Rigorous Derivation of a Nonlinear Diffusion Equation as Fast-Reaction Limit of a continuous Coagulation–Fragmentation Model with Diffusion

J. A. Carrillo, L. Desvillettes, and K. Fellner

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J. A. Carrillo

ICREA (Institució Catalana de Recerca i Estudis Avançats)
and Departament de Matemàtiques
Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain
email: carrillo@mat.uab.es

L. Desvillettes

CMLA, ENS Cachan, IUF & CNRS, PRES UniverSud
61 Av. du Pdt. Wilson, 94235 Cachan Cedex, France
email: desville@cmla.ens-cachan.fr

K. Fellner

DAMTP, CMS, University of Cambridge
Wilberforce Road, Cambridge CB3 0WA, United Kingdom
email: K.Fellner@damtp.cam.ac.uk
On leave from:
Faculty of Mathematics, University of Vienna
Nordbergstr. 15, 1090 Wien, Austria
email: Klemens.Fellner@univie.ac.at

Abstract

Weak solutions of the spatially inhomogeneous (diffusive) Aizenmann-Bak model of coagulation-breakup within a bounded domain with homogeneous Neumann boundary conditions are shown to converge, in the fast reaction limit, towards local equilibria determined by their mass. Moreover, this mass is the solution of a nonlinear diffusion equation whose nonlinearity depends on the (size-dependent) diffusion coefficient. Initial data are assumed to have integrable zero order moment and square integrable first order moment in size, and finite entropy. In contrast to our previous result [CDF2], we are able to show the convergence without assuming uniform bounds from above and below on the number density of clusters.

1   Introduction

In this work, we shall analyze the fast reaction asymptotics of the spatially inhomogeneous Aizenman-Bak model of coagulation-breakup with spatial diffusion given by

$$\partial_t f - a(y) \triangle_x f = Q(f,f). \tag{1}$$

Here, $f = f(t,x,y)$ is the concentration of clusters with size $y \geq 0$ at time $t \geq 0$ which are spatially diffusing in $x \in \Omega \subset \mathbb{R}^d$, $d \geq 1$ with normalized volume, i.e., $|\Omega| = 1$. Homogeneous Neumann boundary condition:

$$\nabla_x f(t,x,y) \cdot \nu(x) = 0 \qquad \text{on } \partial\Omega \tag{2}$$

where ν denotes the outward unit normal to Ω, are imposed in order to preserve the total number of aggregates. As in [CDF, CDF2], we assume that the diffusion coefficient $a(y)$ is non-degenerate (more precisely, bounded from below and above by strictly positive constants)

$$0 < a_* \leq a(y) \leq a^* \tag{3}$$

with $a_*, a^* \in \mathbb{R}^+$. The collision operator $Q(f,f)$ takes into account cluster coagulation and fragmentation/breakup, and it reads as

$$Q(f,f) = Q_c(f,f) + Q_b(f,f) \tag{4}$$

with

$$Q_c(f,f) := \int_0^y f(t,x,y-y')f(t,x,y')\,dy' - 2f(t,x,y)\int_0^\infty f(t,x,y')\,dy'\,,$$

and

$$Q_b(f,f) := Q_b^+(f,f) - Q_b^-(f,f) := 2\int_y^\infty f(t,x,y')\,dy' - y\,f(t,x,y)\,.$$

We refer to [LM, LM2, CDF, CDF2] for a complete set of references and applications in which these models appear.

By noticing that since the reaction term (4) satisfies $\int_0^\infty y\,Q(f,f)\,dy = 0$, we have (formally) for all $t \geq 0$ the conservation of total mass,

$$N_\infty := \int_\Omega N(t,x)\,dx = \int_\Omega N_{in}(x)\,dx, \ \ \text{with } N(t,x) := \int_0^\infty y f(t,x,y)\,dy, \tag{5}$$

where $\int_\Omega N_{in}(x)\,dx$ denotes the total inital mass. The corresponding local conservation law writes

$$\partial_t N - \triangle_x \left(\int_0^\infty ya(y)\,f(t,x,y)\,dy \right) = 0 \,. \qquad (6)$$

The other important property for eq. (1) is the dissipation of the entropy functional. In order to present it, we first write down the weak form of the operator (4) (for a given test function φ, and at the formal level):

$$\int_0^\infty Q(f,f)\,\varphi\,dy = \int_0^\infty \int_0^\infty [\varphi(y'+y) - \varphi(y) - \varphi(y')]\,f(y)f(y')\,dy\,dy' \\ + 2 \int_0^\infty \Phi(y)\,f(y)\,dy - \int_0^\infty y\,\varphi(y)\,f(y)\,dy, \qquad (7)$$

where the function Φ is the primitive of φ (i.e. $\partial_y \Phi = \varphi$) such that $\Phi(0) = 0$.

Let us consider the entropy functional associated to any positive density f as

$$H(f)(t,x) = \int_0^\infty (f\ln f - f)\,dy\,.$$

Then, the entropy formally dissipates as

$$\frac{d}{dt} \int_\Omega H(f)\,dx = - \int_\Omega \int_0^\infty a(y)\frac{|\nabla_x f|^2}{f}\,dy\,dx \\ - \int_\Omega \int_0^\infty \int_0^\infty (f'' - ff')\ln\left(\frac{f''}{ff'}\right)\,dy\,dy'dx := -D_H(f), \qquad (8)$$

where $f'' = f(t,x,y+y')$, $f' = f(t,x,y')$ and $f = f(t,x,y)$.

Our present aim is to study the fast-reaction asymptotics, i.e. the limit $\varepsilon \to 0$ of the rescaled problem:

$$\begin{cases} \partial_t f^\varepsilon - a(y)\triangle_x f^\varepsilon = \dfrac{1}{\varepsilon}Q(f^\varepsilon, f^\varepsilon)\,, & \text{for } x \in \Omega,\ y \in (0,\infty), t > 0\,, \\ \nabla_x f^\varepsilon(t,x,y)\cdot\nu(x) = 0\,, & \text{for } x \in \partial\Omega,\ y \in (0,\infty), t > 0\,, \\ f^\varepsilon(t=0,x,y) = f_{in}(x,y) \geq 0\,, & \text{for } x \in \Omega,\ y \in (0,\infty), \end{cases} \qquad (9)$$

where we shall assume that the zero- and first-order initial moments as well as the initial entropy are finite, i.e. $(1 + y + \ln f_{in})f_{in} \in L^1(\Omega \times (0,\infty))$.

This asymptotic regime is called the fast reaction limit since the reaction term is dominant as ε gets smaller. In fact, letting formally $\varepsilon \to 0$, we expect that $f^\varepsilon \to f$, where $Q(f,f) = 0$, i.e.

$$f^\varepsilon(t,x,y) \to e^{-\frac{y}{\sqrt{N(t,x)}}}\,,$$

where $N(t,x)$ diffuses according to the limit of the moment equation (6):

$$\partial_t N - \triangle_x n(N) = 0\,. \qquad (10)$$

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Here $n(N)$ denotes the function

$$n(N) := \int_0^\infty a(y) y \, e^{-\frac{y}{\sqrt{N}}} \, dy \,. \tag{11}$$

We say that f^ε is a weak solution to eq. (1), see [LM], if $f^\varepsilon \in C([0,T], w - L^1(\Omega \times (0,\infty)))$ such that $y \, f^\varepsilon \in L^\infty(0,T; L^1(\Omega \times (0,\infty)))$ satisfying (1) in the distributional sense with test functions compactly supported in time and in size y, and verifying the conservation of mass. Let us remark that global weak solutions to this problem have been constructed in [LM] satisfying an entropy inequality with a weaker version of the Fisher information as calculated formally in (8):

$$\int_\Omega H(f^\varepsilon(t)) \, dx + \left(\int_0^t \int_\Omega \int_0^\infty a(y)^{1/2} |\nabla_x f^\varepsilon| dx dy dt \right)^2 \left(\int_0^t \int_\Omega M^\varepsilon(t,x) \, dx \, dt \right)^{-1}$$

$$+ \frac{1}{\varepsilon} \int_0^t \tilde{D}_H(f^\varepsilon(s)) \, ds \leq \int_\Omega H(f_{in}) \, dx \tag{12}$$

for all $t \geq 0$ with

$$\tilde{D}_H(f) := \int_\Omega \int_0^\infty \int_0^\infty (f'' - f f') \ln \left(\frac{f''}{f f'} \right) dy \, dy' dx.$$

Our main goal is a complete rigorous justification of this formal limit under the following assumptions on the non trivial initial data $f_{in} \neq 0$:

Hypothesis **(NL2)**, $\quad N_{in} = \int_0^\infty y \, f_{in}(x,y) \, dy \in L^2(\Omega)$,

Hypothesis **(ML1)**, $\quad M_{in} = \int_0^\infty f_{in}(x,y) \, dy \in L^1(\Omega)$,

Hypothesis **(Entr)**, $\quad \int_0^\infty f_{in}(x,y) |\ln(f_{in}(x,y))| \, dy \in L^1(\Omega)$.

The main result of this work is the following:

Theorem 1.1 *Let Ω be a bounded smooth subset of $\mathbb{R}^d$ with normalized volume $|\Omega| = 1$ and let the diffusion coefficient $a(y)$ satisfy (3). We consider (for all $\varepsilon > 0$) the nonnegative initial data $f^\varepsilon(t = 0, x, y) = f_{in}(x,y) \geq 0$ not identically zero satisfying the hypotheses **(NL2)**, **(ML1)** and **(Entr)**.*

Then, as $\varepsilon \to 0$, the mass $N^\varepsilon := N^\varepsilon(t,x)$ associated to any weak solution f^ε to eq. (1) satisfying (12) converges in $L^1((0,T) \times \Omega)$ (strongly, and for all $T > 0$) to the unique solution $N(t,x)$ of the Neumann problem for the nonlinear diffusion equation

$$\begin{cases} \partial_t N - \triangle_x n(N) = 0 \,, \\ \nabla_x N \cdot \nu(x)|_{\partial\Omega} = 0, \end{cases} \tag{13}$$

with initial datum $N_{in}(0,x) = \int_0^\infty y f_{in}(x,y) \, dy$, with $n(N)$ given by (13).

This result improves our previously published *If-Theorem* in [CDF2] since it avoids the assumption on the solution that there exists $0 < \mathcal{M}_* \leq \mathcal{M}^* < \infty$ such that $\mathcal{M}_* \leq M^\varepsilon(t,x) = \int_0^\infty f^\varepsilon(t,x,y)\,dy \leq \mathcal{M}^*$ for all $t \geq 0$, $x \in \Omega$ and $\varepsilon > 0$. These uniform-in-ϵ bounds, although quite plausible, are not yet known to hold even close to global equilibrium or by linearization techniques applied to problem (9). In fact, any proof would be immediately faced with the difficulties of size-dependent diffusion including possible variants of Turing-type instabilities and diffusion-induced blow-up as known for systems of reaction-diffusion equations.

The presented result is fully by-passing these technical problems providing the first rigorous proof of the fast-reaction limit of a continuous diffusive coagulation-fragmentation model. As a trade-off, with the present method we are not able to obtain an estimate on the speed of convergence with respect to the scaling parameter ε.

Our method of proof includes the following ingredients: the estimate of entropy/entropy dissipation as shown in [CDF] using an inequality proven by Aizenman and Bak for the spatially homogeneous problem in [AB], a method of duality for parabolic equations as presented, for instance, by M. Pierre and D. Schmitt in [PSch], the classical Cziszar-Kullback inequality (Cf. [Czi, Kul]), and estimates for the gain of moments taken from [CDF].

Let us comment on the assumptions of Theorem 1.1. Assuming constant coagulation-fragmentation coefficients as in the Aizenman-Bak model [AB] restricts to a particular, yet an important case amongst coagulation-fragmentation models with general detail-balanced coefficients and, thus, with an entropy functional, see e.g. [LM]. Nevertheless, the homogeneous Aizenman-Bak model represents the only case, where a so-called entropy entropy-dissipation estimate is known, i.e. that the entropy dissipation can be controlled below by the relative entropy with respect to local equilibrium states (see estimate (16) below). For general coagulation-fragmentation models an entropy entropy-dissipation estimate is a widely open problem. Let us remark that entropy entropy-dissipation estimates were established in the cases up to four reversible reaction-diffusion equations [DF06, DF], and in case of the discrete homogeneous Becker Döring model [JN].

Moreover, the lower and upper bound on the diffusion coefficient (3) are required for the duality argument. Although there exists a variant of the duality argument avoiding lower bounds of the diffusion coefficients (see, e.g. [CDFpre] in the context of discrete coagulation-fragmentation models) the resulting a-priori bounds are to weak to prove Theorem 1.1 as below.

Finally, we remark that under assumption (3), equation (13) is a wellposed non-degenerate, nonlinear diffusion equation satisfying $0 < a_* N \leq n(N) \leq a^* N$ and $0 < a_* \leq n'(N) \leq a^*$, see [LSU] for the existence theory and [CJG, DGJ] for the uniqueness arguments.

Note that fast reactions asymptotics have been rigorously proven in different contexts, Cf. [BH] for reversible chemical equations, and [ELM] for discrete coagulation-breakup models. We also refer to [LM, CDF] and the references therein, for studies on the related problem of large time behavior of continuous

models of coagulation-fragmentation-diffusion.

Notation 1 *We will use various short-cuts like $L^p_x = L^p(\Omega)$, $L^p_y = L^p((0,\infty))$, and $L^2_t(L^1_{x,y}) = L^2((0,\infty), L^1(\Omega \times (0,\infty)))$. Further we denote*

$$M^\varepsilon(t,x) := \int_0^\infty f^\varepsilon(t,x,y)\,dy\,, \qquad N^\varepsilon(t,x) := \int_0^\infty y\,f^\varepsilon(t,x,y)\,dy\,,$$

where M^ε denotes the number density of polymers. The letter C will denote various constants. Moreover C_T will indicate a dependence of the constant on the time interval $[0,T]$.

The rest of the paper is devoted to the proof of theorem 1.1. Section 2 consists in proving the basic (independent of ε) a-priori estimates for our problem. Then, strong compactness of M^ε and N^ε is obtained in section 3, and moments (up to order 3) are shown to exist in section 4. Finally, the passage to the limit is performed in section 5.

2 A-priori Estimates

In this subsection, we prove ε independent bounds on M^ε and N^ε, which will entail compactness of this sequence as shown in the following subsection. We first observe that the conservation law of mass (6) implies

$$\int_\Omega N^\varepsilon(t,x)\,dx = N_\infty\,, \tag{14}$$

for all $t \geq 0$ with N_∞ defined in (5).

Moreover, integrating the evolution equation for the monomer density M^ε

$$\partial_t M^\varepsilon - \triangle_x \left(\int_0^\infty a(y)\,f^\varepsilon(t,x,y)\,dy \right) = \frac{1}{\varepsilon}\left(N^\varepsilon - (M^\varepsilon)^2 \right),$$

yields

$$\frac{d}{dt}\int_\Omega M^\varepsilon(\tau,x)\,dx \leq \frac{1}{\varepsilon}\left(N_\infty - \left(\int_\Omega M^\varepsilon(\tau,x)\,dx \right)^2 \right)$$

by Hölder's inequality and (14). Therefore, for all $\varepsilon > 0$, we have

$$\sup_{t \in [0,\infty)} \int_\Omega M^\varepsilon(t,x)\,dx \leq \max\left\{ \int_\Omega M_{in}(x)\,dx, \sqrt{N_\infty} \right\}. \tag{15}$$

Entropy Dissipation

The trend to local equilibrium follows from the dissipation of the entropy, which is better understood by using the inequality proven in [AB, Propositions 4.2

and 4.3]: Let $g := g(y)$ be a function of $L^1_+((0,\infty))$ with finite entropy $g\ln g \in L^1((0,\infty))$, then

$$\int_0^\infty \int_0^\infty g(y)g(y')\ln g(y+y')\,dy\,dy' \leq \left(\int_0^\infty g(y)\,dy\right)\left(\int_0^\infty g(y')\ln g(y')\,dy'\right)$$
$$-\left(\int_0^\infty g(y)\,dy\right)^2.$$

Applying this inequality as in [CDF] implies that

$$\int_0^\infty \int_0^\infty (f''-ff')\ln\left(\frac{f''}{ff'}\right)dy\,dy' \geq M\,H(f|e^{-\frac{y}{\sqrt{N}}}) + 2(M-\sqrt{N})^2\,, \quad (16)$$

with the notation $H(f|g) := H(f) - H(g)$.

Thus, from the entropy inequality (12) we obtain the estimate

$$\int_\Omega H(f^\varepsilon(t))\,dx + \left(\int_0^t \int_\Omega \int_0^\infty a(y)^{\frac{1}{2}}|\nabla_x f^\varepsilon|dxdydt\right)^2\left(\int_0^t \int_\Omega M^\varepsilon(t,x)\,dx\,dt\right)^{-1}$$
$$+\frac{1}{\varepsilon}\int_\Omega \left[M^\varepsilon\,H(f^\varepsilon|e^{-\frac{y}{\sqrt{N^\varepsilon}}}) + 2(M^\varepsilon - \sqrt{N^\varepsilon})^2\right]dx \leq \int_\Omega H(f_{in})\,dx \quad (17)$$

for all $t \geq 0$. Taking into account the Csiszar-Kullback inequality for any two functions $f \geq 0$, $g > 0$:

$$\int_0^\infty \left(f\ln\frac{f}{g} - f + g\right)dy \geq C\left(\int_0^\infty (f+g)\,dy\right)^{-1}\left(\int_0^\infty |f-g|\,dy\right)^2,$$

the dissipation of the entropy (17), together with (3) and (15) imply the uniform bounds

$$\sup_{t\in[0,\infty)}\int_\Omega \int_0^\infty |f^\varepsilon\ln f^\varepsilon - f^\varepsilon|\,dydx \leq C\,, \quad (18)$$

$$\int_0^T \int_\Omega \int_0^\infty |\nabla_x f^\varepsilon|\,dydxdt \leq C_T\,, \quad (19)$$

$$\int_0^\infty \int_{\Omega_\varepsilon(t)} \frac{M^\varepsilon}{M^\varepsilon + \sqrt{N^\varepsilon}}\left(\int_0^\infty \left|f^\varepsilon - e^{-\frac{y}{\sqrt{N^\varepsilon}}}\right|dy\right)^2 dxdt \leq C\,\varepsilon\,, \quad (20)$$

$$\int_0^\infty \int_\Omega \left|M^\varepsilon - \sqrt{N^\varepsilon}\right|^2 dxdt \leq C\,\varepsilon\,, \quad (21)$$

for all $T > 0$ with $\Omega_\varepsilon(t) = \text{supp}(M^\varepsilon(t,\cdot))$. We remark that thanks to the a priori bound (21) and the conservation of mass (5), we also deduce a uniform L^2-bound on M^ε, i.e.

$$\|M^\varepsilon(t,x)\|_{L^2((0,T)\times\Omega)} \leq C_T\,.$$

Duality Argument

Finally, a duality argument (see e.g. [PSch]) implies that solutions of (6), i.e.

$$\begin{cases} \partial_t N^\varepsilon - \triangle_x(A^\varepsilon N^\varepsilon) = 0\,, & 0 < a_* \leq A^\varepsilon \leq a^*\,, \\ \nabla_x N^\varepsilon \cdot \nu(x)|_{\partial\Omega} = 0\,, & N^\varepsilon(0,x) = N_{in}(x) \in L^2(\Omega)\,, \end{cases}$$

with

$$A^\varepsilon := \begin{cases} \dfrac{1}{N^\varepsilon} \displaystyle\int_0^\infty y a(y) f^\varepsilon(y)\, dy & , \text{ whenever } N^\varepsilon > 0 \\ 0 & , \text{ otherwise} \end{cases}$$

satisfy a uniform L^2-bound in terms of the initial data and any $T > 0$

$$\|N^\varepsilon(t,x)\|_{L^2((0,T)\times\Omega)} \leq C_T\,. \tag{22}$$

To show (22), it is known that solutions of the dual parabolic problem

$$\begin{cases} -\partial_t w - A^\varepsilon \triangle_x w = H\,, & 0 \leq H \in C_0^\infty([0,T]\times\Omega) \\ \nabla_x w \cdot \nu(x)|_{\partial\Omega} = 0, \end{cases}$$

with end data $w(T) = 0$ and for all non-negative testfunctions $H \in C_0^\infty([0,T]\times\Omega)$, satisfy the following L^2 estimates

$$\begin{aligned} \|\triangle_x w\|_{L^2([0,T]\times\Omega)} &\leq C\|H\|_{L^2([0,T]\times\Omega)}\,, \\ \|w(0)\|_{L^2(\Omega)} &\leq C\|H\|_{L^2([0,T]\times\Omega)}\,. \end{aligned}$$

Let us point out that the choice of L^2 is optimal, i.e. the lowest admissible L^p space to have the above estimates for parabolic equation with discontinuous, but bounded coefficients A^ε; see counterexamples in [PSch].

On a formal level these estimates are obtained by multiplying the equation in w by $-\Delta_x w$, which gives

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla_x w(t)|^2\, dx + \int_\Omega A^\varepsilon(\Delta_x w)^2\, dx = -\int_\Omega H\Delta_x w\, dx$$
$$\leq \int_\Omega \left[\frac{a_*}{2}(\Delta_x w(t))^2 + C(a_*)H^2\right]\, dx\,,$$

where $0 < a_* \leq A^\varepsilon$. After integration in time, it follows that

$$\int_0^T\int_\Omega (\Delta_x w)^2\, dx \leq C\int_0^T\int_\Omega H^2\, dx.$$

Going back to the equation in w, we deduce a bound for $\partial_t w$ in $L^2((0,T)\times\Omega)$ and therefore a bound $w(0)$ in $L^2(\Omega)$ in terms of the $L^2((0,T)\times\Omega)$ norm of H. Standard approximations allow to render this formal arguments rigorous (Cf. [PSch]). The global bound (22) follows finally by duality after integration by parts

$$\int_0^T\int_\Omega N^\varepsilon H\, dxdt = \int_\Omega N_{in}\, w(0)\, dx \leq C\|N_{in}\|_{L^2(\Omega)}\|H\|_{L^2([0,T]\times\Omega)}\,.$$

3 Compactness of M^ε and N^ε

In this section, we show that the bounds derived in subsection 2 imply a.e. convergence of M^ε and N^ε.

Step 1: Using (19) gives

$$\int_0^T\!\!\int_\Omega |\nabla_x M^\varepsilon|dxdt \leq \int_0^T\!\!\int_\Omega\!\int_0^\infty |\nabla_x f^\varepsilon|dxdydt \leq C_T\,,$$

and thus, for all $h \in \mathbb{R}^d$ and any domain $\omega \subset \Omega$, such that $h+\omega \subset \Omega$, we obtain

$$\int_0^T\!\!\int_\omega |M^\varepsilon(t,x+h) - M^\varepsilon(t,x)|\,dxdt \leq C_T\,|h|\,. \tag{23}$$

Step 2: Let us fix any $h \in \mathbb{R}^d$ and any domain $\omega \subset \Omega$, such that $h+\omega \subset \Omega$. As a consequence of (23) and (21) it follows easily that

$$\int_0^T\!\!\int_\omega |\sqrt{N^\varepsilon}(t,x+h) - \sqrt{N^\varepsilon}(t,x)|\,dxdt \leq C_T\,(\varepsilon^{\frac{1}{2}} + |h|)\,.$$

Given any $R > 0$, it is straightforward, using the L^2-bound (22), to check that

$$\int_0^T\!\!\int_\omega |\sqrt{N^\varepsilon}(t,x+h) - \sqrt{N^\varepsilon}(t,x)||\sqrt{N^\varepsilon}(t,x+h) + \sqrt{N^\varepsilon}(t,x)|\,dxdt$$

$$\leq 2RC_T\,(\varepsilon^{\frac{1}{2}} + |h|) + \frac{3}{R^2}\int_0^T\!\!\int_\Omega |N^\varepsilon|^2(x)\,dxdt \leq C_T\left[R\,(\varepsilon^{\frac{1}{2}} + |h|) + \frac{1}{R^2}\right],$$

by dividing the first integral into four different sets, where $\sqrt{N^\varepsilon}(t,x) < R$ and $\sqrt{N^\varepsilon}(t,x) > R$ and where $\sqrt{N^\varepsilon}(t,x+h) < R$ and $\sqrt{N^\varepsilon}(t,x+h) > R$, respectively. Optimizing R, i.e. choosing $R = (\varepsilon^{\frac{1}{2}} + |h|)^{-\frac{1}{3}}$, leads to

$$\int_0^T\!\!\int_\omega |N^\varepsilon(t,x+h) - N^\varepsilon(t,x)|\,dxdt \leq C_T\,(\varepsilon^{\frac{1}{3}} + |h|^{\frac{2}{3}})\,. \tag{24}$$

Step 3: Let us fix now a smooth mollifier $\phi(x) \in C_0^\infty(\mathbb{R}^d)$, $\phi \geq 1$ with unit integral and support on the euclidean ball centered at 0 of unit radius and $\phi(x) = \phi(-x)$. Let us define for any $\delta > 0$, the sequence $\phi_\delta(x) = \delta^{-d}\phi(\frac{x}{\delta})$.

We first remark that N^ε are solutions of equation (6) in the following weak sense

$$\int_\Omega (N^\varepsilon(t+\tau,x) - N^\varepsilon(t,x))\varphi(x)\,dx = \int_t^{t+\tau}\!\!\int_\Omega A^\varepsilon(s,x)N^\varepsilon(s,x)\triangle_x\varphi(x)\,dxds,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ smooth with $\nabla_x\varphi \cdot \nu(x)|_{\partial\Omega} = 0$ and all $t \geq 0$ and $\tau > 0$.

Given any subdomain ω such that $\bar\omega \subset \Omega$, let us consider the characteristic function of the domain ω denoted by χ_ω, then the function $\phi_\delta * (\chi_\omega\,\psi)$ with $\psi(t,x) \in L^\infty((0,T) \times \omega)$ is a suitable test function above since it is a smooth

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function with $\overline{\operatorname{supp}_x(\psi(t,x))} \subset \Omega$ for δ small enough and a.e. $t \in (0,T)$. Thus, we deduce that for $t \in [0,T]$

$$\left| \int_0^T \int_\omega (N^\varepsilon * \phi_\delta(t+\tau, x) - N^\varepsilon * \phi_\delta(t,x)) \psi(t,x)\, dx dt \right|$$

$$\leq \int_0^T \int_\omega \int_t^{t+\tau} \left| \left(\int_0^\infty y a(y) f^\varepsilon(s,x,y)\, dy \right) * (\triangle_x \phi_\delta) \right| |\psi(t,x)|\, ds dx dt$$

$$\leq \int_0^T \int_t^{t+\tau} a^* \|N^\varepsilon\|_{L_x^1} \|\triangle_x \phi_\delta\|_{L_x^1} ds dt \, \|\psi\|_{L_{t,x}^\infty} \leq C_T |\tau| \|\triangle_x \phi_\delta\|_{L_x^1} \|\psi\|_{L_{t,x}^\infty}\,.$$

Using the duality $L^1 - L^\infty$, we finally deduce

$$\int_0^T \int_\omega |N^\varepsilon * \phi_\delta(t+\tau,x) - N^\varepsilon * \phi_\delta(t,x)|\, dx dt \leq C_T |\tau| \|\triangle_x \phi_\delta\|_{L_x^1}.$$

Recalling (24) it follows that

$$\int_0^T \int_\omega |N^\varepsilon(t+\tau,x) - N^\varepsilon(t,x)|\, dx dt \leq C_T |\tau| \|\triangle_x \phi_\delta\|_{L_x^1}$$

$$+ C_T \int_{\mathbb{R}^d} (\varepsilon^{\frac{1}{3}} + |x|^{\frac{2}{3}}) \phi_\delta(x)\, dx\,.$$

Taking $\delta = |\tau|^{\frac{3}{8}}$ with τ small enough, we find

$$\int_0^T \int_\omega |N^\varepsilon(t+\tau,x) - N^\varepsilon(t,x)|\, dx dt \leq C_T (\varepsilon^{\frac{1}{3}} + |\tau|^{\frac{1}{4}})\,,$$

and, finally

$$\int_0^T \int_\omega |N^\varepsilon(t+\tau, x+h) - N^\varepsilon(t,x)|\, dx dt \leq C_T (\varepsilon^{\frac{1}{3}} + |\tau|^{\frac{1}{4}} + |h|^{\frac{2}{3}})\,. \qquad (25)$$

Hence, by the Riesz-Frechet-Kolmogorov compactness theorem, the sequence N^ε is strongly compact in $L^1((0,T) \times \Omega)$, and so $\sqrt{N^\varepsilon}$ is strongly compact in $L^2((0,T) \times \Omega)$, and thus, due to (21), M^ε is strongly compact in $L^2((0,T) \times \Omega)$. In particular, as $\varepsilon \to 0$, for a subsequence that we denote with the same index, there exists a $M \geq 0$ and

$$M^\varepsilon \to M \ a.e.\,, \quad N^\varepsilon \to M^2 \ a.e.\,, \qquad \text{as } \varepsilon \to 0\,, \qquad (26)$$

by the use of (21).

4   Moment estimate

We show that solutions of (9) satisfying the hypotheses **(NL2)** have moments in y up to order three bounded for all $t > 0$.

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We start by observing [MW, Appendix A] that for given nontrivial initial data $y\, f_{in} \in L^1_{x,y}$, there exists a concave function $\Phi(r) : [0, \infty) \longrightarrow \mathbb{R}$, depending on f_{in}, smoothly increasing in r from $\Phi(0) = 0$ to ∞ sublinearly, such that

$$\int_\Omega \int_0^\infty y\, \Phi(y)\, f_{in}(y)\, dy dx < \infty\,.$$

Moreover, the function Φ can be constructed to grow at most linearly in y and to satisfy

$$\Phi(y) - \Phi(y') \geq C \frac{y - y'}{y \ln^2(e + y)} \tag{27}$$

for $0 < y' < y$ with a constant C not depending on f_{in}. We refer to [MW, Appendix] for all the details of this construction, which adapt to the present situation without essential difficulties by integrating in the spatial variable in their argument. We consider then in a first step the evolution of the moment

$$M^\varepsilon_{1+\Phi}(f^\varepsilon)(t) = \int_\Omega \int_0^\infty y\, \Phi(y) f^\varepsilon(x, y)\, dy\, dx.$$

For the fragmentation part, we use (27) for $0 < y' < y$ and estimate

$$\int_0^\infty y\, \Phi(y)\, Q_b(f^\varepsilon)\, dy = 2 \int_0^\infty \int_0^y y'(\Phi(y') - \Phi(y))\, dy'\, f^\varepsilon(y)\, dy$$

$$\leq -C \int_0^\infty \frac{1}{\ln(e + y)y} \int_0^y y'(y - y')\, dy'\, f^\varepsilon(y)\, dy$$

$$= -C \int_0^\infty \frac{y^2}{\ln^{-2}(e + y)} f^\varepsilon(y)\, dy \leq -C_\delta \int_0^\infty y^{2-\delta}\, f^\varepsilon(y)\, dy\,,$$

for all $\delta > 0$ and a positive constant C_δ, where the (t, x)-dependence has been dropped for notational convenience. For the coagulation part, Taylor expansion in the weak formulation (7) shows that for a monotone, at most linearly increasing Φ

$$(y + y')\Phi(y + y') - y\Phi(y) - y'\Phi(y') \leq Cyy'\,,$$

for a constant C.

Altogether, we estimate the evolution of the moment $M^\varepsilon_{1+\Phi}(f^\varepsilon)$ to be

$$\frac{d}{dt} M^\varepsilon_{1+\Phi}(f^\varepsilon) \leq -\frac{C_\delta}{\varepsilon} M^\varepsilon_{2-\delta}(f^\varepsilon) + \frac{C}{\varepsilon} \int_\Omega (N^\varepsilon)^2\, dx\,,$$

where $M^\varepsilon_{2-\delta}(f^\varepsilon)$ denotes the moment $\int_\Omega \int_0^\infty y^{2-\delta} f^\varepsilon(y)\, dy$. Then, integrating in time over the interval $[0, T]$ and using the bounds (22), we see that

$$\int_0^T M^\varepsilon_{2-\delta}(f^\varepsilon)\, dt < \infty,$$

thus the moment $M^\varepsilon_{2-\delta}(f^\varepsilon)$ is finite for a.e. $t \in (0, T]$.

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Next, choosing a time $t_* > 0$ when $M_{2-\delta}^\varepsilon(f^\varepsilon)(t_*) < \infty$, and repeating the above (with $\Phi(y) = y^{1-\delta}$) shows that $\int_{t_*}^T M_{3-\delta}^\varepsilon(f^\varepsilon)\,dt < \infty$. The same argument once more with $\Phi(y) = y$ leads then finally to

$$\int_{t_*}^T \int_\Omega \int_0^\infty y^3\, f^\varepsilon(y)\, dy < \infty\,, \tag{28}$$

where t_* can be chosen arbitrarily small.

5   Passing to the limit

First, let us fix $T > 0$. We observe that due to the estimate (20), up to extraction of a subsequence,

$$\phi_\varepsilon(t,x)\, \frac{M^\varepsilon}{M^\varepsilon + \sqrt{N^\varepsilon}} \left(\int_0^\infty \left| f^\varepsilon - e^{-\frac{y}{\sqrt{N^\varepsilon}}} \right| dy\right)^2 \to 0 \quad \text{as } \varepsilon \to 0\,, \text{ a.e.}$$

with $(t,x) \in (0,T) \times \Omega$ and ϕ_ε the characteristic function of the set $\mathrm{supp}(M^\varepsilon)$, and moreover, thanks to the convergence (26)

$$\int_0^\infty \left| f^\varepsilon - e^{-\frac{y}{\sqrt{N^\varepsilon}}} \right| dy \to 0 \quad \text{as } \varepsilon \to 0 \qquad \text{for a.e. } (t,x) \in B_M\,, \tag{29}$$

where $B_M := \mathrm{supp}(M)$ in $(0,T) \times \Omega$.

Denoting then $h^\varepsilon = f^\varepsilon - e^{-\frac{y}{\sqrt{N^\varepsilon}}}$ we will show that

$$\int_0^T \int_\Omega \int_0^\infty y\, |h^\varepsilon|\, dydxdt \to 0 \quad \text{as } \varepsilon \to 0\,. \tag{30}$$

First, the conservation of mass (5) implies that $\int_0^{t_*}\int_\Omega \int_0^\infty y\,|h^\varepsilon|\,dydxdt = \mathrm{O}(t_*)$ independent of ε and with $t_* > 0$ arbitrarily small as in (28).

We then estimate the remaining part of (30) in three steps: First, for $R > 2$ thanks to (22) and (28) that

$$\int_{t_*}^T \int_\Omega \int_{|y|\geq R} y\,|h^\varepsilon|\,dydxdt \leq \int_{t_*}^T \int_\Omega \int_{|y|\geq R} \frac{y^3}{R^2}\left[f^\varepsilon + e^{-\frac{y}{\sqrt{N^\varepsilon}}} \right] dydxdt$$

$$\leq \frac{1}{R^2}\left(C + \int_{t_*}^T \int_\Omega (N^\varepsilon)^2\,dxdt\right) \leq \frac{C}{R^2}\,.$$

Secondly, for a constant $Q > 1$, we observe that $|h^\varepsilon| \geq Q$ implies (due to $0 \leq e^{-\frac{y}{\sqrt{N^\varepsilon}}} \leq 1$) that $|h^\varepsilon| = h^\varepsilon$ and $f^\varepsilon \geq Q$. Thus, using (18)

$$\int_{t_*}^T \int_\Omega \int_{|y|\leq R} y\,|h^\varepsilon|\,\mathbf{1}_{\{|h^\varepsilon|\geq Q\}}\,dydxdt \leq 2R \int_{t_*}^T \int_\Omega \int_{|y|\leq R} f^\varepsilon\,\mathbf{1}_{\{f^\varepsilon\geq Q\}}\,dydxdt$$

$$\leq \frac{2R}{\ln Q}\int_{t_*}^T \int_\Omega \int_0^\infty f^\varepsilon |\ln f^\varepsilon|\,dydxdt \leq \frac{CR}{\ln Q}\,.$$

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Hence, we have

$$\int_{t_*}^{T}\int_{\Omega}\int_{0}^{\infty} y|h^{\varepsilon}|\,dydxdt \leq \frac{C}{R^2} + \frac{CR}{\ln Q} + R\int_{t_*}^{T}\int_{\Omega}\int_{|y|\leq R} |h^{\varepsilon}|\,\mathbf{1}_{\{|h^{\varepsilon}|\leq Q\}}\,dydxdt\,.$$

We need to show that

$$\int_{t_*}^{T}\int_{\Omega}\int_{|y|\leq R} |h^{\varepsilon}|\,\mathbf{1}_{\{|h^{\varepsilon}|\leq Q\}}\,dydxdt \to 0$$

as $\varepsilon \to 0$. We decompose this integral in the (t,x) variables onto B_M and its complementary set $B_M^c = ([0,T]\times\Omega)/B_M$. In the support of $M(t,x)$, we use (29) to get

$$\int_{|y|\leq R} |h^{\varepsilon}|\,\mathbf{1}_{\{|h^{\varepsilon}|\leq Q\}}\,dy \leq \int_{0}^{\infty} |h^{\varepsilon}|\,dy \to 0 \quad \text{as } \varepsilon \to 0,$$

for a.e. $(t,x) \in B_M$, and moreover, we have the trivial estimate

$$\int_{|y|\leq R} |h^{\varepsilon}|\,\mathbf{1}_{\{|h^{\varepsilon}|\leq Q\}}\,dy \leq Q\,R \in L^{1}([0,T]\times\Omega)\,,$$

and Lebegues' dominated convergence implies that

$$\int_{B_M}\int_{|y|\leq R} |h^{\varepsilon}|\,\mathbf{1}_{\{|h^{\varepsilon}|\leq Q\}}\,dydxdt \to 0 \quad \text{as } \varepsilon \to 0.$$

In the complementary set, we estimate it as

$$\int_{B_M^c}\int_{|y|\leq R} |h^{\varepsilon}|\,\mathbf{1}_{\{|h^{\varepsilon}|\leq Q\}}\,dydxdt \leq \int_{B_M^c} \left(M^{\varepsilon}(t,x) + \sqrt{N^{\varepsilon}}(t,x) \right) dxdt$$

for which we know that $M^{\varepsilon}(t,x) \to M = 0$ and $\sqrt{N^{\varepsilon}}(t,x) \to M = 0$ a.e. in B_M^c. Since $M^{\varepsilon}(t,x) \to M(t,x)$ and $\sqrt{N^{\varepsilon}}(t,x) \to \sqrt{N}(t,x) = M(t,x)$ in $L^2((0,T)\times\Omega)$, then

$$\int_{B_M^c} \left(M^{\varepsilon}(t,x) + \sqrt{N^{\varepsilon}}(t,x) \right) dxdt \to 2\int_{B_M^c} M(t,x)\,dxdt = 0$$

as $\varepsilon \to 0$. Collecting all previous estimates, we have shown (30).

Finally, we consider a weak formulation of

$$\begin{cases} \partial_t N^{\varepsilon} - \triangle_x \left(\int_{0}^{\infty} ya(y)\,f^{\varepsilon}(t,x,y)\,dy \right) = 0\,, \\ \nabla_x N^{\varepsilon} \cdot \nu(x)|_{\partial\Omega} = 0\,, \end{cases}$$

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in the sence that for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ smooth with $\nabla_x \varphi \cdot \nu(x)|_{\partial\Omega} = 0$ and $T > 0$

$$\int_\Omega (N^\varepsilon(T,x) - N_{in}^\varepsilon(x))\varphi(x)\, dx = \int_0^T \int_\Omega \left(\int_0^\infty y a(y)\, f^\varepsilon(t,x,y)\, dy\right) \triangle_x \varphi(x)\, dxdt,$$

with initial data satisfying the hypotheses **(NL2)**, **(ML1)**, and **(Entr)**. Then, due to the pointwise convergence (26), we know that $N^\varepsilon \to N = M^2$ in $L^1((0,T) \times \Omega)$. Further, we consider

$$\int_0^\infty y a(y)\, f^\varepsilon\, dy = \int_0^\infty y a(y)\, e^{-\frac{y}{\sqrt{N^\varepsilon}}}\, dy + \int_0^\infty y a(y)\, h^\varepsilon\, dy\,,$$

where with (30)

$$\int_0^T \int_\Omega \int_0^\infty y a(y)\, |h^\varepsilon|\, dydxdt \le a^* \int_0^T \int_\Omega \int_0^\infty y\, |h^\varepsilon|\, dydxdt \to 0 \quad \text{as } \varepsilon \to 0\,.$$

Therefore, it follows that

$$\int_0^\infty y a(y)\, f^\varepsilon\, dy \to n(N)\,,$$

in $L^1((0,T) \times \Omega)$ as $\varepsilon \to 0$, and moreover N is a weak solution of the problem

$$\begin{cases} \partial_t N - \triangle_x n(N) = 0\,, \\ \nabla_x N \cdot \nu(x)|_{\partial\Omega} = 0\,, \end{cases}$$

in the following sense:

$$\int_\Omega (N(T,x) - N_{in}(x))\varphi(x)\, dx = \int_0^T \int_\Omega n(N(t,x))\triangle_x \varphi(x)\, dxdt,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ smooth with $\nabla_x \varphi \cdot \nu(x)|_{\partial\Omega} = 0$ and $T > 0$. Note that $\partial_t N^\varepsilon$ is bounded in a negative Sobolev space in the x variable allowing to define values of N at a given time $T > 0$. This concludes the proof of Theorem 1.1.

Acknowledgements

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