Regularization Properties of the 2-Dimensional Non Radially Symmetric Non Cutoff Spatially Homogeneous Boltzmann Equation for Maxwellian Molecules

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Abstract

We prove in this work that the solution of the two-dimensional spatially homogeneous Boltzmann equation with Maxwellian molecules lies "almost" in H^1 (in the velocity variable) as soon as the initial datum has finite mass, energy and entropy. No assumption of radial symmetry of the solution is made in this paper.

1 Introduction

The spatially homogeneous Boltzmann equation of rarefied gases (Cf. [Ce], [Ch, Co], [Tr, Mu]) writes

$$\frac{\partial f}{\partial t}(t,v) = Q(f,f)(t,v), \tag{1.1}$$

where $f(t,v)$ is the nonnegative density of particles which at time t move with velocity v, and Q is a quadratic kernel acting only on the velocity variable v.

In the 2-dimensional case, this kernel can be written in the form of

$$Q(f,g)(t,v) = \int_{v_*\in\mathbb{R}^2}\int_{\theta=-\pi}^{\pi}\left\{f\left(t, \frac{v+v_*}{2} + R_\theta(\frac{v-v_*}{2})\right)\right.$$

$$\left. g\left(t, \frac{v+v_*}{2} - R_\theta(\frac{v-v_*}{2})\right) - f(t,v)g(t,v_*)\right\} b(|v-v_*|,\theta)d\theta dv. \tag{1.2}$$

The function b in (1.2) is a nonnegative cross section which depends on the type of interaction between particles.

We concentrate in this work on the case of Maxwellian molecules, which means that

$$b(x,y) = \beta(|y|). \tag{1.3}$$

We do not make here the assumption of angular cutoff of Grad (Cf. [Gr]). It means that β will be singular in 0. More precisely, by analogy with the 3-dimensional case, we shall always make the following assumption :

Assumption 1 : *The cross section b satisfies (1.3). Moreover $\beta \in L^\infty_{loc}(]0,\pi[)$, and there exists $\beta_0, \beta_1 > 0, \gamma \in]1,3[$ such that*

$$\beta_0|\theta|^{-\gamma} \le \beta(\theta) \le \beta_1|\theta|^{-\gamma}. \tag{1.4}$$

Note that β will therefore never be integrable over $[0,\pi]$. However, we do not consider here the very difficult case when $\gamma = 3$, which corresponds (at least in dimension 3) to the Coulombian interaction.

Few results are known for the Boltzmann equation under assumption 1. Uniqueness for example is an open problem. However, a direct application of the methods of [A] (Cf. also [De 1]) yields a theorem of existence.

More precisely, we shall make the

Assumption 2 : *We suppose that* $f_0 : \mathbb{R}^2 \to \mathbb{R}_+$ *is an initial datum satisfying*

$$\int_{v\in\mathbb{R}^2} f_0(v)\{1+|v|^2+|\log f_0(v)|\}dv < +\infty. \tag{1.5}$$

We also define for all $k \geq 0$ the quantities

$$l_{k,f}(t) = \int_{v\in\mathbb{R}^2} f(t,v)|v|^{2k}dv. \tag{1.6}$$

Then, the following theorem holds:

Theorem 1 : *Let* b *be a cross section satisfying assumption 1, and* f_0 *be an initial datum satisfying assumption 2. Then, there exists a weak solution of the Boltzmann equation (1.1), (1.2) such that*

$$\forall t \in \mathbb{R}_+, \quad l_{0,f}(t) = l_{0,f}(0), \tag{1.7}$$

and

$$\sup_{t\in\mathbb{R}_+} l_{1,f}(t) \leq l_{1,f}(0). \tag{1.8}$$

Moreover, for all $T > 0$*, we also have the entropy estimate:*

$$\sup_{t\in[0,T]} \int_{v\in\mathbb{R}^2} f(t,v)|Log f(t,v)|dv < +\infty. \tag{1.9}$$

Finally, if for some $k \geq 2, k \in \mathbb{N}$*,*

$$l_{k,f}(0) < +\infty, \tag{1.10}$$

then

$$\sup_{t\in\mathbb{R}_+} \quad l_{k,f}(t) < +\infty \tag{1.11}$$

and

$$\forall t \in \mathbb{R}_+, \quad l_{1,f}(t) = l_{1,f}(0), \tag{1.12}$$

which means that the conservation of energy holds.

Remark : A weak solution of the Boltzmann equation (1.1), (1.2) is simply a function $f(t,v) \geq 0$ satisfying (1.7), (1.8) and such that for all $\phi \in C_b^2(\mathbb{R}_v^2)$,

$$\frac{\partial}{\partial t} \int_{v \in \mathbb{R}^2} f(t,v)\phi(v)dv = \int_{v \in \mathbb{R}^2} Q(f)(t,v)\phi(v)dv. \qquad (1.13)$$

This work is aimed at proving that for any $t > 0$, a solution $f(t,.)$ of (1.1), (1.2) given by theorem 1 lies in fact in $H^{1-0}(\mathbb{R}_v^2)$.

Note that a similar result was already proven in [De 1] in the particular case when f_0 (and consequently $f(t,.)$ for all $t \geq 0$) is radially symmetric.

The proof was based on the use of the Fourier transform $\hat{f}$ of f with respect to the velocity variable. Note that the use of the Fourier transform in the context of the Boltzmann equation is classical for example when one looks for explicit or semi–explicit solutions (Cf. [Bb]).

The main difference between the present work and that of [De 1] is that in the former, a pointwise analysis of $\hat{f}$ was possible, the relevant quantities being of the form

$$Y_{\alpha,f}(t) = \sup_{\xi \in \mathbb{R}^2} (1 + |\xi|)^\alpha |\hat{f}(t,\xi)|, \qquad (1.14)$$

whereas in the latter, one needs to perform an L^2 analysis of $\hat{f}$, the relevant quantities becoming norms of Sobolev spaces of the form

$$Z_{\alpha,f}(t) = \int_{\xi \in \mathbb{R}^2} (1 + |\xi|)^\alpha |\hat{f}(t,\xi)|^2 d\xi. \qquad (1.15)$$

Note also that quantities like $Z_{\alpha,f}(t)$ were already used in another generalization of [De 1], namely the case when f_0 is radially symmetric, but b depends (smoothly) on $|v - v_*|$ (Cf. [De 2]).

Finally, note that if β were not singular (this is exactly the case of the cutoff equation), no improvement of the regularity of $f(t,.)$ could occur (Cf. [L], [We] or [B, De]).

The structure of this article is as follows: in section 2, we split the quantity

$$l_{\alpha,R}(f,g) = \int_{|\xi| \leq R} (1 + |\xi|)^\alpha \widehat{Q(f,g)}(\xi) \overline{\hat{f}(\xi)} d\xi \qquad (1.16)$$

in order to obtain a "dominant" and a "dominated" term. In section 3, we prove that the "dominant" term is nonnegative and is in fact bigger than some Sobolev norm of f (up to a strictly positive constant). Then, in section 4, we estimate the "dominated" term by some Sobolev norm of lower order, so that the terms "dominant", and "dominated" are justified. Finally, in section 5, we use these estimates to prove our main theorem.

2 The structure of $l_{\alpha,R}$

This section is devoted to the proof of the following lemma:

Lemma 2.1 : *We make assumption 1. Let $f, g \in L^1(I\!R^2_v, I\!R)$ be such that $\hat{f}, \hat{g} \in C^2(I\!R^2_\xi)$ and $\alpha \in I\!R$, $R > 1$. Then the quantity $l_{\alpha,R}(f,g)$ defined in (1.16) can be written in the form of*

$$l_{\alpha,R}(f,g) = p_{\alpha,R}(f,g) + q_{\alpha,R}(f,g), \tag{2.1}$$

where

$$\begin{aligned} p_{\alpha,R}(f,g) = \int_{|\xi|\le R} \int_{\theta=-\pi}^{\pi} (1+|\xi|)^\alpha \Big\{ \frac{1}{2} \hat{f}(R_{\frac{\theta}{2}}\xi)\, \overline{\hat{f}(\xi)}\, \hat{g}\left(\frac{\xi}{2} - R_\theta \frac{\xi}{2}\right) \\ + \frac{1}{2}\, \overline{\hat{f}(R_{\frac{\theta}{2}}\xi)}\, \hat{f}(\xi)\, \overline{\hat{g}\left(\frac{\xi}{2} - R_\theta \frac{\xi}{2}\right)} - \frac{1}{2}\Big(|\hat{f}(\xi)|^2 \\ + |\hat{f}(R_{\frac{\theta}{2}}\xi)|^2\Big)\, \hat{g}(0) \Big\}\, \beta(|\theta|) d\theta d\xi, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} q_{\alpha,R}(f,g) = \int_{|\xi|\le R} \int_{\theta=-\pi}^{\pi} (1+|\xi|)^\alpha \left\{ \hat{f}\left(\frac{\xi}{2} + R_\theta \frac{\xi}{2}\right) - \hat{f}\left(R_{\frac{\theta}{2}}\xi\right) \right\} \\ \times \hat{g}\left(\frac{\xi}{2} - R_\theta \frac{\xi}{2}\right) \overline{\hat{f}(\xi)} \beta(|\theta|) d\theta d\xi. \end{aligned} \tag{2.3}$$

Proof of lemma 2.1 : We compute here the Fourier transform of $Q(f,g)$.

$$\begin{aligned} \widehat{Q(f,g)} = \int_{v\in I\!R^2} \int_{v_*\in I\!R^2} \int_{\theta=-\pi}^{\pi} e^{-iv.\xi} \Big\{ f\Big(\frac{v+v_*}{2} + R_{-\theta}\Big(\frac{v-v_*}{2}\Big)\Big) \\ \times g\Big(\frac{v+v_*}{2} - R_\theta\Big(\frac{v-v_*}{2}\Big)\Big) - f(v)g(v_*) \Big\} \beta(|\theta|) d\theta dv_* dv. \end{aligned} \tag{2.4}$$

Using the change of variables

$$v' = \frac{v+v_*}{2} + R_\theta\left(\frac{v-v_*}{2}\right), \tag{2.5}$$

$$v'_* = \frac{v+v_*}{2} - R_\theta\left(\frac{v-v_*}{2}\right), \tag{2.6}$$

one gets

$$\widehat{Q(f,g)}(\xi) = \int_{v\in\mathbb{R}^2}\int_{v_*\in\mathbb{R}^2}\int_{\theta=-\pi}^{\pi}\left(e^{-i\{\frac{v+v_*}{2}+R_\theta(\frac{v-v_*}{2})\}\cdot\xi} - e^{-iv\cdot\xi}\right)$$

$$\times f(v)g(v_*)\beta(|\theta|)d\theta dv_* dv$$

$$= \int_{\theta=-\pi}^{\pi}\left\{\hat f\left(\frac{\xi}{2}+R_\theta\frac{\xi}{2}\right)\hat g\left(\frac{\xi}{2}-R_\theta\frac{\xi}{2}\right) - \hat f(\xi)\hat g(0)\right\}\beta(|\theta|)d\theta. \tag{2.7}$$

Note that eq. (2.7) is particularly simple. This comes out of assumption (1.3). To get an idea of the corresponding term in the general case, we refer to [De 2].

We now compute

$$l_{\alpha,R}(f,g) = \int_{|\xi|\le R}(1+|\xi|)^\alpha \widehat{Q(f,g)}(\xi)\overline{\hat f}(\xi)d\xi$$

$$= \int_{|\xi|\le R}\int_{\theta=-\pi}^{\pi}(1+|\xi|)^\alpha\left\{\hat f\left(R_{\frac{\theta}{2}}\xi\right)\hat g\left(\frac{\xi}{2}-R_\theta\frac{\xi}{2}\right)\overline{\hat f(\xi)}\right.$$

$$\left. - \hat f(\xi)\overline{\hat f(\xi)}\hat g(0)\right\}\beta(|\theta|)d\theta d\xi + q_{\alpha,R}(f,g). \tag{2.8}$$

Using then the change of variables $\xi \to R_{\frac{\theta}{2}}\xi$, we get

$$l_{\alpha,R}(f,g) - q_{\alpha,R}(f,g) = \int_{|\xi|\le R}\int_{\theta=-\pi}^{\pi}(1+|\xi|)^\alpha$$

$$\times\left\{\hat f\left(R_{\frac{\theta}{2}}\xi\right)\overline{\hat f(\xi)}\hat g\left(\frac{\xi}{2}-R_\theta\frac{\xi}{2}\right) - \frac{1}{2}\left(|\hat f(\xi)|^2 + |\hat f\left(R_{\frac{\theta}{2}}\xi\right)|^2\right)\hat g(0)\right\}\beta(|\theta|)d\theta d\xi. \tag{2.9}$$

Using now the change of variables $\xi \to -\xi$, and the fact that f, g are real functions, we get lemma 2.1.

3 The dominant term

This section is devoted to the proof of a lemma giving the relation between the dominant term $p_{\alpha,R}(f)$ and Sobolev norms like $Z_{\alpha,f}$. More precisely, we prove the

Lemma 3.1 : *With the same hypothesis as in lemma 2.1, the quantity $p_{\alpha,R}(f,g)$ is real and satisfies the following estimate for all $\epsilon \in]0, \frac{\pi}{6}[$, $r > 1$:*

$$- p_{\alpha,R}(f,g) \geq \chi_{\epsilon,r}(g) \int_{1\leq|\xi|\leq R} |\xi|^{\alpha+\gamma-1} |\hat{f}(\xi)|^2 d\xi, \tag{3.1}$$

where

$$\chi_{\epsilon,r}(g) = \beta_0 \frac{18}{\pi^2} \epsilon^2 \left\{ l_{0,g} - \frac{l_{1,g}}{r^2} - \sup_{|A|\leq(\frac{8r^2}{\pi}+\frac{32}{3}\pi r)\,\epsilon} \int_A g(x)dx \right\}. \tag{3.2}$$

Proof of lemma 3.1 : Note first that for all $z_1, z_2, z_3, \in \mathbb{C}$, $x \in \mathbb{R}_+$,

$$(|z_1|^2 + |z_2|^2)x - (z_1\bar{z}_2\bar{z}_3 + \bar{z}_1 z_2 z_3) \geq (|z_1|^2 + |z_2|^2)\,(x - |z_3|). \tag{3.3}$$

Applying this inequality to

$$z_1 = \hat{f}(\xi), \tag{3.4}$$

$$z_2 = \hat{f}\left(R_{\frac{\theta}{2}}\xi\right), \tag{3.5}$$

$$z_3 = \hat{g}\left(\frac{\xi}{2} - R_\theta \frac{\xi}{2}\right) \tag{3.6}$$

$$x = \hat{g}(0), \tag{3.7}$$

we immediately get

$$-p_{\alpha,R}(f,g) \geq \frac{1}{2} \int_{|\xi|\leq R} \int_{\theta=-\pi}^{\pi} (1+|\xi|)^\alpha\, |\hat{f}(\xi)|^2$$
$$\left\{ \hat{g}(0) - \left| \hat{g}\left(\frac{\xi}{2} - R_\theta\frac{\xi}{2}\right)\right| \right\} \beta(|\theta|)d\theta d\xi. \tag{3.8}$$

We now wish to bound from below the quantity

$$k_\xi(g) = \int_{\theta=-\pi}^{\pi} \left\{ \hat{g}(0) - \left| \hat{g}\left(\frac{\xi}{2} - R_\theta\frac{\xi}{2}\right)\right| \right\} \beta(|\theta|)d\theta. \tag{3.9}$$

Note first that using the change of variables $u = \theta|\xi|$, we get

$$k_\xi(g) \geq |\xi|^{\gamma-1} \int_{u=-\pi|\xi|}^{\pi|\xi|} \left\{\hat{g}(0) - \left|\hat{g}\left(\frac{\xi}{2} - R_{\frac{u}{|\xi|}} \frac{\xi}{2}\right)\right|\right\} \beta_0 \, |u|^{-\gamma} du. \tag{3.10}$$

We now suppose that $|\xi| \geq 1$. Then,

$$k_\xi(g) \geq |\xi|^{\gamma-1}\beta_0 \int_{u=-1}^{1} \left\{\hat{g}(0) - \left|\hat{g}\left(\sin\left(\frac{u}{2|\xi|}\right) R_{\frac{u}{2|\xi|}-\frac{\pi}{2}}(\xi)\right)\right|\right\} du$$

$$\geq |\xi|^{\gamma-1}\beta_0 \int_{u=-1}^{1} \inf_{k\in I\!R} \int_{x\in I\!R^2} g(x)$$

$$\times \left\{1 - \cos\left\{k + \sin\left(\frac{u}{2|\xi|}\right)\left(R_{\frac{u}{2|\xi|}-\frac{\pi}{2}}\xi\right)\cdot x\right\}\right\} dx du$$

$$\geq |\xi|^{\gamma-1}\beta_0 \int_{u=-1}^{1} \inf_{k\in I\!R} \int_{x\in I\!R^2} 2\, g(x)$$

$$\times \sin^2\left(\frac{k}{2} + \frac{1}{2}\sin\left(\frac{u}{2|\xi|}\right)\left(R_{\frac{u}{2|\xi|}-\frac{\pi}{2}}\xi\right)\cdot x\right) dx du. \tag{3.11}$$

But for all $\epsilon \in [0, \frac{\pi}{6}[$, $\eta \in I\!R^2$, $k \in I\!R$,

$$L_{k,\eta}(g) = \int_{x\in I\!R^2} g(x)\, \sin^2\left(\frac{k+\eta\cdot x}{2}\right) dx$$

$$\geq (\frac{3\epsilon}{\pi})^2 \int_{\{x\in I\!R^2, \forall p\in Z\!\!Z, |\eta\cdot x + k - 2p\pi| \geq 2\epsilon\}} g(x)\, dx, \tag{3.12}$$

since for $|\epsilon| \leq \frac{\pi}{6}$, we have

$$|\sin\epsilon| \geq \frac{3}{\pi}|\epsilon|. \tag{3.13}$$

Therefore, for all $r > 1$, $\epsilon \in [0, \frac{\pi}{6}[$, $\eta \in I\!R^2$, $k \in I\!R$, (and denoting by $|A|$ the measure of a given set A),

$$L_{k,\eta}(g) \geq (\frac{3\epsilon}{\pi})^2 \left\{ \int_{x\in I\!R^2} g(x)\, dx - \int_{\{x\in I\!R^2, \exists p\in Z\!\!Z, |\frac{\eta}{|\eta|}\cdot x + \frac{k}{|\eta|} - \frac{2p\pi}{|\eta|}| \leq \frac{2\epsilon}{|\eta|}\}} g(x)\, dx\right\}$$

$$\geq (\frac{3\epsilon}{\pi})^2 \left\{l_{0,g} - \int_{|x|\geq r} g(x)\, dx - \int_{\{|x|\leq r, \exists p\in Z\!\!Z, |\frac{\eta}{|\eta|}\cdot x + \frac{k}{|\eta|} - \frac{2p\pi}{|\eta|}| \leq \frac{2\epsilon}{|\eta|}\}} g(x)\, dx\right\}$$

$$\geq (\frac{3\epsilon}{\pi})^2 \left\{l_{0,g} - \frac{l_{1,g}}{r^2} - \sup_{\{|A|\leq \frac{8\epsilon r}{|\eta|}(\frac{r}{\pi}|\eta|+1)\}} \int_A g(x)\, dx\right\}. \tag{3.14}$$

Then,

$$k_\xi(g) \geq |\xi|^{\gamma-1}\beta_0 \int_{u=-1}^{1} 2 \inf_{k\in\mathbb{R}} L_{k,\sin\left(\frac{u}{2|\xi|}\right)}\left(R_{\frac{u}{2|\xi|}-\frac{\pi}{2}}\xi\right)(g)\, du$$

$$\geq |\xi|^{\gamma-1}\beta_0\, 2\,(\frac{3\epsilon}{\pi})^2 \int_{\frac{1}{2}\leq|u|\leq 1} \left\{ l_{0,g} - \frac{l_{1,g}}{r^2} - \sup_{\{|A|\leq \frac{8\epsilon r^2}{\pi} + \frac{8\epsilon r}{|\sin(\frac{u}{2|\xi|})||\xi|}\}} \int_A g(x)\, dx \right\}, \tag{3.15}$$

and lemma 3.1 is an easy consequence of this inequality and of estimate (3.13).

4 The dominated term

We now need to estimate the quantity $q_{\alpha,R}(f,g)$. We begin by proving the following lemma :

Lemma 4.1 : *Under the hypothesis of lemma 2.1 and for all $\epsilon > 0$ (small enough), there exists $C(\epsilon,\alpha,\gamma) > 0$ and $\tau(\epsilon) > 0$ such that*

$$|q_{\alpha,R}(f,g)| \leq C(\epsilon,\alpha,\gamma)\beta_1 \left(\| f \|_{L^1(\mathbb{R}^2_v)} + \| g \|_{L^1(\mathbb{R}^2_v)}\right)$$
$$\left(\int_{|\xi|\leq R} (1+|\xi|)^{\alpha+\gamma-1-\tau(\epsilon)} |\hat f(\xi)|^2 d\xi + \int_{|\xi|\leq R} (1+|\xi|)^{\alpha+\gamma-1-\tau(\epsilon)} |\hat g(\xi)|^2 d\xi \right.$$
$$\left. + \int_{|\xi|\leq R} (1+|\xi|)^{\alpha+\gamma-3+\epsilon} |\nabla \hat f(\xi)|^2 d\xi \right). \tag{4.1}$$

Proof of lemma 4.1 : Note that for all $\delta > 0$ small enough,

$$|q_{\alpha,R}(f,g)| \leq J_1 + J_2, \tag{4.2}$$

where

$$J_1 = \int_{|\xi|\leq R}\int_{|\theta|\geq\frac{\pi}{2}} (1+|\xi|)^\alpha |\hat f\left(\frac{\xi}{2} + R_\theta \frac{\xi}{2}\right) - \hat f\left(R_{\frac{\theta}{2}}\xi\right)|$$
$$\times |\hat g\left(\frac{\xi}{2} - R_\theta\left(\frac{\xi}{2}\right)\right)|\,|\hat f(\xi)|\,\beta(|\theta|) d\theta d\xi, \tag{4.3}$$

and

$$J_2 = \int_{|\xi|\leq R}\int_{|\theta|\leq\frac{\pi}{2}} (1+|\xi|)^\alpha |\hat f\left(\frac{\xi}{2} + R_\theta \frac{\xi}{2}\right) - \hat f\left(R_{\frac{\theta}{2}}\xi\right)|^{\frac{3-\gamma-\delta}{2}}$$

$$\times|\frac{\xi}{2}+R_\theta\frac{\xi}{2}-R_{\frac{\theta}{2}}\xi|^{\frac{\gamma-1+\delta}{2}}\left(\int_{u=0}^{1}|\nabla\hat{f}\left(u\left(\frac{\xi}{2}+R_\theta\frac{\xi}{2}\right)+(1-u)R_{\frac{\theta}{2}}\xi\right)|du\right)^{\frac{\gamma-1+\delta}{2}}$$

$$\times|\hat{g}\left(\frac{\xi}{2}-R_\theta\left(\frac{\xi}{2}\right)\right)||\hat{f}(\xi)|\,\beta(|\theta|)d\theta d\xi. \tag{4.4}$$

Then,

$$J_1\leq\|\hat{f}\|_{L^\infty(I\!R^2_\xi)}\int_{|\xi|\leq R}\int_{|\theta|\geq\frac{\pi}{2}}(1+|\xi|)^\alpha|\hat{g}\left(\frac{\xi}{2}-R_\theta\frac{\xi}{2}\right)|^2\beta_1|\theta|^{-\gamma}d\theta d\xi$$

$$+\|\hat{f}\|_{L^\infty(I\!R^2_\xi)}\int_{|\xi|\leq R}\int_{|\theta|\geq\frac{\pi}{2}}(1+|\xi|)^\alpha|\hat{f}(\xi)|^2\beta_1|\theta|^{-\gamma}d\theta d\xi$$

$$\leq\beta_1\|\hat{f}\|_{L^\infty(I\!R^2_\xi)}\int_{|\xi|\leq R}(1+|\xi|)^\alpha|\hat{g}(\xi)|^2d\xi\int_{|\theta|\geq\frac{\pi}{2}}|\sin\frac{\theta}{2}|^{-2-\alpha}|\theta|^{-\gamma}d\theta$$

$$+\beta_1\|\hat{f}\|_{L^\infty(I\!R^2_\xi)}\int_{|\xi|\leq R}(1+|\xi|)^\alpha|\hat{f}(\xi)|^2d\xi\int_{|\theta|\geq\frac{\pi}{2}}|\theta|^{-\gamma}d\theta. \tag{4.5}$$

Moreover,

$$J_2\leq\|\hat{g}\|_{L^\infty(I\!R^2_\xi)}\int_{|\xi|\leq R}\int_{|\theta|\leq\frac{\pi}{2}}(1+|\xi|)^{\alpha+\gamma-1-\frac{\delta}{2}}|\hat{f}(\xi)|^2\beta_1|\theta|^{-1+\delta}d\theta d\xi$$

$$+\|\hat{g}\|_{L^\infty(I\!R^2_\xi)}\int_{|\xi|\leq R}\int_{|\theta|\leq\frac{\pi}{2}}(1+|\xi|)^{\alpha+\frac{3}{2}\delta}$$

$$\times|\hat{f}\left(\frac{\xi}{2}+R_\theta\frac{\xi}{2}\right)-\hat{f}\left(R_{\frac{\theta}{2}}\xi\right)|^{3-\gamma-\delta}$$

$$\times\left(\int_{u=0}^{1}|\nabla\hat{f}\left(u\left(\frac{\xi}{2}+R_\theta\frac{\xi}{2}\right)+(1-u)R_{\frac{\theta}{2}}\xi\right)|du\right)^{\gamma-1+\delta}\beta_1|\theta|^{-1+\delta}d\theta d\xi$$

$$\leq\beta_1\|\hat{g}\|_{L^\infty(I\!R^2_\xi)}\int_{|\xi|\leq R}(1+|\xi|)^{\alpha+\gamma-1-\frac{\delta}{2}}|\hat{f}(\xi)|^2d\xi\int_{|\theta|\leq\frac{\pi}{2}}|\theta|^{-1+\delta}d\theta$$

$$+\beta_1\|\hat{g}\|_{L^\infty(I\!R^2_\xi)}\int_{|\xi|\leq R}\int_{|\theta|\leq\frac{\pi}{2}}(1+|\xi|)^{\alpha+\gamma-1-\frac{\delta}{2}}$$

$$\times|\hat{f}\left(\frac{\xi}{2}+R_\theta\frac{\xi}{2}\right)-\hat{f}\left(R_{\frac{\theta}{2}}\xi\right)|^2\beta_1|\theta|^{-1+\delta}d\theta d\xi$$

$$+\beta_1\|\hat{g}\|_{L^\infty(I\!R^2_\xi)}\int_{|\xi|\leq R}\int_{|\theta|\leq\frac{\pi}{2}}(1+|\xi|)^{\alpha+\gamma-3+\frac{\delta}{2}\left\{\frac{-\gamma+13-\delta}{\gamma-1+\delta}\right\}}$$

$$\times \left(\int_{u=0}^{1} |\nabla \hat{f} \left(u \left(\frac{\xi}{2} + R_\theta \frac{\xi}{2} \right) + (1-u) R_{\frac{\theta}{2}} \xi \right) |du \right)^2 |\theta|^{-1+\delta} d\theta d\xi. \tag{4.6}$$

Finally,

$$|q_{\alpha,R}(f,g)| \leq C\beta_1 \left(\|\hat{f}\|_{L^\infty(\mathbb{R}^2_\xi)} + \|\hat{g}\|_{L^\infty(\mathbb{R}^2_\xi)} \right)$$

$$\times \int_{|\xi|\leq R} (1+|\xi|)^{\alpha+\gamma-1-\frac{\delta}{2}} \left(|\hat{f}(\xi)|^2 + |\hat{g}(\xi)|^2 \right) d\xi$$

$$+ C\beta_1 \|\hat{g}\|_{L^\infty(\mathbb{R}^2_\xi)} \int_{|\xi|\leq R} (1+|\xi|)^{\alpha+\gamma-3+\frac{\delta}{2}\left\{\frac{-\gamma+13-\delta}{\gamma-1+\delta}\right\}} |\nabla \hat{f}(\xi)|^2 d\xi \tag{4.7}$$

for some constant $C > 0$ depending only on δ.

But when $\delta \to 0^+$, $\frac{\delta}{2}\left\{\frac{-\gamma+13-\delta}{\gamma-1+\delta}\right\} \to 0^+$, and therefore lemma 4.1 holds.

Lemma 4.2 : *Under the hypothesis of lemma 2.1 and for all $\epsilon > 0$ (small enough), there exists $C(\epsilon,\alpha,\gamma) > 0$ and $\tau(\epsilon) > 0$ such that*

$$|q_{\alpha,R}(f,g)| \leq C(\epsilon,\alpha,\gamma)\beta_1 \left(\|f\|_{L^1(\mathbb{R}^2)} + \|g\|_{L^1(\mathbb{R}^2)} \right)$$

$$\left\{ \int_{|\xi|\leq R} (1+|\xi|)^{\sup(\alpha+\gamma-1-\tau(\epsilon), 2\alpha+2\gamma-4+3\epsilon)} \left(|\hat{f}(\xi)|^2 + |\hat{g}(\xi)|^2 \right) d\xi + \right.$$

$$\left. \|(1+x^2)f\|_{L^1(\mathbb{R}^2)} + \int_{|\xi|=R} (1+|\xi|)^{\alpha+\gamma-3+\epsilon} |\hat{f}(\xi)||\nabla \hat{f}(\xi)| d\xi \right\}. \tag{4.8}$$

Proof of lemma 4.2 : According to lemma 4.1, we only have to estimate the quantity

$$W_{\alpha,\gamma,\epsilon,R}(f) = \int_{|\xi|\leq R} (1+|\xi|)^{\alpha+\gamma-3+\epsilon} |\nabla \hat{f}(\xi)|^2 d\xi. \tag{4.9}$$

Integrating by parts, we get

$$|W_{\alpha,\gamma,\epsilon,R}(f)| = -\int_{|\xi|\leq R} (1+|\xi|)^{\alpha+\gamma-3+\epsilon} \hat{f}(\xi) \Delta \hat{f}(\xi) d\xi$$

$$-\int_{|\xi|\leq R} (\alpha+\gamma-3+\epsilon)(1+|\xi|)^{\alpha+\gamma-4+\epsilon} \hat{f}(\xi) \left(\nabla \hat{f}(\xi) \cdot \frac{\xi}{|\xi|} \right) d\xi$$

$$+ \int_{|\xi|=R} (1+|\xi|)^{\alpha+\gamma-3+\epsilon} \hat{f}(\xi) \left(\nabla \hat{f}(\xi) \cdot \frac{\xi}{|\xi|} \right) d\xi. \tag{4.10}$$

Using now Hölder's inequality,

$$|W_{\alpha,\gamma,\epsilon,R}(f)| \leq C(\epsilon,\alpha,\gamma)\Big\{\|\nabla \hat{f}\|_{L^\infty(I\!R^2)} + \|\Delta \hat{f}\|_{L^\infty(I\!R^2)}$$

$$+ \int_{|\xi|\leq R} (1+|\xi|)^{2\alpha+2\gamma-4+3\epsilon} |\hat{f}(\xi)|^2 d\xi$$

$$+ \int_{|\xi|=R} (1+|\xi|)^{\alpha+\gamma-3+\epsilon} |\hat{f}(\xi)|\, |\nabla \hat{f}(\xi)| d\xi\Big\}, \tag{4.11}$$

whence lemma 4.2.

5 Properties of Regularization for the Boltzmann Equation

We are now able to give our main theorem :

Theorem 2 : *Let f_0 be an initial datum satisfying assumption 2 and b be a cross section satisfying assumption1.*

Then, a solution f of the Boltzmann equation (1.1), (1.2) together with the initial datum f_0 given by theorem 1 is such that for all $t_0 > 0, \epsilon > 0$, $f \in L^1_{loc}([t_0,+\infty[; H^{1-\epsilon}(I\!R^2_v)) \cap L^\infty_{loc}([t_0,+\infty[; H^{\frac{3-\gamma}{2}-\epsilon}(I\!R^2_v))$ or, in abridged notation :

$$f \in L^1_{loc}(]0;+\infty[; H^{1-0}(I\!R^2_v)) \quad \cap \quad L^\infty_{loc}(]0;+\infty[; H^{\frac{3-\gamma}{2}-0}(I\!R^2_v)). \tag{5.1}$$

Before giving the proof of this theorem , we shall state the following miscellaneous results, which enable us to apply the lemmas of sections 2,3 and 4 to the present situation :

Lemma 5.1 : *Let f be a solution of the Boltzmann equation given by theorem 1. Then, for all $T > 0$, there exists $C_T > 0$, $\epsilon_0 \in]0, \frac{\pi}{6}[$ and $r_0 > 1$ such that*

$$\inf_{0\leq t\leq T} \chi_{\epsilon_0,r_0}(f(t,.)) \geq C_T, \tag{5.2}$$

where $\chi_{\epsilon,r}$ is defined in (3.2).

Proof of lemma 5.1 : Note that because of estimates (1.7), (1.8) and (1.9), the family $(f(t,\cdot))_{t\in[0,T]}$ is equiintegrable (or weakly compact in

$L^1(\mathbb{R}_v^2))$. Therefore, for any given $r > 1$,

$$\lim_{\epsilon \to 0} \sup_{t \in [0,T]} \sup_{|A| \le (\frac{8r^2}{\pi} + \frac{32}{3}\pi r)\,\epsilon} \int_A f(t,v)\, dv = 0. \tag{5.3}$$

Lemma 5.1 is then an easy consequence of this estimate and definition (3.2).

Lemma 5.2 : *Let $\alpha \in \mathbb{R}, R > 1$, and f be a solution of the Boltzmann equation given by theorem 1. Then, for all $T > 0$, there exists $D(\alpha,\gamma,T) > 0, E(\alpha,\gamma,T) > 0$, such that when $t \in [0,T]$,*

$$-p_{\alpha,R}(f(t,.), f(t,.)) \ge D(\alpha,\gamma,T) \int_{|\xi| \le R} (1+|\xi|)^{\alpha+\gamma-1} |\hat{f}(t,\xi)|^2 d\xi$$

$$-E(\alpha,\gamma,T) \left\{ \|f\|_{L^1}^2 + 1_{\{\alpha+\gamma-1>0\}} \int_{|\xi| \le R} |\hat{f}(t,\xi)|^2 d\xi \right\}. \tag{5.4}$$

Proof of lemma 5.2 : Note first that for any $t \in [0,T]$, estimates (1.7), (1.8) and (1.9) ensure that $f(t,\cdot) \in L^1(\mathbb{R}_v^2)$, $\hat{f}(t,\cdot) \in C^2(\mathbb{R}_\xi^2)$. Therefore, we can apply lemma 3.1 and get, thanks to lemma 5.1,

$$-p_{\alpha,R}(f(t,.), f(t,.)) \ge C_T \int_{1 \le |\xi| \le R} |\xi|^{\alpha+\gamma-1} |\hat{f}(t,\xi)|^2 d\xi. \tag{5.5}$$

Note then that

$$(1+|\xi|)^{\alpha+\gamma-1} \le \sup(1, 2^{\alpha+\gamma-2})\,(1+|\xi|^{\alpha+\gamma-1}). \tag{5.6}$$

Therefore,

$$-p_{\alpha,R}(f(t,.), f(t,.)) \ge D(\alpha,\gamma,T) \int_{1 \le |\xi| \le R} (1+|\xi|)^{\alpha+\gamma-1} |\hat{f}(t,\xi)|^2 d\xi$$

$$-1_{\{\alpha+\gamma-1>0\}} C_T \int_{1 \le |\xi| \le R} |\hat{f}(t,\xi)|^2 d\xi$$

$$\ge D(\alpha,\gamma,T) \int_{|\xi| \le R} (1+|\xi|)^{\alpha+\gamma-1} |\hat{f}(t,\xi)|^2 d\xi$$

$$-E(\alpha,\gamma,T) \left\{ \| \hat{f} \|_{L^\infty}^2 + 1_{\{\alpha+\gamma-1>0\}} \int_{|\xi| \le R} |\hat{f}(t,\xi)|^2 d\xi \right\}, \tag{5.7}$$

which concludes the proof of lemma 5.2.

Lemma 5.3 : *Let $\alpha \in I\!R$, $R > 1$, and f be a solution of the Boltzmann equation given by theorem 1. Then, for all $\epsilon > 0$ (small enough), there exists $C(\epsilon, \alpha, \gamma) > 0$ and $\tau(\epsilon) > 0$ such that for all $t \in I\!R_+$,*

$$|q_{\alpha,R}(f(t,.), f(t,.))| \leq C(\epsilon,\alpha,\gamma)\beta_1 \parallel f \parallel_{L^1(I\!R^2)} \{l_{0,f} + l_{1,f}$$
$$+ \int_{|\xi|\leq R} (1+|\xi|)^{\sup(\alpha+\gamma-1-\tau(\epsilon), 2\alpha+2\gamma-4+3\epsilon)} |\hat{f}(t,\xi)|^2 d\xi$$
$$+ \int_{|\xi|=R} (1+|\xi|)^{\alpha+\gamma-3+\epsilon} |\hat{f}(t,\xi)|\, |\nabla \hat{f}(t,\xi)| d\xi \Big\}. \quad (5.8)$$

Proof of lemma 5.3 : Just make f = g in lemma 4.2 and observe that $f(t,.)$ satifies for all $t > 0$ the hypothesis of lemma 2.1

We now give the proof of theorem 2 :

Proof of theorem 2 : Note first that since f is a weak solution of the Boltzmann equation (1.1), (1.2) and $v \to e^{-iv.\xi}$ in $C^2(I\!R^2_v)$ for all $\xi \in I\!R^2$, one can write

$$\frac{\partial \hat{f}}{\partial t}(t,\xi) = \widehat{Q(f,f)}(t,\xi). \quad (5.9)$$

Therefore, for all $\alpha \in I\!R$, and according to lemma 2.1,

$$\frac{\partial}{\partial t}\int_{|\xi|\leq R}(1+|\xi|)^\alpha |\hat{f}(t,\xi)|^2 d\xi = 2Re\left(\int_{|\xi|\leq R}(1+|\xi|^\alpha)\,\widehat{Q(f,f)}(t,\xi)\overline{\hat{f}(t,\xi)}d\xi\right)$$
$$= 2p_{\alpha,R}(f,f) + 2Re[q_{\alpha,R}(f,f)]. \quad (5.10)$$

Let us introduce the notation,

$$Z_{\alpha,R}(f)(t) = \int_{|\xi|\leq R}(1+|\xi|)^\alpha |\hat{f}(t,\xi)|^2 d\xi. \quad (5.11)$$

Then, Using lemma 5.2 and lemma 5.3, we get the following estimate for all $\alpha \in I\!R$, $R > 1$, $\epsilon > 0$ (small enough):

$$\frac{\partial}{\partial t}\{Z_{\alpha,R}(f)\}(t) + 2D(\alpha,\gamma,T) Z_{\alpha+\gamma-1,R}(f)(t)$$

$$\leq 2E(\alpha,\gamma,T)\left\{\| f \|^2_{L^1(\mathbb{R}^2)} + 1_{\{\alpha+\gamma-1>0\}} Z_{0,R}(f)(t)\right\}$$
$$+ C(\epsilon,\alpha,\gamma)\beta_1 \| f \|_{L^1(\mathbb{R}^2)} \left\{ l_{0,f} + l_{1,f} + \sup(Z_{\alpha+\gamma-1-\tau(\epsilon),R}(f)(t),\right.$$
$$\left. Z_{2\alpha+2\gamma-4+3\epsilon,R}(f)(t)) + \int_{|\xi|=R} (1+|\xi|)^{\alpha+\gamma-3+\epsilon} |\hat{f}(t,\xi)|\,|\nabla \hat{f}(t,\xi)| d\xi \right\}. \tag{5.12}$$

Let us now describe in a few words the main steps of what we still have to prove. We first show that all the terms in the right-hand side of (5.12) can be estimated by $Z_{\alpha+\gamma-1,R}(f)(t)$. Therefore, we get an estimate (that is (5.17)) for $Z_{\alpha,R}(f)(t^*)$ and for $\int_t^{t^*} Z_{\alpha+\gamma-1,R}(f)(s)\, ds$ in term of $Z_{\alpha,R}(f)(t)$ (for any $t^* > t$). Since $\gamma - 1 > 0$, we can conclude by induction (we gain $\gamma - 1$ derivatives in the space of velocities at each step).

We now come back to the details of the proof. When $\alpha < 3 - \gamma$,

$$2\alpha + 2\gamma - 4 < \alpha + \gamma - 1. \tag{5.13}$$

Moreover, Hölder's inequality ensures that for all $\delta > 0$, $\theta > -2 + \delta$,

$$Z_{\theta-\delta,R}(f) \leq Z^{\frac{\delta}{2+\delta+\theta}}_{-2-\delta,R}(f)\, Z^{\frac{2+\theta}{2+\delta+\theta}}_{\theta,R}(f). \tag{5.14}$$

Therefore, when $\alpha < 3 - \gamma$, we can find for all $\epsilon > 0$ two constant real numbers $F(\alpha,\gamma,T,\epsilon) > 0$ and $G(\alpha,\gamma,T,\epsilon) > 0$ such that for all $t > 0$,

$$\frac{\partial}{\partial t}\{Z_{\alpha,R}(f)\}(t) + F(\alpha,\gamma,T,\epsilon)\, Z_{\alpha+\gamma-1,R}(f)(t)$$
$$\leq G(\alpha,\gamma,T,\epsilon)\{1 + \int_{|\xi|=R} (1+|\xi|)^{\alpha+\gamma-3+\epsilon} |\hat{f}(t,\xi)|\,|\nabla \hat{f}(t,\xi)| d\xi\}. \tag{5.15}$$

Therefore, when $\alpha < 3 - \gamma$, and for any $\epsilon > 0$ small enough,

$$\frac{\partial}{\partial t}\{Z_{\alpha,R}(f)\}(t) + F(\alpha,\gamma,T,\epsilon)\, Z_{\alpha+\gamma-1,R}(f)(t)$$
$$\leq G(\alpha,\gamma,T,\epsilon)\{1 + \pi R(1+R)^{\alpha+\gamma-3+\epsilon} \| f \|_{L^1} \| xf \|_{L^1}\}. \tag{5.16}$$

Integrating (5.16) with respect to the time variable between t and t^*, one gets (for $\alpha < 3 - \gamma$ and $\epsilon > 0$ small enough),

$$Z_{\alpha,R}(f)(t^*) + F(\alpha,\gamma,T,\epsilon) \int_t^{t^*} Z_{\alpha+\gamma-1,R}(f)(s) ds$$

$$\leq Z_{\alpha,R}(f)(t) + G(\alpha,\gamma,T,\epsilon)\{1+\pi R(1+R)^{\alpha+\gamma-3+\epsilon} \parallel f \parallel_{L^1} \parallel xf \parallel_{L^1}\}t^*. \tag{5.17}$$

We prove then by induction that for any $t_0 > 0, \alpha < 2-\gamma$,

$$\sup_{R>1} \sup_{s\in[t_0,T]} Z_{\alpha,R}(f)(s) < +\infty, \tag{5.18}$$

and

$$\sup_{R>1} \int_{t_0}^{T} Z_{\alpha+\gamma-1,R}(f)(s)ds < +\infty. \tag{5.19}$$

Note before all that (if $\alpha < 2-\gamma$),

$$\lim_{R\to+\infty} R\,(1+R)^{\alpha+\gamma-3+\epsilon} = 0. \tag{5.20}$$

Then, because of the conservation of mass, estimate (5.18) holds for $\alpha < -2$. Therefore, when $\alpha < -2$,

$$\int_0^{t_0} \sup_{R>1} Z_{\alpha+\gamma-1,R}(f)(s)ds < +\infty. \tag{5.21}$$

Then, one can find $t_1 \in [0, t_0[$ such that

$$\sup_{R>1} Z_{\alpha+\gamma-1,R}(f)(t_1) < +\infty. \tag{5.22}$$

Using then (5.17) for $t = t_1$, and $t^* \in]t_1, T]$, we get for any β such that $\beta < 2-\gamma$, $\beta < -3+\gamma$,

$$\sup_{R>1} \sup_{s\in[t_1,T]} Z_{\beta,R}(f)(s) < +\infty, \tag{5.23}$$

$$\sup_{R>1} \int_{t_1}^{T} Z_{\beta+\gamma-1,R}(f)(s)ds < +\infty. \tag{5.24}$$

One can in this way build a sequence

$$0 \leq t_1 \leq ... \leq t_N < t_0 \tag{5.25}$$

such that when $\beta < 2-\gamma$, $\beta < -2+j(\gamma-1)$,

$$\sup_{R>1} \sup_{s\in[t_j,T]} Z_{\beta,R}(f)(s) < +\infty, \tag{5.26}$$

$$\sup_{R>1} \int_{tj}^{T} Z_{\beta+\gamma-1,R}(f)(s)ds < +\infty. \tag{5.27}$$

We finally get (5.18), (5.19).

Note that we already have proven (by letting R go to $+\infty$) that

$$f \in L^{\infty}_{loc}(]0;+\infty[\,;\ H^{1-\frac{\gamma}{2}-0}(\mathbb{R}^2_v)) \cap L^1_{loc}(]0;+\infty[\,;\ H^{\frac{1}{2}-0}(\mathbb{R}^2_v)). \tag{5.28}$$

In order to prove theorem 2, we need to estimate more thoroughly the last term of (5.15).

Note first that according to estimate (5.19), for any t_0, $T > 0, \delta > 0$,

$$\int_{t_0}^{T} \int_{\xi \in \mathbb{R}^2} (1+|\xi|)^{-\frac{1}{2}-\delta} |\hat{f}(t,\xi)| d\xi dt$$

$$\leq \left(\int_{t_0}^{T} \int_{\xi \in \mathbb{R}^2} (1+|\xi|)^{-2-\delta} d\xi dt \right)^{1/2} \times \left(\int_{t_0}^{T} \int_{\xi \in \mathbb{R}^2} (1+|\xi|)^{1-\delta} |\hat{f}(t,\xi)|^2 d\xi dt \right)^{1/2}$$

$$\leq K_{T,\delta} \parallel f \parallel^{\frac{1}{2}}_{L^1([t_0,T];H^{\frac{1-\delta}{2}}(\mathbb{R}^2_v))} \tag{5.29}$$

for some $K_{T,\delta} > 0$.

Therefore, one can find a sequence $R_n \to +\infty$ and a constant $L_{\delta,T} > 0$ such that

$$\int_{t_0}^{T} \int_{|\xi|=R_n} (1+|\xi|)^{-\frac{1}{2}-\delta} |\hat{f}(t,\xi)| d\xi dt \leq \frac{L_{\delta,T}}{R_n}. \tag{5.30}$$

Using this estimate and integrating (5.15) between t_0 and $t \in]t_0, T]$, one gets

$$Z_{\alpha,R_n}(f)(t) + F(\alpha,\gamma,T,\epsilon) \int_{t_0}^{t} Z_{\alpha+\gamma-1,R_n}(f)(s) ds$$

$$\leq Z_{\alpha,R_n}(f)(t_0) + G(\alpha,\gamma,T,\epsilon)\{T + \parallel vf \parallel_{L^1(\mathbb{R}^2_v)} L_{\epsilon,T} R_n^{\alpha+\gamma-\frac{7}{2}+\epsilon}\}, \tag{5.31}$$

as soon as $\alpha < 3 - \gamma$.

Then $\alpha < \frac{7}{2} - \gamma$ and therefore one can apply the same induction as before to get theorem 2.

Remark: The results obtained here can probably be improved if one assumes that (1.9) holds for some $k > 1$ (Cf. [De 1] for a similar situation).

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