (including hard spheres) as well as in the case of hard potentials (but not including Maxwellian molecules) and under assumption (1.13), the moments $l_q(t)$ (for $q > 2$) are bounded on $[\bar{t}, +\infty[$ (for any $\bar{t} > 0$). In other words, every polynomial moments of f exist for $t > 0$ as soon as one of them (of order strictly higher than 2) exists initially.

In section 3, we give some estimates for the solution f of eq. (1.1) with soft potentials. We write the cross section B under the form

$$B(x, y) = x^{-\gamma}\, \beta(y), \tag{1.17}$$

with $\gamma > 0$ ($\gamma = -\alpha$ in eq. (1.6)).

We prove that as soon as $l_r(0)$ exists (with $r > 2$), we can find $K_0 > 0$ such that

$$l_r(t) \leq K_0\, t + K_0. \tag{1.18}$$

This estimate is a little more explicit than that of [A 2]. Moreover, we get also

$$\int_0^t l_{r-\gamma}(s)\, ds \leq K_0\, t + K_0, \tag{1.19}$$

which means that $l_{r-\gamma}$ is bounded in the Cesaro sense. Note that the same kind of estimates can be found in [Pe 1] and [Pe 2], in a linear context. Note also that the estimates can be derived from the works of Elmroth (Cf. [El 1] and [El 2]). However, we give here for the sake of completness a self–contained proof.

These estimates are then used to prove partial results of convergence towards the equilibrium when t goes to infinity (the reader can find a survey on this subject in [De 2]).

Finally, in section 4, we introduce Kac's model (Cf. [K], [MK]) and, using monotony results, we prove exponential and Maxwellian estimates for its solution.

2 Hard potentials

The bounds that we present in this work are based on formula (1.9). The exploitation of this estimate is called "method of moments". We begin by putting (1.9) under a new form.

Writing

$$\omega = \cos\theta \, \frac{v_1 - v}{|v_1 - v|} + \sin\theta \, (\cos\phi \, i_{v,v_1} + \sin\phi \, j_{v,v_1}), \tag{2.1}$$

where

$$\Big(\frac{v_1 - v}{|v_1 - v|}, i_{v,v_1}, j_{v,v_1} \Big) \tag{2.2}$$

is an orthonormal basis of $I\!R^3$, estimate (1.9) becomes

$$\int_{v\in I\!R^3} Q(f)(t,v)\, \psi(v)\, dv = \int_{v\in I\!R^3} \int_{v_1\in I\!R^3} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2}$$

$$\Big\{ \psi(v + \cos\theta \, |v - v_1| \, \{\cos\theta \, \frac{v_1 - v}{|v_1 - v|} + \sin\theta \, (\cos\phi \, i_{v,v_1} + \sin\phi \, j_{v,v_1})\}) - \psi(v) \Big\}$$

$$f(t,v)\, f(t,v_1)\, 2\sin\theta \, B(|v - v_1|, \cos\theta)\, d\theta d\phi dv_1 dv. \tag{2.3}$$

Introducing in eq. (2.3) the change of variables $\theta = \frac{\delta}{2}$, and defining

$$R_{\delta,\phi}(\frac{v_1 - v}{|v_1 - v|}) = \cos\delta \, \frac{v_1 - v}{|v_1 - v|} + \sin\delta \, (\cos\phi \, i_{v,v_1} + \sin\phi \, j_{v,v_1}), \tag{2.4}$$

one obtains

$$\int_{v\in I\!R^3} Q(f)(t,v)\, \psi(v)\, dv$$

$$= \int_{v\in I\!R^3} \int_{v_1\in I\!R^3} \int_{\phi=0}^{2\pi} \int_{\delta=0}^{\pi} \Big\{ \psi\Big(\frac{v + v_1}{2} + \frac{|v - v_1|}{2} R_{\delta,\phi}(\frac{v_1 - v}{|v_1 - v|}) \Big) - \psi(v) \Big\}$$

$$f(t,v)\, f(t,v_1)\, \sin\frac{\delta}{2} \, B(|v - v_1|, \cos\frac{\delta}{2})\, d\delta d\phi dv_1 dv, \tag{2.5}$$

which is in fact a classical form for the Boltzmann collision term (Cf. [Bo] or [De 3] for example). We state now three useful lemmas:

Lemma 1: *Assume that $\epsilon > 0$ and that Λ is a strictly positive function of $L^\infty([0,\pi])$. Then, there exists $K_1 > 0$ and two functions $T_1(v,v_1), T_2(v,v_1)$ such that*

$$W(v,v_1) = \int_{\phi=0}^{2\pi} \int_{\delta=0}^{\pi} \Big| 1 + \frac{|v - v_1||v + v_1|}{v^2 + v_1^2}$$

$$\times \left(R_{\delta,\phi}(\frac{v_1 - v}{|v_1 - v|}) \cdot \frac{v + v_1}{|v + v_1|} \right) \Big|^{1+\epsilon} \Lambda(\delta)\, d\delta d\phi$$

$$= T_1(v, v_1) + T_2(v, v_1), \tag{2.6}$$

with

$$T_1(v, v_1) = -T_1(v_1, v) \tag{2.7}$$

and

$$0 \le T_2(v, v_1) \le K_1 < 2^{1+\epsilon}\, \pi \int_{\delta=0}^{\pi} \Lambda(\delta)\, d\delta. \tag{2.8}$$

Proof of lemma 1: We take the following notations for $i = 1, 2$:

$$T_i(v, v_1) = \int_{\phi=0}^{2\pi} \int_{\delta=0}^{\pi} \chi_i \Big(\frac{|v - v_1||v + v_1|}{v^2 + v_1^2} \{ R_{\delta,\phi}(\frac{v_1 - v}{|v_1 - v|}) \cdot \frac{v + v_1}{|v + v_1|} \} \Big) \Lambda(\delta) d\delta d\phi, \tag{2.9}$$

with

$$\chi_i(x) = \frac{(1 + x)^{1+\epsilon} + (-1)^i (1 - x)^{1+\epsilon}}{2}. \tag{2.10}$$

We can see that

$$W(v, v_1) = T_1(v, v_1) + T_2(v, v_1), \tag{2.11}$$

and

$$T_1(v, v_1) = -T_1(v_1, v). \tag{2.12}$$

But χ_2 is even, strictly increasing from $x = 0$ to $x = 1$, and

$$\chi_2(0) = 1, \qquad \chi_2(1) = 2^{\epsilon}. \tag{2.13}$$

Therefore, using the inequality

$$|v - v_1||v + v_1| \le v^2 + v_1^2, \tag{2.14}$$

we obtain the estimate

$$0 \le T_2(v, v_1) \le 2^{1+\epsilon}\, \pi \int_{\delta=0}^{\pi} \Lambda(\delta)\, d\delta. \tag{2.15}$$

Then, a simple argument of compactness ensures that lemma 1 holds.

We now prove the second lemma.

Lemma 2: *Assume that $\epsilon > 0$ and that the cross section B in (1.2) satisfies*

$$B(x,y) = B_0(x) B_1(y), \tag{2.16}$$

where $B_1 \in L^\infty([0,\pi])$ is strictly positive. Then, there exists $K_2 > 0$ and $K_3 \in]0,1[$ such that

$$\int_{v\in \mathbb{R}^3} Q(f)(t,v)\,|v|^{2+2\epsilon} dv \le K_2 \int_{v\in \mathbb{R}^3}\int_{v_1\in \mathbb{R}^3} \{\frac{K_3}{2}(v^2+v_1^2)^{1+\epsilon} - |v|^{2+2\epsilon}\}$$
$$\times f(t,v)\, f(t,v_1)\, B_0(|v-v_1|)\, dv_1 dv. \tag{2.17}$$

Proof of lemma 2: According to eq. (2.5), for $\epsilon > 0$,

$$\int_{v\in \mathbb{R}^3} Q(f)(t,v)\,|v|^{2+2\epsilon} dv = \int_{v\in \mathbb{R}^3}\int_{v_1\in \mathbb{R}^3} \Big\{ (\frac{v^2+v_1^2}{2})^{1+\epsilon} \int_{\phi=0}^{2\pi}\int_{\delta=0}^{\pi}$$
$$\left| 1 + \frac{|v-v_1||v+v_1|}{v^2+v_1^2} \Big(R_{\delta,\phi}(\frac{v_1-v}{|v_1-v|}) \cdot \frac{v+v_1}{|v+v_1|}\Big) \right|^{1+\epsilon} \sin\frac{\delta}{2}\, B_1(\cos\frac{\delta}{2})\, d\delta d\phi$$
$$- |v|^{2+2\epsilon} \int_{\phi=0}^{2\pi}\int_{\delta=0}^{\pi} \sin\frac{\delta}{2}\, B_1(\cos\frac{\delta}{2})\, d\delta d\phi \Big\}\, f(t,v)\, f(t,v_1)\, B_0(|v-v_1|)\, dv_1 dv. \tag{2.18}$$

Moreover, using lemma 1 with

$$\Lambda(\delta) = \sin\frac{\delta}{2}\, B_1(\cos\frac{\delta}{2}), \tag{2.19}$$

we have

$$\int_{v\in \mathbb{R}^3} Q(f)(t,v)\,|v|^{2+2\epsilon} dv$$
$$\le \int_{v\in \mathbb{R}^3}\int_{v_1\in \mathbb{R}^3} \Big\{ (\frac{v^2+v_1^2}{2})^{1+\epsilon} (K_1 + T_1(v,v_1))$$
$$- |v|^{2+2\epsilon} \int_{\phi=0}^{2\pi}\int_{\delta=0}^{\pi} \sin\frac{\delta}{2}\, B_1(\cos\frac{\delta}{2})\, d\delta d\phi \Big\}\, f(t,v)\, f(t,v_1)\, B_0(|v-v_1|)\, dv_1 dv, \tag{2.20}$$

with K_1 and $T_1(v,v_1)$ as in lemma 1. Therefore, taking

$$K_2 = 2\pi \int_{\delta=0}^{\pi} \sin\frac{\delta}{2}\, B_1(\cos\frac{\delta}{2})\, d\delta, \tag{2.21}$$

$$K_3 = \frac{K_1}{2^{1+\epsilon}\pi} \Big(\int_{\delta=0}^{\pi} \sin\frac{\delta}{2}\, B_1(\cos\frac{\delta}{2})\, d\delta \Big)^{-1} < 1, \tag{2.22}$$

and using the change of variables $(v, v_1) \longrightarrow (v_1, v)$, we obtain lemma 2.

We prove now the last lemma.

Lemma 3: *Let B and ϵ be as in lemma 2. Then, there exist $K_4, K_5 > 0$ such that*

$$\int_{v \in I\!\!R^3} Q(f)(t,v)\, |v|^{2+2\epsilon} dv$$

$$\leq -K_4 \int_{v \in I\!\!R^3} \int_{v_1 \in I\!\!R^3} |v|^{2+2\epsilon} f(t,v)\, f(t,v_1)\, B_0(|v-v_1|)\, dv_1 dv$$

$$+K_5 \int_{v \in I\!\!R^3} \int_{v_1 \in I\!\!R^3} |v|^2 |v_1|^{2\epsilon} f(t,v)\, f(t,v_1)\, B_0(|v-v_1|)\, dv_1 dv. \tag{2.23}$$

Proof of lemma 3: Note that there exists $K_6 > 0$ such that

$$(v^2 + v_1^2)^{1+\epsilon} \leq |v|^{2+2\epsilon} + |v_1|^{2+2\epsilon} + K_6\, (|v|^2 |v_1|^{2\epsilon} + |v|^{2\epsilon} |v_1|^2). \tag{2.24}$$

Using lemma 2 and the change of variables $(v, v_1) \longrightarrow (v_1, v)$, one easily obtains lemma 3 with $K_4 = K_2\,(1 - K_3)$ and $K_5 = K_2\, K_3\, K_6$.

We now come to the main theorem of this section.

Theorem 1: *Let f_0 satisfying (1.13) be a nonnegative initial datum for the Boltzmann equation (1.1) with hard potentials (but not with Maxwellian molecules) or with the VHS model (including hard spheres). We denote by $f(t,v)$ a solution of the equation with this initial datum.*

Then, for all $r' > 0, \bar{t} > 0$, there exists $C(r', \bar{t}\,) > 0$ such that

$$\int_{v \in I\!\!R^3} f(t,v)\, |v|^{r'}\, dv \leq C(r', \bar{t}\,) \tag{2.25}$$

when $t \geq \bar{t}$.

Proof of theorem 1: According to (1.6) and (1.8), the cross section for hard potentials (but not Maxwellian molecules) or for the VHS model (including hard spheres) is of the form (2.16) with $B_0(x) = |x|^{\alpha}$, and $\alpha \in]0, 1]$. Therefore, we can apply lemma 3. For $\epsilon > 0$, we write

$$\int_{v\in\mathbb{R}^3} Q(f)(t,v)\,|v|^{2+2\epsilon}\,dv$$

$$\le -K_4 \int_{v\in\mathbb{R}^3}\int_{v_1\in\mathbb{R}^3} |v|^{2+2\epsilon}\,|v-v_1|^{\alpha}\,f(t,v)\,f(t,v_1)\,dv_1 dv$$

$$+ K_5 \int_{v\in\mathbb{R}^3}\int_{v_1\in\mathbb{R}^3} |v|^{2}\,|v_1|^{2\epsilon}\,|v-v_1|^{\alpha}\,f(t,v)\,f(t,v_1)\,dv_1 dv \tag{2.26}$$

$$\le -K_4\, 2^{-\alpha}\, l_{2+2\epsilon+\alpha}(t)\, l_0(t) + K_4\, 2^{2-\alpha}\, l_2(t)\, l_{2\epsilon+\alpha}(t)$$

$$+ K_5\, l_{2+\alpha}(t)\, l_{2\epsilon}(t) \;+\; K_5\, l_{2\epsilon+\alpha}(t)\, l_2(t), \tag{2.27}$$

with the notation (1.16).

Since f is solution of eq. (1.1), the conservations of mass and energy (1.14) ensure that for $\theta \in [0,2]$, $l_\theta(t)$ is bounded (for $t \ge 0$). Therefore, there exist $K_7, K_8, K_9 > 0$ such that

$$\int_{v\in\mathbb{R}^3} Q(f)(t,v)\,|v|^{2+2\epsilon}\,dv \le -K_7\, l_{2+2\epsilon+\alpha}(t) + K_8\, l_{2+\alpha}(t)\, l_{2\epsilon}(t) + K_9\, l_{2\epsilon+\alpha}(t). \tag{2.28}$$

Remember that $\bar t > 0$ and $r > 2$ are given in the hypothesis of theorem 1 (r is defined in (1.13)). We can always suppose that $r \le 4$. We prove in a first step that there exists $t_0 \in]0, \bar t\,[$ such that $l_{r+\alpha}(t)$ is bounded on $[t_0, +\infty[$.

According to Hölder's inequality, when $0 < \mu < \nu$,

$$l_\mu(t) \le l_0^{1-\mu/\nu}(t)\, l_\nu^{\mu/\nu}(t). \tag{2.29}$$

Therefore, using estimate (2.28) with $\epsilon = \frac{r}{2} - 1$, one obtains

$$\int_{v\in\mathbb{R}^3} Q(f)(t,v)\,|v|^{r}\,dv \le -\, K_7\, l_{r+\alpha}(t)$$

$$+ K_8\, l_{r+\alpha}^{\frac{2+\alpha}{r+\alpha}}(t)\, l_0^{1-\frac{2+\alpha}{r+\alpha}}(t)\; l_{r-2}(t) \;+\; K_9\, l_{r+\alpha}^{\frac{r+\alpha-2}{r+\alpha}}(t)\, l_0^{1-\frac{r+\alpha-2}{r+\alpha}}(t). \tag{2.30}$$

Remember that since $r-2 \in]0,2]$, the moments $l_0(t)$ and $l_{r-2}(t)$ are bounded on $[0,+\infty[$. Moreover, we can find $K_{10}, K_{11} > 0$ such that when $x \ge 0$, $t \ge 0$,

$$-K_7\, x + K_8\, x^{\frac{2+\alpha}{r+\alpha}}\, l_0^{1-\frac{2+\alpha}{r+\alpha}}(t) l_{r-2}(t) + K_9\, x^{\frac{r+\alpha-2}{r+\alpha}}\, l_0^{1-\frac{r+\alpha-2}{r+\alpha}}(t) \le -K_{10}\, x + K_{11}. \tag{2.31}$$

Therefore,

$$\int_{v\in\mathbb{R}^3} Q(f)(t,v)\,|v|^{r}\,dv \le -K_{10}\, l_{r+\alpha}(t) + K_{11}. \tag{2.32}$$

Integrating the Boltzmann equation (1.1) on $[0,\bar{t}\,] \times I\!\!R^3$ against $|v|^r$ and using estimate (2.32), one gets

$$l_r(\bar{t}\,) + K_{10} \int_0^{\bar{t}} l_{r+\alpha}(s)\, ds \le K_{11}\, \bar{t} + l_r(0). \qquad (2.33)$$

According to (1.13) and (2.33), we can see that there exists $t_0 \in]0,\bar{t}\,[$ such that $l_{r+\alpha}(t_0) < +\infty$. But it is well known that if a moment exists at a given time t_0, then it is bounded for $t \ge t_0$ (Cf. [El 1] or the remark at the end of section 2), therefore $l_{r+\alpha}(t)$ is bounded for $t \ge t_0$.

We now come back to estimate (2.28). Using equation (2.29), an estimate similar to (2.31) and the result of boundedness for $l_{r+\alpha}(t)$, one obtains $K_{12}, K_{13} > 0$ such that

$$\int_{v \in I\!\!R^3} Q(f)(t,v)\, |v|^{2+2\epsilon}\, dv \le -K_{12}\, l_{2+2\epsilon+\alpha}(t) + K_{13} \qquad (2.34)$$

for $t \ge t_0$.

We now integrate (when $t_0 \le t_- < \bar{t}\,$) the Boltzmann equation (1.1) on $[t_-,\bar{t}\,] \times I\!\!R^3$ against $|v|^{2+2\epsilon}$ and we use estimate (2.34) to obtain

$$l_{2+2\epsilon}(\bar{t}\,) + K_{12} \int_{t_-}^{\bar{t}} l_{2+2\epsilon+\alpha}(s)\, ds \le K_{13}\, (\,\bar{t} - t_-) + l_{2+2\epsilon}(t_-). \qquad (2.35)$$

Therefore, if $t_0 \le t_-$ and if $l_{2+2\epsilon}(t_-) < +\infty$, there exists $\tau \in [t_-,\bar{t}\,[$ such that $l_{2+2\epsilon+\alpha}(\tau) < +\infty$.

Finally, we note that any moment is bounded on $[\tau, +\infty[$ as soon as it is defined at time τ (Cf. [El 1]), and we use a proof by induction to get theorem 1.

Remark: Note that using eq. (2.35), we can produce explicitly the maximum principle for $l_{2+2\epsilon}$. Namely when $\epsilon > 0$, estimate (2.29) ensures that there exists $K_{14} > 0$ such that

$$\frac{d}{dt} l_{2+2\epsilon}(t) \le -K_{14}\, l_{2+2\epsilon}^{\frac{2+2\epsilon+\alpha}{2+\epsilon}}(t) + K_{13}, \qquad (2.36)$$

which gives

$$l_{2+2\epsilon}(t) \le \sup \Big(l_{2+2\epsilon}(t_-), (\frac{K_{13}}{K_{14}})^{\frac{2+2\epsilon}{2+2\epsilon+\alpha}} \Big), \qquad (2.37)$$

for $t \geq t_-$ (this is another proof of the result of [El 1]).

Remark: Theorem 1 can be applied in a lot of situations. For example, it allows to simplify the results on exponential convergence towards equilibrium stated in [A 3]. Namely, the hypothesis used in [A 3] is that there exists enough moments initially bounded.

Note also the recent application of this theorem by Wennberg in [We] to the problem of uniqueness of the solution of the Boltzmann equation with hard potentials.

Finally, note that in the same work, Wennberg proves a similar theorem in an $L^p \cap L^1$ setting.

3 Soft potentials

We consider in this section the Boltzmann equation (1.1) with a cross section B of the form

$$B(x,y) = x^{-\gamma}\,\beta(y), \tag{3.1}$$

with $\gamma > 0$ ($\gamma = -\alpha$ in formula (1.6)), and β satisfying (1.7). This is exactly the hypothesis of soft potentials.

We begin by proving the

Theorem 2: *We consider the operator Q defined in (1.2) with B satisfying (3.1). Then for $\epsilon > 0$, there exist $K_{20}, K_{21} > 0$ such that*

$$\int_{v\in I\!\!R^3} Q(f)(t,v)\,|v|^{2+2\epsilon}\,dv \leq K_{20} - K_{21}\int_{v\in I\!\!R^3} f(t,v)\,|v|^{2+2\epsilon-\gamma}\,dv \tag{3.2}$$

when $f(t,v)$ satisfies the conservations of mass and energy (1.14). (The constants K_{20} and K_{21} depend in fact of this mass and this energy).

Proof of theorem 2: According to eq. (2.5), for $\epsilon > 0$,

$$\int_{v\in I\!\!R^3} Q(f)(t,v)|v|^{2+2\epsilon}dv$$

$$= \int_{v\in I\!\!R^3}\int_{v_1\in I\!\!R^3}\int_{\phi=0}^{2\pi}\int_{\delta=0}^{\pi}\left\{\left|\frac{v+v_1}{2}+\frac{|v-v_1|}{2}R_{\delta,\phi}(\frac{v_1-v}{|v_1-v|})\right|^{2+2\epsilon} - |v|^{2+2\epsilon}\right\}$$

$$f(t,v)\,f(t,v_1)\,|v-v_1|^{-\gamma}\,\sin\frac{\delta}{2}\,\beta(\cos\frac{\delta}{2})\,d\delta d\phi dv_1 dv. \tag{3.3}$$

We make the change of variables $u = v_1 - v$, and consider the integral in (3.3) when $|u| \geq \frac{1}{2}$ and when $|u| \leq \frac{1}{2}$. Then, we use lemma 3 for the first term and get

$$\int_{v\in \mathbb{R}^3} Q(f)(t,v)\,|v|^{2+2\epsilon}\,dv$$

$$\leq - K_4 \int_{v\in \mathbb{R}^3}\int_{v_1\in \mathbb{R}^3} |v|^{2+2\epsilon}\, f(t,v)\, f(t,v_1)\, B_0(|v-v_1|)\, dv_1 dv$$

$$+ K_5 \int_{v\in \mathbb{R}^3}\int_{v_1\in \mathbb{R}^3} |v|^2\, |v_1|^{2\epsilon}\, f(t,v)\, f(t,v_1)\, B_0(|v-v_1|)\, dv_1 dv$$

$$+ (\frac{1}{2})^{1-\gamma}\,(2+2\epsilon) \int_{v\in \mathbb{R}^3}\int_{|u|\leq \frac{1}{2}}\int_{\phi=0}^{2\pi}\int_{\delta=0}^{\pi} (|v| + \frac{1}{2})^{1+2\epsilon}$$

$$\times f(t,v)\, f(t,v+u) \sin\frac{\delta}{2}\, \beta(\cos\frac{\delta}{2})\, d\delta d\phi du dv, \qquad (3.4)$$

where

$$B_0(x) = 1_{x\geq \frac{1}{2}}\, x^{-\gamma}. \qquad (3.5)$$

With the notation (1.16), one obtains after computations:

$$\int_{v\in \mathbb{R}^3} Q(f)(t,v)\,|v|^{2+2\epsilon}\,dv \leq - \frac{K_4}{4}\, l_{2+2\epsilon-\gamma}(t)\, \frac{(l_0(t))^2}{l_0(t)+l_2(t)}$$

$$+ \frac{K_4}{2}\,(l_0(t))^2 + 2\, K_4\, l_{2\epsilon-\gamma}(t)\,(l_2(t) + \frac{1}{4} l_0(t)) \;+\; K_5\, 2^\gamma\, l_2(t)\, l_{2\epsilon}(t)$$

$$+ 2^{\gamma+2\epsilon}\,(2+2\epsilon)\,\pi^2\beta_1\, l_{1+2\epsilon}(t)\, l_0(t) \;+\; 2^{\gamma-1}\,(2+2\epsilon)\,\pi^2\beta_1\,(l_0(t))^2. \qquad (3.6)$$

Since we supposed that $l_0(t) = l_0(0)$ and $l_2(t) = l_2(0)$, there exist K_{15}, K_{16}, K_{17}, K_{18}, $K_{19} > 0$ such that

$$\int_{v\in \mathbb{R}^3} Q(f)(t,v)\,|v|^{2+2\epsilon}\,dv \leq -K_{15}\, l_{2+2\epsilon-\gamma}(t) \;+\; K_{16}\, l_{1+2\epsilon}(t)$$

$$+ K_{17}\, l_{2\epsilon}(t) \;+\; K_{18}\, l_{2\epsilon-\gamma}(t) \;+\; K_{19}. \qquad (3.7)$$

Using estimate (2.29) and working as in (2.31), we obtain theorem 2.

We give now the main corollaries of this theorem

Corollary 2.1: *We suppose that $f(t,v)$ is a solution of the Boltzmann equation (1.1) with a cross section B satisfying (3.1)(i-e in the case of soft* *potentials), such that* $f(0,v) = f_0(v) \geq 0$ *and* f_0 *satisfies (1.13). Then, there exists* $K_0 > 0$ *such that*

$$\int_{v\in I\!R^3} f(t,v)\,|v|^r\,dv \leq K_0\,t + K_0 \tag{3.8}$$

(with r *defined in (1.13)).*

Proof of corollary 2.1: Integrating the Boltzmann equation (1.1) on $[0,t] \times I\!R^3$ against $|v|^r$ and using theorem 2 with $\epsilon = \frac{r}{2} - 1$, one obtains

$$\int_{v\in I\!R^3} f(t,v)|v|^r dv - \int_{v\in I\!R^3} f_0(v)|v|^r dv$$

$$\leq K_{20}\,t - K_{21}\int_0^t \int_{v\in I\!R^3} f(s,v)|v|^{r-\gamma} dvds, \tag{3.9}$$

which yields estimate (3.8) for $K_0 = \sup\,(K_{20}, l_r(0))$.

Corollary 2.2: *We suppose that* $f(t,v)$ *is a solution of the Boltzmann equation (1.1) with a cross section* B *satisfying (3.1)(i-e in the case of soft potentials), such that* $f(0,v) = f_0(v) \geq 0$ *and*

$$\int_{v\in I\!R^3} f_0(v)\,(1 + |v|^r + |\log f_0(v)|)\,dv < +\infty \tag{3.10}$$

for some $r > 2 + \gamma$*. Then, there exist* $K_0, K_{22} > 0$ *such that*

$$\int_0^t \int_{v\in I\!R^3} f(s,v)\,|v|^{r-\gamma}\,dvds \leq K_0\,t + K_0, \tag{3.11}$$

and

$$\frac{d}{dt}\int_{v\in I\!R^3} f(t,v)\,|v|^{r-\gamma}\,dv \leq K_{22}. \tag{3.12}$$

Proof of corollary 2.2: Estimate (3.11) comes out of eq. (3.9). Moreover, injecting $\epsilon = \frac{r}{2} - \frac{\gamma}{2} - 1$ in theorem 2, we immediately obtain estimate (3.12).

We now give a corollary of formulas (3.11) and (3.12), relative to the convergence towards equilibrium for the Boltzmann equation (1.1) with soft potentials.

Corollary 2.3: *We suppose that $f(t,v)$ is a solution of the Boltzmann equation (1.1) with a cross section B satisfying (3.1)(i-e in the case of soft potentials), such that $f(0,v) = f_0(v) \geq 0$ and*

$$\int_{v\in I\!\!R^3} f_0(v)(1+|v|^r + |\log f_0(v)|)\, dv < +\infty \tag{3.13}$$

*for some $r > 2+\gamma$. Then, there exists a sequence $(t_n)_{n\in I\!\!N}$ going to infinity such that for all $T>0$, $f_n(t,v) = f(t+t_n, v)$ converges in $L^\infty([0,T]; L^1(I\!\!R^3))$ weak * to the time–independant Maxwellian*

$$m(v) = \frac{\tilde{\rho}}{(2\pi\tilde{T})^{3/2}} e^{-\frac{|v-\tilde{u}|^2}{2\tilde{T}}}, \tag{3.14}$$

with

$$\tilde{\rho} = \int_{v\in I\!\!R^3} f_0(v)\, dv, \tag{3.15}$$

$$\tilde{\rho}\,\tilde{u} = \int_{v\in I\!\!R^3} v\, f_0(v)\, dv, \tag{3.16}$$

and

$$\tilde{\rho}\,\frac{|\tilde{u}|^2}{2} + \frac{3}{2}\,\tilde{\rho}\,\tilde{T} = \int_{v\in I\!\!R^3} \frac{|v|^2}{2}\, f_0(v)\, dv. \tag{3.17}$$

Proof of corollary 2.3: We first note that the solution f of the Boltzmann equation (1.1) with soft potentials satisfies the following entropy estimate:

$$\sup_{t\in[0,+\infty[} \int_{v\in I\!\!R^3} f(t,v)\,|\log f(t,v)|\, dv$$
$$+ \int_{s=0}^{+\infty} \int_{v\in I\!\!R^3} \int_{v_1\in I\!\!R^3} \int_{\omega\in S^2} \{f(s,v')\, f(s,v_1') - f(s,v)\, f(s,v_1)\}$$
$$\log\{\frac{f(s,v')\, f(s,v_1')}{f(s,v)\, f(s,v_1)}\}\, |v-v_1|^{-\gamma}\, \beta(|\omega\cdot\frac{v-v_1}{|v-v_1|}|)\, d\omega dv_1 dv ds < +\infty. \tag{3.18}$$

This inequality is obtained from (1.12), (1.14), (1.15) and (3.13) as in the space–dependant case (Cf. [DP, L 1] and [DP, L 2]).

Now according to corollary 2.2, there exists a sequence $(t_n)_{n\in I\!\!N}$ going to infinity and $\tilde{r} = r-\gamma > 2$ such that

$$\int_{v\in I\!\!R^3} f(t_n,v)\,|v|^{\tilde{r}}\, dv \leq K_0 + 1. \tag{3.19}$$

Moreover, because of estimate (3.12), we have for $t \in [0,T]$,

$$\int_{v\in I\!R^3} f(t_n+t,v)\,|v|^{\tilde r}\,dv \le K_0 + 1 + K_{22}\,T. \tag{3.20}$$

Denoting

$$\Gamma(x,y) = (x-y)\log\,(\frac{x}{y}), \tag{3.21}$$

and using estimates (3.18), (3.20) and the conservation of mass (1.14), we can find $K_{23} > 0$ such that $f_n(t,v) = f(t+t_n,v)$ satisfies:

$$\sup_{t\in[0,T]} \int_{v\in I\!R^3} f_n(t,v)\,\{1+|v|^{\tilde r} + |\log f_n(t,v)|\}\,dv \le K_{23}, \tag{3.22}$$

and

$$\int_{s=0}^{T}\int_{v\in I\!R^3}\int_{v_1\in I\!R^3}\int_{\omega\in S^2} \Gamma(f_n(s,v')\,f_n(s,v_1'), f_n(s,v)\,f_n(s,v_1))$$

$$|v-v_1|^{-\gamma}\,\beta(|\omega\cdot\frac{v-v_1}{|v-v_1|}|)\,d\omega dv_1 dv ds \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.23}$$

According to estimate (3.22), there exists a subsequence of f_n (still denoted by f_n) which converges to a limit $m(t,v)$ in $L^\infty([0,T];L^1(I\!R^3))$ weak *.

To prove that m is a Maxwellian function of v which does not depend on t, one can proceed essentially as in [De 1].

Then, one must identify $\tilde\rho, \tilde u$, and $\tilde T$.

Using the conservations of mass, impulsion and energy (1.14), one gets for all $t \in [0,T]$,

$$\int_{v\in I\!R^3} f_n(t,v)\,(1,v,\frac{|v|^2}{2})\,dv = \int_{v\in I\!R^3} f_0(v)\,(1,v,\frac{|v|^2}{2})\,dv. \tag{3.24}$$

But because of estimate (3.22),

$$\int_{t=0}^{T}\int_{v\in I\!R^3} (1,v,|v|^2)\,f_n(t,v)\,dvdt \underset{n\to+\infty}{\longrightarrow} T\int_{v\in I\!R^3}(1,v,|v|^2)\,m(v)\,dv, \tag{3.25}$$

and therefore the parameters $\tilde\rho$,$\tilde u$, $\tilde T$ are given by formulas (3.15) – (3.17).

Remark: This is only a partial result. One would expect in fact that the whole function tends when $t \longrightarrow +\infty$ to the Maxwellian given in (3.14) – (3.17). Note that this is the case when hard potentials are concerned, the convergence being even strong and exponential under suitable assumptions (Cf. [A 3]). Note also that the existence of a converging subsequence for any sequence t_n going to infinity can be derived from the papers of Arkeryd (Cf. [A 2]), but the limits in that case may have less energy than the initial datum.

3.1 The Kac equation

We introduce now the one–dimensional homogeneous Kac model (Cf. [K], [MK]), where all collisions have the same probability. The density $f(t,v) > 0$ of particles which at time t move with velocity v satisfies

$$\frac{\partial f}{\partial t}(t,v) = Q'(f)(t,v), \tag{4.1}$$

where Q' is a quadratic collision kernel:

$$Q'(f)(t,v) = \int_{v_1 \in I\!R} \int_{\theta=-\pi}^{\pi} \{f(t,v^*)f(t,v_1^*) - f(t,v)f(t,v_1)\} \frac{d\theta}{2\pi} dv_1, \tag{4.2}$$

with

$$v^* = \sqrt{v^2 + v_1^2}\,\cos\theta, \tag{4.3}$$

$$v_1^* = \sqrt{v^2 + v_1^2}\,\sin\theta. \tag{4.4}$$

It is easy to prove (at least at the formal level) the conservation of mass and energy

$$\int_{v \in I\!R} f(t,v)\,(1, \frac{|v|^2}{2})\,dv = \int_{v \in I\!R} f(0,v)\,(1, \frac{|v|^2}{2})\,dv, \tag{4.5}$$

and the entropy estimate

$$\int_{v \in I\!R} f(t,v)\, \log f(t,v)\,dv \le \int_{v \in I\!R} f(0,v)\, \log f(0,v)\,dv. \tag{4.6}$$

Adapting for example the proof of Arkeryd (Cf. [A 1] or [De 4]) for the Boltzmann equation, one can prove that as soon as $f_0 \ge 0$ satisfies

$$\int_{v \in I\!R} f_0(v)\,(1 + |v|^2 + |\log f_0(v)|)\,dv < +\infty, \tag{4.7}$$

there exists a solution of the Kac equation (4.1) such that $f(0,v) = f_0(v)$. Moreover, this solution satisfies estimates (4.5) and (4.6).

It is also easy to adapt the theorems of Truesdell (Cf. [Tr] and [Tr, Mu]) for this equation. Namely, one can give an explicit induction formula to compute the moments

$$L_n(t) = \int_{v\in \mathbb{R}} f(t,v)\, v^n\, dv \tag{4.8}$$

when $n \in \mathbb{N}$, as soon as these moments exist initially. Therefore, we do not deal in this work with the polynomial moments of f, but rather with the Maxwellian moments

$$\mathcal{M}_f(t,\lambda) = \int_{v\in \mathbb{R}} f(t,v)\, e^{\lambda v^2}\, dv, \tag{4.9}$$

for $\lambda > 0$.

We begin by proving the following theorem:

Theorem 3: *Let $f_0 \geq 0$ satisfy (4.7), and consider a solution $f(t,v)$ of the Kac equation (4.1) such that $f(0,v) = f_0(v)$.*
Suppose moreover that there exists $\lambda_0 > 0$ such that $\mathcal{M}_f(0,\lambda_0) < +\infty$. Then, there exists $\overline{\lambda} > 0$ and $K_{24} > 0$ such that when $t \geq 0$, $\mathcal{M}_f(t,\overline{\lambda}) \leq K_{24}$.

Proof of theorem 3: We look for an equation satisfied by $\mathcal{M}_f(t,\lambda)$.

$$\frac{\partial}{\partial t}\mathcal{M}_f(t,\lambda) = \int_{v\in \mathbb{R}} Q'(f)(t,v)\, e^{\lambda v^2}\, dv$$

$$= \int_{v\in \mathbb{R}} \int_{v_1\in \mathbb{R}} \int_{\theta=-\pi}^{\pi} f(t,v)\, f(t,v_1)\, \{e^{\lambda v^{*2}} - e^{\lambda v^2}\}\, \frac{d\theta}{2\pi} dv_1 dv$$

$$= \int_{v\in \mathbb{R}} \int_{v_1\in \mathbb{R}} \int_{\theta=-\pi}^{\pi} f(t,v)\, f(t,v_1)\, \{e^{\lambda (v^2+v_1^2)\,\cos^2\theta} - e^{\lambda v^2}\}\, \frac{d\theta}{2\pi} dv_1 dv$$

$$= \int_{\theta=-\pi}^{\pi} (\mathcal{M}_f^2(t,\lambda\cos^2\theta) - \mathcal{M}_f(t,\lambda)\,\mathcal{M}_f(0,0))\, \frac{d\theta}{2\pi}, \tag{4.10}$$

since the conservation of mass (4.5) holds.

For any $\overline{\rho}, \overline{T} > 0$, we denote by $m_{\overline{\rho},\overline{T}}$ the steady Maxwellian of density $\overline{\rho}$ and temperature $\overline{T}$,

$$m_{\overline{\rho},\overline{T}}(t,v) = \frac{\overline{\rho}}{(2\pi\overline{T})^{1/2}} e^{-\frac{|v|^2}{2\overline{T}}}. \tag{4.11}$$

It is easy to see that $m_{\overline{\rho},\overline{T}}$ is a steady solution of the Kac equation (4.1). Therefore

$$\mathcal{M}_{m_{\overline{\rho},\overline{T}}}(t,\lambda) = \frac{\overline{\rho}}{\sqrt{1-2\lambda\overline{T}}} \tag{4.12}$$

is a steady solution of equation (4.10) on $[0,+\infty[\times[0,\frac{1}{2\overline{T}}[$ (this can be seen directly on equation (4.10)).

We now prove that under the hypothesis of theorem 3, there exist $\tilde{\lambda} > 0, \overline{T} > 0$, such that

$$\forall\lambda \in [0,\tilde{\lambda}], \qquad \mathcal{M}_f(0,\lambda) \le \mathcal{M}_{m_{\overline{\rho},\overline{T}}}(0,\lambda), \tag{4.13}$$

with

$$\overline{\rho} = \int_{v\in I\!R} f(0,v)dv. \tag{4.14}$$

In order to prove (4.13), we use a development around 0 of $\mathcal{M}_f(0,\lambda)$:

$$\mathcal{M}_f(0,\lambda) = \int_{v\in I\!R} f(0,v)\, e^{\lambda v^2}\, dv$$

$$= \int_{v\in I\!R} f(0,v)\,(1+\lambda v^2 + \lambda^2 v^4 \int_{u=0}^{1} (1-u)e^{\lambda u v^2} du\,)\, dv$$

$$= \int_{v\in I\!R} f(0,v)\, dv + \lambda \int_{v\in I\!R} f(0,v)\,|v|^2\, dv + O(\lambda^2), \tag{4.15}$$

since

$$\int_{v\in I\!R} f(0,v)\, v^4\, (\int_{u=0}^{1} (1-u)\, e^{\lambda u v^2}\, du\,)\, dv \le \int_{v\in I\!R} f(0,v)\, v^4\, e^{\lambda v^2}\, dv < +\infty \tag{4.16}$$

when $\lambda < \lambda_0$.

But

$$\mathcal{M}_{m_{\overline{\rho},\overline{T}}}(0,\lambda) = \overline{\rho} + \lambda\,\overline{\rho}\,\overline{T} + O(\lambda^2), \tag{4.17}$$

and therefore (4.13) holds as soon as we take $\tilde{\lambda}$ small enough and

$$\overline{T} > \frac{1}{\overline{\rho}} \int_{v\in I\!R} f(0,v)\,|v|^2\, dv. \tag{4.18}$$

But equation (4.10) satisfies clearly the following monotony property: If $a(0,\lambda)$ and $b(0,\lambda)$ are two initial data for (4.10) and $\overline{\lambda}$ is a strictly positive number such that

$$\forall \lambda \in [0, \overline{\lambda}\,[, \qquad a(0,\lambda) \le b(0,\lambda), \tag{4.19}$$

and

$$a(0,0) = b(0,0), \tag{4.20}$$

then for all $t \ge 0$, the solutions $a(t,\lambda)$ and $b(t,\lambda)$ of (4.10) satisfy

$$\forall \lambda \in [0, \overline{\lambda}\,[, \qquad a(t,\lambda) \le b(t,\lambda). \tag{4.21}$$

According to (4.12), (4.13), (4.21), and taking

$$0 < \overline{\lambda} < \inf\,(\tilde{\lambda}, \frac{1}{2\overline{T}}), \tag{4.22}$$

we obtain theorem 3.

We now give estimates for the exponential moments:

$$\mathcal{N}_f(t,\lambda) = \int_{v\in I\!R} f(t,v)\, e^{\lambda v}\, dv, \tag{4.23}$$

for $\lambda \in I\!R$.

We can prove the following theorem:

Theorem 4: *Let $f_0 \ge 0$ satisfy (4.7), and consider a solution $f(t,v)$ of the Kac equation (4.1) such that $f(0,v) = f_0(v)$.*

Suppose moreover that there exists $\lambda_0 > 0$ such that $\mathcal{N}_f(0,\lambda_0) < +\infty$, $\mathcal{N}_f(0,-\lambda_0) < +\infty$, and

$$\int_{v\in I\!R} f_0(v)\, v\, dv = 0. \tag{4.24}$$

Then, there exists $\overline{\lambda} > 0$ and $K_{25} > 0$ such that $\mathcal{N}_f(t,\overline{\lambda}) + \mathcal{N}_f(t,-\overline{\lambda}) \le K_{25}$ for $t \ge 0$.

Proof of theorem 4: It is easy to see that

$$\frac{\partial}{\partial t}\mathcal{N}_f(t,\lambda) = \int_{\theta=-\pi}^{\pi} \{\mathcal{N}_f(t,\lambda\cos\theta)\,\mathcal{N}_f(t,\lambda\sin\theta) - \mathcal{N}_f(t,\lambda)\,\mathcal{N}_f(0,0)\}\,\frac{d\theta}{2\pi}. \tag{4.25}$$

Moreover, since $m_{\overline{\rho},\overline{T}}$ is a steady solution of the Kac equation (4.1),

$$\mathcal{N}_{m_{\overline{\rho},\overline{T}}}(t,\lambda) = \overline{\rho}\, e^{\frac{\lambda^2}{2}\overline{T}} \tag{4.26}$$

is a steady solution of equation (4.25) on $[0,+\infty[\times \mathbb{R}$ (this can be seen directly on equation (4.25)).

Then, the proof is quite similar to the proof of theorem 3.

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