Homogenization of the discrete diffusive coagulation-fragmentation equations in perforated domains.

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Abstract

The asymptotic behavior of the solution of an infinite set of Smoluchowski's discrete coagulation-fragmentation-diffusion equations with non-homogeneous Neumann boundary conditions, defined in a periodically perforated domain, is analyzed. Our homogenization result, based on Nguetseng-Allaire two-scale convergence, is meant to pass from a microscopic model (where the physical processes are properly described) to a macroscopic one (which takes into account only the effective or averaged properties of the system). When the characteristic size of the perforations vanishes, the information given on the microscale by the non-homogeneous Neumann boundary condition is transferred into a global source term appearing in the limiting (homogenized) equations. Furthermore, on the macroscale, the geometric structure of the perforated domain induces a correction in the diffusion coefficients.

Keywords: coagulation; fragmentation; Smoluchowski equations; homogenization; perforated domain

1 Introduction

This paper is devoted to the homogenization of an infinite set of Smoluchowski's discrete coagulation-fragmentation-diffusion equations in a periodically perforated domain. The system of evolution equations considered describes the dynamics of cluster growth, that is the mechanisms allowing clusters to coalesce to form larger clusters or break apart into smaller ones. These clusters can diffuse in space with a diffusion constant which depends on their size. Since the size of clusters is not limited *a priori*, the system of reaction-diffusion equations that we consider consists of an infinite number of equations. The structure of the chosen equations, defined in a perforated medium with a non-homogeneous Neumann condition on the boundary of the perforations, is useful for investigating several phenomena arising in porous media [14], [8], [13] or in the field of biomedical research [11].

Typically, in a porous medium, the domain consists of two parts: a fluid phase where colloidal species or chemical substances, transported by diffusion, are dissolved and a solid skeleton (formed by grains or pores) on the boundary of which deposition processes or chemical reactions take place. In recent years, the Smoluchowski equation has been also considered in biomedical research to model the aggregation and diffusion of β-amyloid peptide (Aβ) in the cerebral tissue, a process thought to be associated with the development of Alzheimer's disease. One can define a perforated geometry, obtained by removing from a fixed domain (which represents the cerebral tissue) infinitely many small holes (the neurons). The production of Aβ in monomeric form from the neuron membranes can be modeled by coupling the Smoluchowski equation for the concentration of monomers with a non-homogeneous Neumann condition on the boundaries of the holes.

The results of this paper constitute a generalization of some of the results contained in [14], [11], by considering an infinite system of equations where both the coagulation and fragmentation processes are taken into account. Unlike previous theoretical works, where existence and uniqueness of solutions for an infinite system of coagulation-fragmentation equations (with homogeneous Neumann boundary conditions) have been studied [19], [15], we focus in this paper on a distinct aspect, that is, the averaging of the system of Smoluchowski's equations over arrays of

periodically-distributed microstructures.

Our homogenization result, based on Nguetseng-Allaire two-scale convergence [17], [1], is meant to pass from a microscopic model (where the physical processes are properly described) to a macroscopic one (which takes into account only the effective or averaged properties of the system).

1.1 Setting of the problem

Let Ω be a bounded open set in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Let Y be the unit periodicity cell $[0,1[^3$ (having the paving property). We perforate Ω by removing from it a set T_ϵ of periodically distributed holes defined as follows. Let us denote by T an open subset of Y with a smooth boundary Γ, such that $\overline{T} \subset \operatorname{Int} Y$. Set $Y^* = Y \setminus T$ which is called in the literature the solid or material part. We define $\tau(\epsilon\overline{T})$ to be the set of all translated images of $\epsilon\overline{T}$ of the form $\epsilon(k+\overline{T})$, $k \in \mathbb{Z}^3$. Then,

$$T_\epsilon := \Omega \cap \tau(\epsilon\overline{T}).$$

Introduce now the periodically perforated domain Ω_ϵ defined by

$$\Omega_\epsilon = \Omega \setminus \overline{T}_\epsilon.$$

For the sake of simplicity, we make the following standard assumption on the holes [6], [9]: there exists a 'security' zone around $\partial\Omega$ without holes, that is the holes do not intersect the boundary $\partial\Omega$, so that Ω_ϵ is a connected set.

The boundary $\partial\Omega_\epsilon$ of Ω_ϵ is then composed of two parts. The first one is the union of the boundaries of the holes strictly contained in Ω. It is denoted by Γ_ϵ and is defined by

$$\Gamma_\epsilon := \cup\Big\{\partial(\epsilon(k+\overline{T})) \mid \epsilon(k+\overline{T}) \subset \Omega\Big\}.$$

The second part of $\partial\Omega_\epsilon$ is its fixed exterior boundary denoted by $\partial\Omega$. It is easily seen that (see [2], Eq. (3))

$$\lim_{\epsilon\to 0} \epsilon \mid \Gamma_\epsilon|_2 = \mid \Gamma|_2 \frac{\mid \Omega \mid_3}{\mid Y \mid_3}, \tag{1}$$

where $|\cdot|_N$ is the N-dimensional Hausdorff measure.

The previous definitions and Assumptions on Ω (and, T, Γ, Ω_ϵ, T_ϵ, Γ_ϵ, $\partial\Omega$) will be denoted in the rest of the paper as **Assumption 0**.

Throughout this paper, we will abuse notations by denoting by ϵ a sequence of positive real numbers which converges to zero. We will consider in the following a discrete coagulation-fragmentation-diffusion model for the evolution of clusters [3], [4]. Denoting by $u_i^\epsilon := u_i^\epsilon(t,x) \geq 0$ the density of clusters with integer size $i \geq 1$ at position $x \in \Omega_\epsilon$ and time $t \geq 0$, and by $d_i > 0$ the diffusion constant for clusters of size i, the corresponding system can be written as a family of equations in Ω_ϵ, the first one being:

$$\begin{cases} \dfrac{\partial u_1^\epsilon}{\partial t} - d_1\,\Delta_x u_1^\epsilon + u_1^\epsilon \sum_{j=1}^\infty a_{1,j} u_j^\epsilon = \sum_{j=1}^\infty B_{1+j}\,\beta_{1+j,1}\,u_{1+j}^\epsilon & \text{in } [0,T]\times\Omega_\epsilon, \\ \\ \dfrac{\partial u_1^\epsilon}{\partial \nu} := \nabla_x u_1^\epsilon \cdot n = 0 & \text{on } [0,T]\times\partial\Omega, \\ \\ \dfrac{\partial u_1^\epsilon}{\partial \nu} := \nabla_x u_1^\epsilon \cdot n = \epsilon\,\psi(t,x,\frac{x}{\epsilon}) & \text{on } [0,T]\times\Gamma_\epsilon, \\ \\ u_1^\epsilon(0,x) = U_1 & \text{in } \Omega_\epsilon. \end{cases} \tag{2}$$

We shall systematically make the following assumption on ψ and U_1:

Assumption A: We suppose that ψ is a given (bounded) function satisfying the following conditions:

(i) $\psi \in C^1([0,T];B)$ with $B = C^1[\overline{\Omega}; C^1_\#(Y)]$ ($C^1_\#(Y)$ being the space of periodic C^1 functions with period relative to Y),

(ii) $\psi(t=0,x,\frac{x}{\epsilon}) = 0$ for $x \in \Omega_\epsilon$,

and U_1 is a constant such that $0 \leq U_1 \leq \|\psi\|_{L^\infty([0,T];B)}$.

In addition, if $i \geq 2$, we introduce the following equations:

$$
\begin{cases}
\dfrac{\partial u_i^\epsilon}{\partial t} - d_i\,\Delta_x u_i^\epsilon = Q_i^\epsilon + F_i^\epsilon & \text{in } [0,T]\times\Omega_\epsilon, \\[2ex]
\dfrac{\partial u_i^\epsilon}{\partial \nu} := \nabla_x u_i^\epsilon \cdot n = 0 & \text{on } [0,T]\times\partial\Omega, \\[2ex]
\dfrac{\partial u_i^\epsilon}{\partial \nu} := \nabla_x u_i^\epsilon \cdot n = 0 & \text{on } [0,T]\times\Gamma_\epsilon, \\[2ex]
u_i^\epsilon(0,x) = 0 & \text{in } \Omega_\epsilon,
\end{cases}
\tag{3}
$$

where the terms Q_i^ϵ, F_i^ϵ due to coagulation and fragmentation, respectively, are given by

$$
Q_i^\epsilon := \frac{1}{2}\sum_{j=1}^{i-1} a_{i-j,j}\, u_{i-j}^\epsilon\, u_j^\epsilon - \sum_{j=1}^{\infty} a_{i,j}\, u_i^\epsilon\, u_j^\epsilon, \tag{4}
$$

$$
F_i^\epsilon := \sum_{j=1}^{\infty} B_{i+j}\, \beta_{i+j,i}\, u_{i+j}^\epsilon - B_i\, u_i^\epsilon. \tag{5}
$$

The parameters B_i, $\beta_{i,j}$ and $a_{i,j}$, for integers $i,j \geq 1$, represent the total rate B_i of fragmentation of clusters of size i, the average number $\beta_{i,j}$ of clusters of size j produced by fragmentation of a cluster of size i, and the coagulation rate $a_{i,j}$ of clusters of size i with clusters of size j. These parameters represent rates, so they are always nonnegative; single particles do not break up further, and mass should be conserved when a cluster breaks up into smaller pieces, so one always imposes the:

Assumption B: The coagulation and fragmentation coefficients satisfy:

$$
a_{i,j} = a_{j,i} \geq 0, \quad \beta_{i,j} \geq 0, \quad (i,j \geq 1), \tag{6}
$$

$$
B_1 = 0, \quad B_i \geq 0, \quad (i \geq 2), \tag{7}
$$

$$
i = \sum_{j=1}^{i-1} j\,\beta_{i,j}, \quad (i \geq 2). \tag{8}
$$

In order to prove the bounds presented in the sequel, we need to impose additional restrictions on the coagulation and fragmentation coefficients, together with constraints on the diffusion coefficients. They are summarized in the:

Assumption C: There exists $C > 0$, $\zeta \in]0,1]$ such that

$$a_{ij} \leq C\,(i+j)^{1-\zeta}. \tag{9}$$

Moreover, for each $m \geq 1$, there exists $\gamma_m > 0$ such that

$$B_j\,\beta_{j,m} \leq \gamma_m\,a_{m,j} \qquad \text{for } \; j \geq m+1. \tag{10}$$

Finally, there exist constants $D_0, D_1 > 0$ such that

$$\forall i \in \mathbb{N} - \{0\}, \qquad 0 < D_0 \leq d_i \leq D_1. \tag{11}$$

Note that the assumption (9) on the coagulation coefficients a_{ij} is quite standard: it enables to show that no gelation occurs in the considered system of coagulation-fragmentation equations, provided that the diffusion coefficients satisfy the bound (11), cf. [4]. For a set of alternative assumptions (more stringent on the coagulation coefficients, but less stringent on the diffusion coefficients), see [3]. Finally, assumption (10) is used in existence proofs for systems where both coagulation and fragmentation are considered, see [19].

1.2 Main statement and comments

Our aim is to study the homogenization of the set of equations (2)-(3) as $\epsilon \to 0$, i.e., to study the behaviour of u_i^ϵ $(i \geq 1)$, as $\epsilon \to 0$, and obtain the equations satisfied by the limit. Since there is no obvious notion of convergence for the sequence u_i^ϵ $(i \geq 1)$ (which is defined on a varying set Ω_ϵ: this difficulty is specific to the case of perforated domains), we use the natural tool of two-scale convergence as elaborated by Nguetseng-Allaire, [17], [1]. Our main statement shows that it is indeed possible to homogenize the equations:

Theorem 1.1. *For $\epsilon > 0$ small enough, there exists a strong solution*

$$u_i^\epsilon := u_i^\epsilon(t,x) \in L^2([0,T]; H^2(\Omega_\epsilon)) \cap H^1(]0,T[; L^2(\Omega_\epsilon)) \quad (i \geq 1)$$

to system (2) - (3)*, which is moreover nonnegative, that is*

$$u_i^\epsilon(t,x) \geq 0 \quad \textit{for } (t,x) \in (0,T) \times \Omega_\epsilon.$$

We now introduce the notation $\widetilde{\ }$ *for the extension by zero outside* Ω_ϵ*, and we denote by* $\chi := \chi(y)$ *the characteristic function of* Y^**.*

Then the sequences $\widetilde{u_i^\epsilon}$ *and* $\widetilde{\nabla_x u_i^\epsilon}$ *(*$i \geq 1$*) two-scale converge (up to a subsequence) to* $(t,x,y) \mapsto [\chi(y)\, u_i(t,x)]$ *and* $(t,x,y) \mapsto [\chi(y)(\nabla_x u_i(t,x) + \nabla_y u_i^1(t,x,y))]$ *(*$i \geq 1$*), respectively, where the limiting functions* $[(t,x) \mapsto u_i(t,x), (t,x,y) \mapsto u_i^1(t,x,y)]$ *(*$i \geq 1$*) are weak solutions lying in* $L^2(0,T;H^1(\Omega)) \times L^2([0,T]\times\Omega; H^1_\#(Y)/\mathbb{R})$ *of the following two-scale homogenized systems:*

If $i = 1$*:*

$$\begin{cases} \theta\, \dfrac{\partial u_1}{\partial t}(t,x) - d_1\, \nabla_x \cdot \Big[\, A\, \nabla_x u_1(t,x) \Big] + \theta\, u_1(t,x) \sum_{j=1}^\infty a_{1,j}\, u_j(t,x) & \\ = \theta \sum_{j=1}^\infty B_{1+j}\, \beta_{1+j,1}\, u_{1+j}(t,x) + d_1 \displaystyle\int_\Gamma \psi(t,x,y)\, d\sigma(y) & \text{in } [0,T]\times\Omega, \\ [A\, \nabla_x u_1(t,x)] \cdot n = 0 & \text{on } [0,T]\times\partial\Omega, \\ u_1(0,x) = U_1 & \text{in } \Omega; \end{cases} \tag{12}$$

If $i \geq 2$*:*

$$\begin{cases} \theta\, \dfrac{\partial u_i}{\partial t}(t,x) - d_i\, \nabla_x \cdot \Big[\, A\, \nabla_x u_i(t,x) \Big] + \theta\, u_i(t,x) \sum_{j=1}^\infty a_{i,j}\, u_j(t,x) & \\ +\theta\, B_i\, u_i(t,x) = \dfrac{\theta}{2} \sum_{j=1}^{i-1} a_{j,i-j} u_j(t,x)\, u_{i-j}(t,x) & \\ +\theta \sum_{j=1}^\infty B_{i+j}\, \beta_{i+j,i}\, u_{i+j}(t,x) & \text{in } [0,T]\times\Omega, \\ [A\, \nabla_x u_i(t,x)] \cdot n = 0 & \text{on } [0,T]\times\partial\Omega, \\ u_i(0,x) = 0 & \text{in } \Omega, \end{cases} \tag{13}$$

where $\theta = \int_Y \chi(y) dy = |Y^*|$ *is the volume fraction of material, and* A *is a matrix (with constant coefficients) defined by*

$$A_{jk} = \int_{Y^*} (\nabla_y w_j + \hat{e}_j) \cdot (\nabla_y w_k + \hat{e}_k)\, dy,$$

with $\hat{e}_j$ *being the* j*-th unit vector in* $\mathbb{R}^3$*, and* $(w_j)_{1\leq j\leq 3}$ *the family of solutions of the*

cell problem

$$\begin{cases} -\nabla_y \cdot [\nabla_y w_j + \hat{e}_j] = 0 & \text{in } Y^*, \\ (\nabla_y w_j + \hat{e}_j) \cdot n = 0 & \text{on } \Gamma, \\ y \mapsto w_j(y) \quad Y - \text{periodic.} \end{cases} \tag{14}$$

Finally,

$$u_i^1(t,x,y) = \sum_{j=1}^{3} w_j(y) \frac{\partial u_i}{\partial x_j}(t,x) \quad (i \geq 1).$$

1.3 Structure of the rest of the paper

The paper is organized as follows. In Section 2, we derive all the *a priori* estimates needed for two-scale homogenization. In particular, in order to prove the uniform L^2-bound on the infinite sums appearing in our set of Eqs. (2)-(3), we extend to the case of non-homogeneous Neumann boundary conditions a duality method first devised by M. Pierre and D. Schmitt [18] and largely exploited afterwards [3], [4]. Then, Section 3 is devoted to the proof of our main results on the homogenization of the infinite Smoluchowski discrete coagulation-fragmentation-diffusion equations in a periodically perforated domain. Finally, Appendix A and Appendix B are introduced to summarize, respectively, some fundamental inequalities in Sobolev spaces tailored for perforated media, and some basic results on the two-scale convergence method (used to perform the homogenization procedure).

2 Estimates

We first obtain the *a priori* estimates for the sequences u_i^ϵ, $\nabla_x u_i^\epsilon$, $\partial_t u_i^\epsilon$ in $[0,T]\times\Omega_\epsilon$, that are independent of ϵ. We start with an adapted duality lemma in the style of [18].

Lemma 2.1. *Let Ω_ϵ be an open set satisfying Assumption 0. We suppose that Assumptions A, B, C hold. Then, for all $T > 0$, classical solutions to system (2)-(3) satisfy the following bound:*

$$\int_0^T \int_{\Omega_\epsilon} \left[\sum_{i=1}^{\infty} i\, u_i^\epsilon(t,x) \right]^2 dt\, dx \leq C, \tag{15}$$

where C is a positive constant independent of ϵ.

Proof. Let us consider the following fundamental identity (or weak formulation) of the coagulation and fragmentation operators [3], [4]:

$$\sum_{i=1}^{\infty} \varphi_i \, Q_i^\epsilon = \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} \, u_i^\epsilon \, u_j^\epsilon \, (\varphi_{i+j} - \varphi_i - \varphi_j), \tag{16}$$

$$\sum_{i=1}^{\infty} \varphi_i \, F_i^\epsilon = - \sum_{i=2}^{\infty} B_i \, u_i^\epsilon \left(\varphi_i - \sum_{j=1}^{i-1} \beta_{i,j} \, \varphi_j \right), \tag{17}$$

which holds for any sequence of numbers $(\varphi_i)_{i \geq 1}$ such that the sums are defined. By choosing $\varphi_i := i$ above and thanks to (8), we have the mass conservation property for the operators Q_i^ϵ and F_i^ϵ:

$$\sum_{i=1}^{\infty} i \, Q_i^\epsilon = \sum_{i=1}^{\infty} i \, F_i^\epsilon = 0. \tag{18}$$

Therefore, summing together Eq. (2) and Eq. (3) multiplied by i, taking into account the identity (18), we get the (local in x) mass conservation property for the system:

$$\frac{\partial}{\partial t} \left[\sum_{i=1}^{\infty} i \, u_i^\epsilon \right] - \Delta_x \left[\sum_{i=1}^{\infty} i \, d_i \, u_i^\epsilon \right] = 0. \tag{19}$$

Denoting

$$\rho^\epsilon(t,x) = \sum_{i=1}^{\infty} i \, u_i^\epsilon(t,x), \tag{20}$$

and

$$A^\epsilon(t,x) = [\rho^\epsilon(t,x)]^{-1} \sum_{i=1}^{\infty} i \, d_i \, u_i^\epsilon(t,x), \tag{21}$$

the following system can be derived from Eqs. (2), (3) and (19):

$$\begin{cases} \dfrac{\partial \rho^\epsilon}{\partial t} - \Delta_x (A^\epsilon \, \rho^\epsilon) = 0 & \text{in } [0,T] \times \Omega_\epsilon, \\[2ex] \nabla_x (A^\epsilon \, \rho^\epsilon) \cdot n = 0 & \text{on } [0,T] \times \partial\Omega, \\[2ex] \nabla_x (A^\epsilon \, \rho^\epsilon) \cdot n = d_1 \, \epsilon \, \psi(t,x,\frac{x}{\epsilon}) & \text{on } [0,T] \times \Gamma_\epsilon, \\[2ex] \rho^\epsilon(0,x) = U_1 & \text{in } \Omega_\epsilon. \end{cases} \tag{22}$$

We observe that (for all $t \in [0,T]$)

$$\|A^\epsilon(t,\cdot)\|_{L^\infty(\Omega_\epsilon)} \leq \sup_i d_i. \tag{23}$$

Multiplying the first equation in (22) by the function w^ϵ defined by the following dual problem:

$$\begin{cases} -\left(\dfrac{\partial w^\epsilon}{\partial t} + A^\epsilon\,\Delta_x w^\epsilon\right) = A^\epsilon\,\rho^\epsilon & \text{in } [0,T]\times\Omega_\epsilon, \\ \\ \nabla_x w^\epsilon \cdot n = 0 & \text{on } [0,T]\times\partial\Omega, \\ \\ \nabla_x w^\epsilon \cdot n = 0 & \text{on } [0,T]\times\Gamma_\epsilon, \\ \\ w^\epsilon(T,x) = 0 & \text{in } \Omega_\epsilon, \end{cases} \tag{24}$$

and integrating by parts on $[0,T]\times\Omega_\epsilon$, we end up with the identity

$$\begin{aligned} &\int_0^T \int_{\Omega_\epsilon} A^\epsilon(t,x)\,(\rho^\epsilon(t,x))^2\,dt\,dx = \int_{\Omega_\epsilon} w^\epsilon(0,x)\,\rho^\epsilon(0,x)\,dx \\ &+ \epsilon\,d_1 \int_0^T \int_{\Gamma_\epsilon} \psi(t,x,\frac{x}{\epsilon})\,w^\epsilon(t,x)\,dt\,d\sigma_\epsilon(x) := I_1 + I_2, \end{aligned} \tag{25}$$

where $d\sigma_\epsilon$ is the measure on Γ_ϵ.

Let us now estimate the terms I_1 and I_2. From Hölder's inequality we obtain

$$I_1 = \int_{\Omega_\epsilon} w^\epsilon(0,x)\,\rho^\epsilon(0,x)\,dx \leq U_1\,|\Omega_\epsilon|^{1/2}\,\|w^\epsilon(0,\cdot)\|_{L^2(\Omega_\epsilon)}. \tag{26}$$

Applying once more Hölder's inequality and using estimate (23), we get

$$\begin{aligned} &\int_{\Omega_\epsilon} |w^\epsilon(0,x)|^2\,dx = \int_{\Omega_\epsilon} \left|\int_0^T \sqrt{A^\epsilon}\,\frac{\partial_t\,w^\epsilon}{\sqrt{A^\epsilon}}\,dt\right|^2 dx \\ &\leq T\,\|A^\epsilon\|_{L^\infty(\Omega_\epsilon)} \int_0^T \int_{\Omega_\epsilon} (A^\epsilon)^{-1} \left|\frac{\partial}{\partial t} w^\epsilon(t,x)\right|^2 dt\,dx \\ &\leq T\,(\sup_i d_i) \int_0^T \int_{\Omega_\epsilon} (A^\epsilon)^{-1} \left|\frac{\partial}{\partial t} w^\epsilon(t,x)\right|^2 dt\,dx. \end{aligned} \tag{27}$$

By exploiting the dual problem (24), Eq. (27) becomes

$$\begin{aligned} &\int_{\Omega_\epsilon} |w^\epsilon(0,x)|^2\,dx \leq T\,(\sup_i d_i) \int_0^T \int_{\Omega_\epsilon} (A^\epsilon)^{-1}\,|A^\epsilon\,\Delta_x w^\epsilon + A^\epsilon\,\rho^\epsilon|^2\,dt\,dx \\ &\leq T\,(\sup_i d_i) \int_0^T \int_{\Omega_\epsilon} (A^\epsilon)^{-1} \left[2\,(A^\epsilon)^2\,(\Delta_x w^\epsilon)^2 + 2\,(A^\epsilon)^2\,(\rho^\epsilon)^2\right] dt\,dx. \end{aligned} \tag{28}$$

Let us now estimate the first term on the right-hand side of (28). Multiplying the first equation in (24) by $(\Delta_x w^\epsilon)$, we see that

$$\int_{\Omega_\epsilon} (\Delta_x w^\epsilon) \left(\frac{\partial w^\epsilon}{\partial t}\right) dx + \int_{\Omega_\epsilon} A^\epsilon \, (\Delta_x w^\epsilon)^2 \, dx = - \int_{\Omega_\epsilon} A^\epsilon \, \rho^\epsilon \, (\Delta_x w^\epsilon) \, dx, \tag{29}$$

and integrating by parts on Ω_ϵ, we get

$$-\frac{\partial}{\partial t} \int_{\Omega_\epsilon} \frac{|\nabla_x w^\epsilon|^2}{2} \, dx + \int_{\Omega_\epsilon} A^\epsilon \, (\Delta_x w^\epsilon)^2 \, dx = - \int_{\Omega_\epsilon} A^\epsilon \, \rho^\epsilon (\Delta_x w^\epsilon) \, dx. \tag{30}$$

Then, integrating once more over the time interval $[0, T]$ and using Young's inequality for the right-hand side of (30), one finds that

$$\int_{\Omega_\epsilon} |\nabla_x w^\epsilon(0, x)|^2 \, dx + \int_0^T \int_{\Omega_\epsilon} A^\epsilon \, (\Delta_x w^\epsilon)^2 \, dt \, dx \leq \int_0^T \int_{\Omega_\epsilon} (\rho^\epsilon)^2 \, A^\epsilon \, dt \, dx. \tag{31}$$

Since the first term of the left-hand side of (31) is nonnegative, we conclude that

$$\int_0^T \int_{\Omega_\epsilon} A^\epsilon \, (\Delta_x w^\epsilon)^2 \, dt \, dx \leq \int_0^T \int_{\Omega_\epsilon} (\rho^\epsilon)^2 \, A^\epsilon \, dt \, dx. \tag{32}$$

Inserting Eq. (32) into Eq. (28), one obtains

$$\int_{\Omega_\epsilon} |w^\epsilon(0, x)|^2 \, dx \leq 4 \, T \, (\sup_i \, d_i) \int_0^T \int_{\Omega_\epsilon} A^\epsilon \, (\rho^\epsilon)^2 \, dt \, dx. \tag{33}$$

Therefore, we end up with the estimate

$$I_1 \leq 2 \, U_1 \left[|\Omega_\epsilon| \, T \, \sup_i \, d_i \right]^{1/2} \left[\int_0^T \int_{\Omega_\epsilon} A^\epsilon \, (\rho^\epsilon)^2 \, dt \, dx \right]^{1/2}. \tag{34}$$

By using Lemma A.1 of Appendix A and Hölder's inequality, the term I_2 in (25) can be rewritten as

$$\begin{aligned} I_2 &= \epsilon \, d_1 \int_0^T \int_{\Gamma_\epsilon} \psi(t, x, \frac{x}{\epsilon}) \, w^\epsilon(t, x) \, dt \, d\sigma_\epsilon(x) \\ &\leq \sqrt{C_1 \, \tilde{C}} \, d_1 \int_0^T \|\psi(t)\|_B \left\{ \left[\int_{\Omega_\epsilon} |w^\epsilon|^2 \, dx \right]^{1/2} + \epsilon \left[\int_{\Omega_\epsilon} |\nabla_x w^\epsilon|^2 \, dx \right]^{1/2} \right\}, \end{aligned} \tag{35}$$

where we have taken into account the following estimate (see Lemma B.7 of Appendix B):

$$\epsilon \int_{\Gamma_\epsilon} |\psi(t, x, \frac{x}{\epsilon})|^2 \, d\sigma_\epsilon(x) \leq \tilde{C} \, \|\psi(t)\|_B^2 \tag{36}$$

(with $\tilde{C}$ being a positive constant independent of ϵ and $B = C^1[\overline{\Omega}; C^1_\#(Y)]$). Note that we do not really need that ψ be of class C^1 in the estimate above, continuity would indeed be sufficient.

Since $\psi \in L^\infty([0,T];B)$, using the Cauchy-Schwarz inequality, Eq. (35) reads

$$I_2 \leq C\, d_1\, \|w^\epsilon\|_{L^2(0,T;L^2(\Omega_\epsilon))} + C\, d_1\, \epsilon\, \|\nabla_x w^\epsilon\|_{L^2(0,T;L^2(\Omega_\epsilon))} := J_1 + J_2, \tag{37}$$

where $C > 0$ is a constant independent of ϵ. Let us now estimate the terms J_1 and J_2. Using Hölder's inequality and estimate (23), by following the same strategy as the one leading to (33), we get

$$\begin{aligned}
&\int_0^T \int_{\Omega_\epsilon} |w^\epsilon(t,x)|^2\, dt\, dx = \int_0^T \int_{\Omega_\epsilon} \left| \int_t^T \sqrt{A^\epsilon}\, \frac{\partial_s w^\epsilon(s,x)}{\sqrt{A^\epsilon}}\, ds \right|^2 dt\, dx \\
&\leq T^2\, (\sup_i\, d_i) \int_0^T \int_{\Omega_\epsilon} (A^\epsilon)^{-1} \left| \frac{\partial w^\epsilon}{\partial t}(t,x) \right|^2 dt\, dx \\
&\leq 4\, T^2\, (\sup_i\, d_i) \int_0^T \int_{\Omega_\epsilon} A^\epsilon\, (\rho^\epsilon)^2\, dt\, dx,
\end{aligned} \tag{38}$$

so that

$$\begin{aligned}
J_1 &= C\, d_1 \left[\int_0^T \int_{\Omega_\epsilon} |w^\epsilon(t,x)|^2\, dt\, dx \right]^{1/2} \\
&\leq 2\, C\, d_1\, T\, (\sup_i\, d_i)^{1/2} \left[\int_0^T \int_{\Omega_\epsilon} A^\epsilon\, (\rho^\epsilon)^2\, dt\, dx \right]^{1/2}.
\end{aligned} \tag{39}$$

In order to estimate J_2, we go back to Eq. (30). Integrating over $[t,T]$, one obtains

$$\begin{aligned}
&\frac{1}{2} \int_t^T \int_{\Omega_\epsilon} \frac{\partial}{\partial s} |\nabla_x w^\epsilon(s,x)|^2\, ds\, dx - \int_t^T \int_{\Omega_\epsilon} A^\epsilon\, (\Delta_x w^\epsilon)^2\, ds\, dx \\
&= \int_t^T \int_{\Omega_\epsilon} A^\epsilon\, \rho^\epsilon\, (\Delta_x w^\epsilon)\, ds\, dx.
\end{aligned} \tag{40}$$

Young's inequality applied to the right-hand side of Eq. (40) leads to

$$\int_{\Omega_\epsilon} |\nabla_x w^\epsilon(t,x)|^2\, dx + \int_t^T \int_{\Omega_\epsilon} A^\epsilon\, (\Delta_x w^\epsilon)^2\, ds\, dx \leq \int_t^T \int_{\Omega_\epsilon} A^\epsilon\, (\rho^\epsilon)^2\, ds\, dx. \tag{41}$$

Taking into account that the second term on the left-hand side of (41) is nonnegative and integrating once more over time, we get

$$\int_0^T \int_{\Omega_\epsilon} |\nabla_x w^\epsilon(t,x)|^2\, dt\, dx \leq T \int_0^T \int_{\Omega_\epsilon} A^\epsilon\, (\rho^\epsilon)^2\, dt\, dx. \tag{42}$$

Therefore, we conclude that

$$\begin{aligned}
J_2 =& C\, d_1\, \epsilon \left[\int_0^T \int_{\Omega_\epsilon} |\nabla_x w^\epsilon(t,x)|^2\, dt\, dx \right]^{1/2} \\
&\leq C\, d_1\, \epsilon\, (T)^{1/2} \left[\int_0^T \int_{\Omega_\epsilon} A^\epsilon\, (\rho^\epsilon)^2\, dt\, dx \right]^{1/2}.
\end{aligned} \tag{43}$$

By combining (39) and (43), we end up with the estimate

$$I_2 \leq d_1 \left[2\, C\, T \sqrt{\sup_i d_i} + C\, \epsilon \sqrt{T} \right] \left[\int_0^T \int_{\Omega_\epsilon} A^\epsilon\, (\rho^\epsilon)^2\, dt\, dx \right]^{1/2}. \tag{44}$$

Hence, inserting estimates (34) and (44) in Eq. (25), one obtains

$$\int_0^T \int_{\Omega_\epsilon} A^\epsilon(t,x)\, (\rho^\epsilon(t,x))^2\, dt\, dx \leq C_3^2, \tag{45}$$

where

$$C_3 = \max \left(2\, U_1 \sqrt{|\Omega_\epsilon|\, T\, \sup_i d_i}, d_1\, [2\, C\, T \sqrt{\sup_i d_i} + C \sqrt{T}] \right). \tag{46}$$

Thus, recalling the definitions of A^ϵ and ρ^ϵ, and using the lower bound on the diffusion rates in Assumption C, the assertion of the Lemma immediately follows.

□

Corollary 2.2. *Let Ω_ϵ be an open set satisfying Assumption 0. Under Assumptions A, B and C, the following bound holds for all classical solutions of (2), (3), when $i \geq 1$:*

$$\int_0^T \int_{\Omega_\epsilon} \left| \sum_{j=1}^\infty a_{i,j}\, u_j^\epsilon(t,x) \right|^2 dt\, dx \leq C_i, \tag{47}$$

where C_i does not depend on ϵ (but may depend on i).

Proof. Thanks to estimate (15), we see that

$$\int_0^T \int_{\Omega_\epsilon} \left| \sum_{j=1}^\infty j\, u_j^\epsilon(t,x) \right|^2 dt\, dx \leq C.$$

We conclude using estimate (9) of Assumption C. □

Remark 2.3. *We first notice that in order to get Corollary 2.2 (and the results of this section which use it), it would be sufficient to assume that $a_{i,j} \leq C\,(i+j)$. We will however need the more stringent estimate (9) of Assumption C in the proof of the homogenization result in next section. Note that this Assumption ensures that no gelation occurs in the coagulation-fragmentation process that we consider (cf. [4]).*

We also could relax the hypothesis that the diffusion rates d_i be bounded below (and replace it by the assumption that d_i behaves as a (negative) power law), provided that the assumption on the growth coefficients $a_{i,j}$ be made more stringent (cf. [3]). In that situation, the duality lemma reads

$$\int_0^T \int_{\Omega_\epsilon} \left[\sum_{i=1}^\infty i\, d_i\, u_i^\epsilon(t,x) \right] \left[\sum_{i=1}^\infty i\, u_i^\epsilon(t,x) \right] dt\, dx \leq C.$$

We now turn to L^∞ estimates. We start with the

Lemma 2.4. *Let Ω_ϵ be an open set satisfying Assumption 0. We also suppose that Assmptions A, B, and C hold. We finally consider $T > 0$, and a classical solution u_i^ϵ $(i \geq 1)$, of (2) - (3). Then, the following estimate holds:*

$$\|u_1^\epsilon\|_{L^\infty(0,T;L^\infty(\Omega_\epsilon))} \leq |U_1| + \|u_1^\epsilon\|_{L^\infty(0,T;L^\infty(\Gamma_\epsilon))} + \gamma_1 + 1. \tag{48}$$

Proof. Let us test the first equation of (2) with the function

$$\phi_1 := p\,(u_1^\epsilon)^{(p-1)} \quad p \geq 2.$$

We stress that the function ϕ_1 is strictly positive and continuously differentiable on $[0,t] \times \overline{\Omega}$, for all $t > 0$. Integrating, the divergence theorem yields

$$\begin{aligned}
&\int_0^t ds \int_{\Omega_\epsilon} \frac{\partial}{\partial s}(u_1^\epsilon)^p(s)\,dx + d_1\,p\,(p-1) \int_0^t ds \int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon|^2\,(u_1^\epsilon)^{(p-2)}\,dx \\
&= -p \int_0^t ds \int_{\Omega_\epsilon} a_{1,1}\,(u_1^\epsilon)^{(p+1)}\,dx - p \int_0^t ds \int_{\Omega_\epsilon} (u_1^\epsilon)^p \sum_{j=2}^{\infty} a_{1,j}\,u_j^\epsilon\,dx \\
&+ p \int_0^t ds \int_{\Omega_\epsilon} (u_1^\epsilon)^{(p-1)} \sum_{j=2}^{\infty} B_j\,\beta_{j,1}\,u_j^\epsilon\,dx + \epsilon\,d_1\,p \int_0^t ds \int_{\Gamma_\epsilon} \psi(s,x,\frac{x}{\epsilon})\,(u_1^\epsilon)^{(p-1)}\,d\sigma_\epsilon(x) \\
&\leq -p \int_0^t ds \int_{\Omega_\epsilon} \sum_{j=2}^{\infty} [a_{1,j}\,u_1^\epsilon - B_j\,\beta_{j,1}]\,u_j^\epsilon\,(u_1^\epsilon)^{(p-1)}\,dx \\
&+ \epsilon\,d_1\,p \int_0^t ds \int_{\Gamma_\epsilon} \psi(s,x,\frac{x}{\epsilon})\,(u_1^\epsilon)^{(p-1)}\,d\sigma_\epsilon(x).
\end{aligned} \tag{49}$$

Exploiting Assumption C, we end up with the estimate

$$\begin{aligned}
&\int_0^t ds \int_{\Omega_\epsilon} \frac{\partial}{\partial s}(u_1^\epsilon)^p(s)\,dx + d_1\,p\,(p-1) \int_0^t ds \int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon|^2\,(u_1^\epsilon)^{(p-2)}\,dx \\
&\leq \epsilon\,d_1\,p \int_0^t ds \int_{\Gamma_\epsilon} \psi(s,x,\frac{x}{\epsilon})\,(u_1^\epsilon)^{(p-1)}\,d\sigma_\epsilon(x) \\
&+ p\,\gamma_1^p \int_0^t ds \int_{\Omega_\epsilon} \sum_{j=2}^{\infty} a_{1,j}\,u_j^\epsilon\,dx.
\end{aligned} \tag{50}$$

Hölder's inequality applied to the right-hand side of (50), together with the duality estimate (47), leads to

$$\int_{\Omega_\epsilon} (u_1^\epsilon(t,x))^p\,dx + d_1\,p\,(p-1)\int_0^t ds\int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon|^2\,(u_1^\epsilon)^{(p-2)}\,dx$$
$$\leq \int_{\Omega_\epsilon} U_1^p\,dx + \epsilon\,d_1\,p\,\|\psi\|_{L^\infty(0,T;L^\infty(\Gamma_\epsilon))}\int_0^t ds\int_{\Gamma_\epsilon}(u_1^\epsilon)^{(p-1)}\,d\sigma_\epsilon(x) + C\,p\,\gamma_1^p\,|\Omega_\epsilon|^{1/2}. \tag{51}$$

Since the second term of the left-hand side of (51) is nonnegative, one gets

$$\begin{aligned}
&\int_{\Omega_\epsilon} (u_1^\epsilon(t,x))^p\,dx \leq \int_{\Omega_\epsilon} U_1^p\,dx \\
&+\epsilon\,d_1\,p\,\|\psi\|_{L^\infty(0,T;L^\infty(\Gamma_\epsilon))}\int_0^t ds\int_{\Gamma_\epsilon}[1+(u_1^\epsilon)^p]\,d\sigma_\epsilon(x) + C\,p\,\gamma_1^p\,|\Omega_\epsilon|^{1/2} \\
&\leq \int_{\Omega_\epsilon} U_1^p\,dx + \epsilon\,d_1\,p\,\|\psi\|_{L^\infty(0,T;L^\infty(\Gamma_\epsilon))}\,T\,|\Gamma_\epsilon| \\
&+\epsilon\,d_1\,p\,\|\psi\|_{L^\infty(0,T;L^\infty(\Gamma_\epsilon))}\int_0^t ds\int_{\Gamma_\epsilon}(u_1^\epsilon)^p\,d\sigma_\epsilon(x) + C\,p\,\gamma_1^p\,|\Omega|^{1/2}.
\end{aligned} \tag{52}$$

Hence, we conclude that

$$\sup_{t\in[0,T]}\lim_{p\to\infty}\left[\int_{\Omega_\epsilon}(u_1^\epsilon(t,x))^p\,dx\right]^{1/p} \leq |U_1| + \|u_1^\epsilon\|_{L^\infty(0,T;L^\infty(\Gamma_\epsilon))} + \gamma_1 + 1. \tag{53}$$

□

The boundedness of u_1^ϵ in $L^\infty([0,T]\times\Gamma_\epsilon)$, uniformly in ϵ, can then be immediately deduced from Lemma 2.5 below.

Lemma 2.5. *Let Ω_ϵ be an open set satisfying Assumption 0. We also suppose that Assmptions A, B, and C hold. We finally consider $T>0$, and a classical solution u_i^ϵ ($i\geq 1$), of (2) - (3). Then, for $\epsilon>0$ small enough,*

$$\|u_1^\epsilon\|_{L^\infty(0,T;L^\infty(\Gamma_\epsilon))} \leq C, \tag{54}$$

where C does not depend on ϵ.

In order to establish Lemma 2.5, we will first need the following preliminary result, proven in [11]:

Proposition 2.6 ([11], Theorem 5.2, p.730-732)**.** *Let Ω_ϵ be an open set satisfying Assumption 0, and $T>0$. We consider a sequence $w^\epsilon := w^\epsilon(t,x)\geq 0$ defined on $[0,T]\times\Omega_\epsilon$ such that, for some $\hat{k}>0$, $\beta>0$, and all $k\geq\hat{k}$,*

$$\|(w^\epsilon-k)_+\|^2_{Q_\epsilon(T)} := \sup_{0\leq t\leq T}\int_{\Omega_\epsilon}|(w^\epsilon-k)_+|^2\,dx + \int_0^T dt\int_{\Omega_\epsilon}|\nabla[(w^\epsilon-k)_+]|^2\,dx \tag{55}$$

$$\leq \epsilon\, \beta\, k^2 \int_0^T dt \int_{\Gamma_\epsilon} 1_{\{w^\epsilon > k\}}\, dx.$$

Then

$$||w^\epsilon||_{L^\infty([0,T]\times\Gamma_\epsilon)} \leq C(\beta, T, \Omega)\, \hat{k}, \tag{56}$$

where the positive constant $C(\beta, T, \Omega)$ may depend on β, but not on $\hat{k}$ and ϵ.

Proof. of Lemma 2.5 :

Since this proof is close to the proof of Lemma 5.2 in [11], we only sketch it. Let $T > 0$ and $k \geq 0$ be fixed. We define: $u_\epsilon^{(k)}(t) := (u_1^\epsilon(t) - k)_+$ for $t \geq 0$. Its derivatives are

$$\frac{\partial u_\epsilon^{(k)}}{\partial t} = \frac{\partial u_1^\epsilon}{\partial t}\, \mathbb{1}_{\{u_1^\epsilon > k\}}, \tag{57}$$

$$\nabla_x u_\epsilon^{(k)} = \nabla_x u_1^\epsilon\, \mathbb{1}_{\{u_1^\epsilon > k\}}. \tag{58}$$

Moreover,

$$u_\epsilon^{(k)} \mid_{\partial\Omega} = (u_1^\epsilon \mid_{\partial\Omega} - k)_+, \tag{59}$$

$$u_\epsilon^{(k)} \mid_{\Gamma_\epsilon} = (u_1^\epsilon \mid_{\Gamma_\epsilon} - k)_+. \tag{60}$$

We define $\hat{k} := \max(\|\psi\|_{L^\infty(0,T;B)}, \gamma_1)$, and consider $k \geq \hat{k}$. Then,

$$u_1^\epsilon(0, x) = U_1 \leq \hat{k} \leq k. \tag{61}$$

For $t \in [0, T_1]$ with $T_1 \leq T$, we get therefore

$$\begin{aligned} \frac{1}{2}\int_{\Omega_\epsilon} |u_\epsilon^{(k)}(t)|^2\, dx &= \int_0^t \frac{d}{ds}\left[\frac{1}{2}\int_{\Omega_\epsilon} |u_\epsilon^{(k)}(s)|^2\, dx\right] ds \\ &= \int_0^t \int_{\Omega_\epsilon} \frac{\partial u_\epsilon^{(k)}(s)}{\partial s}\, u_\epsilon^{(k)}(s)\, dx ds. \end{aligned} \tag{62}$$

Taking into account Eq. (57) and Eq. (2), we obtain that for all $s \in [0, T_1]$:

$$
\begin{aligned}
&\int_{\Omega_\epsilon} \frac{\partial u_\epsilon^{(k)}(s)}{\partial s}\, u_\epsilon^{(k)}(s)\, dx = \int_{\Omega_\epsilon} \frac{\partial u_1^\epsilon(s)}{\partial s}\, u_\epsilon^{(k)}(s)\, dx \\
&= \int_{\Omega_\epsilon} \left[d_1\, \Delta_x u_1^\epsilon - u_1^\epsilon \sum_{j=1}^{\infty} a_{1,j} u_j^\epsilon + \sum_{j=1}^{\infty} B_{1+j}\, \beta_{1+j,1}\, u_{1+j}^\epsilon \right] u_\epsilon^{(k)}(s)\, dx \\
&= \epsilon\, d_1 \int_{\Gamma_\epsilon} \psi\left(s, x, \frac{x}{\epsilon}\right) u_\epsilon^{(k)}(s)\, d\sigma_\epsilon(x) - d_1 \int_{\Omega_\epsilon} \nabla_x u_1^\epsilon(s) \cdot \nabla_x u_\epsilon^{(k)}(s)\, dx \\
&- \int_{\Omega_\epsilon} (u_1^\epsilon(s))^2\, a_{1,1}\, u_\epsilon^{(k)}(s)\, dx - \int_{\Omega_\epsilon} u_1^\epsilon(s) \sum_{j=2}^{\infty} \left[a_{1,j}\, u_j^\epsilon(s) \right] u_\epsilon^{(k)}(s)\, dx \\
&+ \int_{\Omega_\epsilon} \left[\sum_{j=2}^{\infty} B_j\, \beta_{j,1}\, u_j^\epsilon(s) \right] u_\epsilon^{(k)}(s)\, dx \\
&\leq \epsilon\, d_1 \int_{\Gamma_\epsilon} \psi\left(s, x, \frac{x}{\epsilon}\right) u_\epsilon^{(k)}(s)\, d\sigma_\epsilon(x) - d_1 \int_{\Omega_\epsilon} \nabla_x u_1^\epsilon(s) \cdot \nabla_x u_\epsilon^{(k)}(s)\, dx \\
&- \int_{\Omega_\epsilon} \sum_{j=2}^{\infty} \left[a_{1,j}\, u_1^\epsilon(s) - B_j\, \beta_{j,1} \right] u_j^\epsilon(s)\, u_\epsilon^{(k)}(s)\, dx.
\end{aligned}
\tag{63}
$$

By using Assumption C, Lemma A.1 and Young's inequality, one has, remembering that $k \geq \gamma_1$,

$$
\begin{aligned}
&\int_{\Omega_\epsilon} \frac{\partial u_\epsilon^{(k)}(s)}{\partial s}\, u_\epsilon^{(k)}(s)\, dx \leq \frac{\epsilon\, d_1}{2} \int_{B_k^\epsilon(s)} \left| \psi\left(s, x, \frac{x}{\epsilon}\right) \right|^2 d\sigma_\epsilon(x) \\
&+ \frac{C_1\, d_1}{2} \int_{A_k^\epsilon(s)} |u_\epsilon^{(k)}(s)|^2\, dx - d_1 \left(1 - \frac{C_1 \epsilon^2}{2}\right) \int_{\Omega_\epsilon} |\nabla_x u_\epsilon^{(k)}(s)|^2\, dx,
\end{aligned}
\tag{64}
$$

where we denote by $A_k^\epsilon(t)$ and $B_k^\epsilon(t)$ the set of points in Ω_ϵ and on Γ_ϵ, respectively, at which $u_1^\epsilon(t, x) > k$. We observe that

$$
|A_k^\epsilon(t)| \leq |\Omega_\epsilon|, \qquad |B_k^\epsilon(t)| \leq |\Gamma_\epsilon|,
$$

where $|\cdot|$ is the (resp. 3-dimensional and 2-dimensional) Lebesgue measure. Inserting Eq. (64) into Eq. (62) and varying over t, we end up with the estimate:

$$
\begin{aligned}
&\sup_{0 \leq t \leq T_1} \left[\frac{1}{2} \int_{\Omega_\epsilon} |u_\epsilon^{(k)}(t)|^2\, dx \right] + d_1 \left(1 - \frac{C_1\, \epsilon^2}{2}\right) \int_0^{T_1} dt \int_{\Omega_\epsilon} |\nabla_x u_\epsilon^{(k)}(t)|^2\, dx \\
&\leq \frac{C_1\, d_1}{2} \int_0^{T_1} dt \int_{A_k^\epsilon(t)} |u_\epsilon^{(k)}(t)|^2\, dx + \frac{\epsilon\, d_1}{2} \int_0^{T_1} dt \int_{B_k^\epsilon(t)} \left| \psi\left(t, x, \frac{x}{\epsilon}\right) \right|^2 d\sigma_\epsilon(x).
\end{aligned}
\tag{65}
$$

Introducing the norm (as in the Prop. above):

$$
\|u\|_{Q_\epsilon(T)}^2 := \sup_{0 \leq t \leq T} \int_{\Omega_\epsilon} |u(t)|^2\, dx + \int_0^T dt \int_{\Omega_\epsilon} |\nabla u(t)|^2\, dx, \tag{66}
$$

inequality (65) can be rewritten as follows:

$$\min\left\{\frac{1}{2}, d_1\left(1-\frac{C_1\,\epsilon^2}{2}\right)\right\}\|u_\epsilon^{(k)}\|^2_{Q_\epsilon(T_1)} \le \frac{C_1\,d_1}{2}\int_0^{T_1} dt\int_{A_k^\epsilon(t)} |u_\epsilon^{(k)}(t)|^2\,dx + \frac{\epsilon\,d_1}{2}\int_0^{T_1} dt\int_{B_k^\epsilon(t)}\left|\psi\left(t,x,\frac{x}{\epsilon}\right)\right|^2 d\sigma_\epsilon(x). \tag{67}$$

Let us estimate the right-hand side of (67). From Hölder's inequality, we obtain

$$\int_0^{T_1} dt\int_{A_k^\epsilon(t)} |u_\epsilon^{(k)}(t)|^2\,dx \le \|u_\epsilon^{(k)}\|^2_{L^{\overline{r}_1}(0,T_1;L^{\overline{q}_1}(\Omega_\epsilon))}\,\|\mathbb{1}_{A_k^\epsilon}\|_{L^{r_1'}(0,T_1;L^{q_1'}(\Omega_\epsilon))}, \tag{68}$$

with $r_1' = \frac{r_1}{r_1-1}$, $q_1' = \frac{q_1}{q_1-1}$, $\overline{r}_1 = 2\,r_1$, $\overline{q}_1 = 2\,q_1$, where $\overline{r}_1 \in (2,\infty)$ and $\overline{q}_1 \in (2,6)$ have been chosen in such a way that

$$\frac{1}{\overline{r}_1} + \frac{3}{2\,\overline{q}_1} = \frac{3}{4}.$$

In particular, $r_1', q_1' < \infty$, so that (68) yields

$$\int_0^{T_1} dt\int_{A_k^\epsilon(t)} |u_\epsilon^{(k)}(t)|^2\,dx \le \|u_\epsilon^{(k)}\|^2_{L^{\overline{r}_1}(0,T_1;L^{\overline{q}_1}(\Omega_\epsilon))}\,|\Omega|^{1/q_1'}\,T_1^{1/r_1'}. \tag{69}$$

If we choose (for $\epsilon > 0$ small enough)

$$T_1^{1/r_1'} < \frac{\min\{1,d_1\}}{2C_1 d_1 c^2}\,|\Omega|^{-1/q_1'} \le \frac{\min\left\{\frac{1}{2}, d_1\left(1-\frac{C_1\,\epsilon^2}{2}\right)\right\}}{C_1 d_1 c^2}\,|\Omega|^{-1/q_1'},$$

then from Lemma A.3 (i) (and c being the constant appearing in formula (118) of this Lemma) it follows that

$$\frac{C_1\,d_1}{2}\int_0^{T_1} dt\int_{A_k^\epsilon(t)} |u_\epsilon^{(k)}(t)|^2\,dx \le \frac{1}{2}\min\left\{\frac{1}{2}, d_1\left(1-\frac{C_1\,\epsilon^2}{2}\right)\right\}\|u_\epsilon^{(k)}\|^2_{Q_\epsilon(T_1)}. \tag{70}$$

Analogously, from Hölder's inequality, we have (remember that $k \ge \hat{k}$)

$$\begin{aligned}\frac{\epsilon\,d_1}{2}\int_0^{T_1} dt\int_{B_k^\epsilon(t)}\left|\psi\left(t,x,\frac{x}{\epsilon}\right)\right|^2 d\sigma_\epsilon(x) &\le \frac{\epsilon\,d_1\,k^2}{2}\left(\frac{\hat{k}^2}{k^2}\right)\|\mathbb{1}_{B_k^\epsilon}\|_{L^1(0,T_1;L^1(\Gamma_\epsilon))}\\ &\le \frac{\epsilon\,d_1\,k^2}{2}\int_0^{T_1} dt\,|B_k^\epsilon(t)|.\end{aligned} \tag{71}$$

Thus, estimate (67) yields

$$\|u_\epsilon^{(k)}\|^2_{Q_\epsilon(T_1)} \le \epsilon\,\beta\,k^2\int_0^{T_1} dt\,|B_k^\epsilon(t)|, \tag{72}$$

with $\beta := \max(1,d_1) + 1/2$.

Hence, using Prop. 2.6 for $w^\epsilon := u_1^\epsilon$, we obtain

$$\|u_1^\epsilon\|_{L^\infty(0,T_1;L^\infty(\Gamma_\epsilon))} \leq C(\Omega,\beta,T_1)\,\hat{k},$$

where the positive constant $C(\Omega,\beta,T_1)$ does not depend on ϵ or $\hat{k}$.

The same argument can be repeated on the cylinder $[T_1, 2\,T_1]$ with $k \geq \hat{k_1} := \max(\gamma_1, C(\Omega,\beta,T_1)\,\hat{k})$, yielding

$$\|u_1^\epsilon\|_{L^\infty(0,2\,T_1;L^\infty(\Gamma_\epsilon))} \leq C(\Omega,\beta,T_1)\,\hat{k}_1.$$

Thanks to a straightforward induction, one gets the bound for u_1^ϵ in $L^\infty(0,T;L^\infty(\Gamma_\epsilon))$.

□

We finally write the following L^∞ bound for all u_i^ϵ:

Lemma 2.7. *Let Ω_ϵ be an open set satisfying Assumption 0. We also suppose that Assmptions A, B, and C hold. We finally consider $T > 0$, and a classical solution u_i^ϵ ($i \in \mathbb{N} - \{0\}$) of (2), (3). Then, the following uniform with respect to $\epsilon > 0$ (small enough) estimate holds for all $i \in \mathbb{N} - \{0\}$:*

$$\|u_i^\epsilon\|_{L^\infty(0,T;L^\infty(\Omega_\epsilon))} \leq K_i, \tag{73}$$

where K_1 is given by Lemma 2.4, estimate (48) and Lemma 2.5, estimate (54), and, for $i \geq 2$,

$$K_i = 1 + \frac{\left[\sum_{j=1}^{i-1} a_{j,i-j} K_j K_{i-j}\right]}{(B_i + a_{i,i})} + \gamma_i. \tag{74}$$

Proof. The Lemma can be proved directly by induction following the proof reported in [19] (Lemma 2.2, p. 284). Since we have a zero initial condition for the system (3), we have chosen a function slightly different from the one used in [19] to test the i-th equation of (3), namely

$$\phi_i := p\,(u_i^\epsilon)^{(p-1)} \quad p \geq 2.$$

We stress that the functions ϕ_i are strictly positive and continuously differentiable on $[0,t] \times \overline{\Omega}$, for all $t > 0$.

Therefore, multiplying the i-th equation in system (3) by ϕ_i and reorganizing the terms appearing in the sums, we can write the estimate

$$||u_i^\epsilon||^p_{L^p(\Omega_\epsilon)} + d_i\, p\,(p-1)\int_0^t\int_{\Omega_\epsilon} |\nabla_x u_i^\epsilon|^2\,(u_i^\epsilon)^{p-2}dxds$$

$$\leq \int_0^t\int_{\Omega_\epsilon}\left[\frac{1}{2}\sum_{j=1}^{i-1} a_{i-j,j}\,u_j^\epsilon\,u_{i-j}^\epsilon - a_{i,i}\,|u_i^\epsilon|^2 - B_i\,u_i^\epsilon\right] p\,(u_i^\epsilon)^{p-1}dxds$$

$$-\int_0^t\int_{\Omega_\epsilon}\left[\sum_{j=1}^{i-1} a_{i,j}\,u_i^\epsilon\,u_j^\epsilon + \sum_{j=i+1}^{\infty}\left(a_{i,j}\,u_i^\epsilon - B_j\,\beta_{j,i}\right)\,u_j^\epsilon\right] p\,(u_i^\epsilon)^{p-1}dxds.$$

We now work using an induction on i. Supposing that we already know that $\|u_j^\epsilon\|_{L^\infty(0,T;L^\infty(\Omega_\epsilon))} \leq K_j$ for all $j < i$, and using assumption C, the previous estimate leads to

$$||u_i^\epsilon||^p_{L^p(\Omega_\epsilon)} \leq \int_0^t\int_{\Omega_\epsilon}\left[\frac{1}{2}\sum_{j=1}^{i-1} a_{i-j,j}\,K_j\,K_{i-j} - a_{i,i}\,|u_i^\epsilon|^2 - B_i\,u_i^\epsilon\right] p\,(u_i^\epsilon)^{p-1}dxds$$

$$+\int_0^t\int_{\Omega_\epsilon}\sum_{j=i+1}^{\infty} a_{i,j}\,(-u_i^\epsilon+\gamma_i)\;u_j^\epsilon\,p\,(u_i^\epsilon)^{p-1}dxds =: I_1 + I_2.$$

Then, optimizing w.r.t. u_i^ϵ,

$$I_1 \leq \left[\left(\sum_{j=1}^{i-1} a_{i-j,j}\,K_j\,K_{i-j}\right)^p (B_i + a_{i,i})^{1-p}\right]|\Omega_\epsilon|\,T + p\,a_{i,i}\,|\Omega_\epsilon|\,T,$$

and

$$I_2 \leq \int_0^t\int_{\Omega_\epsilon}\sum_{j=i+1}^{\infty} a_{i,j}\,(\gamma_i - u_i^\epsilon)\,u_j^\epsilon\,1_{\{u_i^\epsilon\leq\gamma_i\}}\,p\,(u_i^\epsilon)^{p-1}dxds$$

$$\leq p\,\gamma_i^p\int_0^t\int_{\Omega_\epsilon}\left(\sum_{j=i+1}^{\infty} a_{i,j}\,u_j^\epsilon\right)dxds$$

$$\leq C\,p\,\gamma_i^p(|\Omega_\epsilon|\,T)^{1/2},$$

where Cauchy-Schwarz inequality and the duality Lemma (more precisely Eq. (47)) have been exploited.

Using these estimates for bounding $||u_i^\epsilon||_{L^p(\Omega_\epsilon)}$ and letting $p\to\infty$, we end up with the desired estimate.

□

We end up this section with bounds for the derivatives of u_i^ϵ.

Lemma 2.8. *Let Ω_ϵ be an open set satisfying Assumption 0. We also suppose that Assmptions A, B, and C hold. We finally consider $T > 0$, and a classical solution u_i^ϵ ($i \in \mathbb{N} - \{0\}$, $\epsilon > 0$ small enough) of (2), (3). Then, the family $\partial_t u_i^\epsilon$ is bounded in $L^2([0,T] \times \Omega_\epsilon)$, and the family $\nabla_x u_i^\epsilon$ is bounded in $L^\infty([0,T]; L^2(\Omega_\epsilon))$, uniformly in ϵ (but not in i).*

Proof. Since this proof is close to the proof of Lemma 5.9 in [11], we only sketch it.

Case $i = 1$: Let us multiply the first equation in (2) by the function $\partial_t u_1^\epsilon(t,x)$. Integrating, the divergence theorem yields

$$\begin{aligned}
&\int_{\Omega_\epsilon} \left| \frac{\partial u_1^\epsilon(t,x)}{\partial t} \right|^2 dx + \frac{d_1}{2} \int_{\Omega_\epsilon} \frac{\partial}{\partial t} (|\nabla_x u_1^\epsilon(t,x)|^2)\, dx \\
&= \epsilon\, d_1 \int_{\Gamma_\epsilon} \psi\left(t, x, \frac{x}{\epsilon}\right) \frac{\partial u_1^\epsilon}{\partial t}\, d\sigma_\epsilon(x) - \int_{\Omega_\epsilon} u_1^\epsilon \left(\sum_{j=1}^{\infty} a_{1,j}\, u_j^\epsilon \right) \frac{\partial u_1^\epsilon}{\partial t}\, dx \\
&+ \int_{\Omega_\epsilon} \left(\sum_{j=1}^{\infty} B_{1+j}\, \beta_{1+j,1}\, u_{1+j}^\epsilon \right) \frac{\partial u_1^\epsilon}{\partial t}\, dx.
\end{aligned} \tag{75}$$

Using Young's inequality and exploiting the boundedness of u_1^ϵ in $L^\infty(0,T; L^\infty(\Omega_\epsilon))$, one gets

$$\begin{aligned}
&C_1 \int_{\Omega_\epsilon} \left| \frac{\partial u_1^\epsilon(t,x)}{\partial t} \right|^2 dx + \frac{d_1}{2} \int_{\Omega_\epsilon} \frac{\partial}{\partial t} (|\nabla_x u_1^\epsilon(t,x)|^2)\, dx \\
&\leq \epsilon\, d_1 \int_{\Gamma_\epsilon} \psi\left(t, x, \frac{x}{\epsilon}\right) \frac{\partial u_1^\epsilon}{\partial t}\, d\sigma_\epsilon(x) + C_2 \int_{\Omega_\epsilon} \left| \sum_{j=1}^{\infty} a_{1,j}\, u_j^\epsilon \right|^2 dx \\
&+ C_3 \int_{\Omega_\epsilon} \left| \sum_{j=2}^{\infty} B_j\, \beta_{j,1}\, u_j^\epsilon \right|^2 dx,
\end{aligned} \tag{76}$$

where C_1, C_2 and C_3 are positive constants which do not depend on ϵ. Integrating over $[0,t]$ with $t \in [0,T]$, thanks to estimate (47) and Assumption C, we end up with the estimate

$$\begin{aligned}
C_1 \int_0^t ds \int_{\Omega_\epsilon} \left| \frac{\partial u_1^\epsilon}{\partial s} \right|^2 dx + \frac{d_1}{2} \int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon(t,x)|^2\, dx &\leq C_4 \\
&+ \epsilon\, d_1 \int_{\Gamma_\epsilon} \psi\left(t, x, \frac{x}{\epsilon}\right) u_1^\epsilon(t,x)\, d\sigma_\epsilon(x) \\
&- \epsilon\, d_1 \int_0^t ds \int_{\Gamma_\epsilon} \frac{\partial}{\partial s} \psi\left(s, x, \frac{x}{\epsilon}\right) u_1^\epsilon(s,x)\, d\sigma_\epsilon(x),
\end{aligned} \tag{77}$$

since $\psi\left(t = 0, x, \frac{x}{\epsilon}\right) \equiv 0$.

Applying once more Young's inequality and taking into account estimate (36) and Lemma A.1, estimate (77) can be rewritten as follows

$$C_1 \int_0^t ds \int_{\Omega_\epsilon} \left| \frac{\partial u_1^\epsilon}{\partial s} \right|^2 dx + \frac{d_1}{2}(1 - \epsilon^2 C_5) \int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon(t,x)|^2 \, dx \tag{78}$$

$$\leq C_6 + C_1 \frac{d_1}{2} \epsilon^2 \int_0^t \int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon(s,x)|^2 \, dxds,$$

where the positive constants C_1, C_5, C_6 do not depend on ϵ, since $\psi \in L^\infty(0,T;B)$, u_1^ϵ is bounded in $L^\infty(0,T;L^\infty(\Omega_\epsilon))$, and the following inequality holds:

$$\epsilon \int_{\Gamma_\epsilon} \left| \partial_t \psi\left(t, x, \frac{x}{\epsilon}\right) \right|^2 d\sigma_\epsilon(x) \leq C_7 \, \|\partial_t \psi(t)\|_B^2 \leq C_8, \tag{79}$$

with C_7 and C_8 which do not depend on ϵ. Then, using Gronwall's lemma,

$$\|\partial_t u_1^\epsilon\|^2_{L^2(0,T;L^2(\Omega_\epsilon))} \leq C, \tag{80}$$

and

$$\|\nabla_x u_1^\epsilon\|^2_{L^\infty(0,T;L^2(\Omega_\epsilon))} \leq C, \tag{81}$$

where $C \geq 0$ is a constant which does not depend on ϵ.

Case $i \geq 2$: Let us multiply the first equation in (3) by the function $\partial_t u_i^\epsilon(t,x)$. Integrating, the divergence theorem yields

$$\begin{aligned} &\int_{\Omega_\epsilon} \left| \frac{\partial u_i^\epsilon(t,x)}{\partial t} \right|^2 dx + \frac{d_i}{2} \int_{\Omega_\epsilon} \frac{\partial}{\partial t}(|\nabla_x u_i^\epsilon(t,x)|^2) \, dx \\ &= \frac{1}{2} \int_{\Omega_\epsilon} \left(\sum_{j=1}^{i-1} a_{i-j,j} \, u_{i-j}^\epsilon \, u_j^\epsilon \right) \frac{\partial u_i^\epsilon}{\partial t} \, dx - \int_{\Omega_\epsilon} u_i^\epsilon \left(\sum_{j=1}^{\infty} a_{i,j} \, u_j^\epsilon \right) \frac{\partial u_i^\epsilon}{\partial t} \, dx \\ &+ \int_{\Omega_\epsilon} \left(\sum_{j=1}^{\infty} B_{i+j} \, \beta_{i+j,i} \, u_{i+j}^\epsilon \right) \frac{\partial u_i^\epsilon}{\partial t} \, dx - \int_{\Omega_\epsilon} B_i \, u_i^\epsilon \, \frac{\partial u_i^\epsilon}{\partial t} \, dx. \end{aligned} \tag{82}$$

Using Young's inequality and exploiting the boundedness of u_i^ϵ in $L^\infty(0,T;L^\infty(\Omega_\epsilon))$, one gets

$$\begin{aligned} &C_1 \int_{\Omega_\epsilon} \left| \frac{\partial u_i^\epsilon(t,x)}{\partial t} \right|^2 dx + \frac{d_i}{2} \int_{\Omega_\epsilon} \frac{\partial}{\partial t}(|\nabla_x u_i^\epsilon(t,x)|^2) \, dx \\ &\leq C_2 + C_3 \int_{\Omega_\epsilon} \left| \sum_{j=1}^{\infty} a_{i,j} \, u_j^\epsilon \right|^2 dx + C_4 \int_{\Omega_\epsilon} \left| \sum_{j=i+1}^{\infty} B_j \, \beta_{j,i} \, u_j^\epsilon \right|^2 dx, \end{aligned} \tag{83}$$

where C_1, C_2, C_3 and C_4 are positive constants which do not depend on ϵ.

Integrating over $[0,t]$ with $t \in [0,T]$, thanks to estimate (47) and Assumption C, we end up with the estimate

$$C_1 \int_0^t ds \int_{\Omega_\epsilon} \left| \frac{\partial u_i^\epsilon}{\partial s} \right|^2 dx + \frac{d_i}{2} \int_{\Omega_\epsilon} |\nabla_x u_i^\epsilon(t,x)|^2 \, dx \leq C_5, \tag{84}$$

with $C_5 \geq 0$ independent of ϵ (but not on i). We conclude that

$$\|\partial_t u_i^\epsilon\|^2_{L^2(0,T;L^2(\Omega_\epsilon))} \leq C, \tag{85}$$

and

$$\|\nabla_x u_i^\epsilon\|^2_{L^\infty(0,T;L^2(\Omega_\epsilon))} \leq C, \tag{86}$$

where $C \geq 0$ is a constant independent of ϵ (but not on i).

□

This concludes the section devoted to *a priori* estimates which are uniform w.r.t. the homogenization parameter ϵ.

3 Proof of the main result

We start here the proof of our main Theorem 1.1.

3.1 Existence of solutions for a given $\epsilon > 0$

We first explain how to get a proof of existence, for a given $\varepsilon > 0$, of a (strong) solution to system (2) - (3). We state the:

Proposition 3.1. *Let $\epsilon > 0$ small enough be given, Ω_ϵ be a bounded regular open set of $\mathbb{R}^3$, and consider data satisfying Assumptions A, B and C. Then there exists a solution $(u_i^\epsilon)_{i\geq 1}$ to system* (2) - (3)*, which is strong in the following sense: For all $T > 0$ and $i \geq 1$, $u_i^\epsilon \in L^\infty([0,T]\times\Omega_\epsilon)$, $\frac{\partial u_i^\epsilon}{\partial t} \in L^2([0,T]\times\Omega_\epsilon)$, $\frac{\partial^2 u_i^\epsilon}{\partial x_k \partial x_l} \in L^2([0,T]\times\Omega_\epsilon)$ for all $k,l \in \{1,..,3\}$.*

Proof. We introduce a finite size truncation of this system, which writes (once the notation of the dependence w.r.t. ε of the unknowns has been eliminated for read-

ability):

$$\begin{cases} \dfrac{\partial u_1^n}{\partial t} - d_1\,\Delta_x u_1^n + u_1^n\,\sum_{j=1}^n a_{1,j} u_j^n = \sum_{j=1}^{n-1} B_{1+j}\,\beta_{1+j,1}\,u_{1+j}^n & \text{in } [0,T]\times\Omega_\epsilon, \\ \\ \dfrac{\partial u_1^n}{\partial \nu} := \nabla_x u_1^n \cdot n = 0 & \text{on } [0,T]\times\partial\Omega, \\ \\ \dfrac{\partial u_1^n}{\partial \nu} := \nabla_x u_1^n \cdot n = \epsilon\,\psi(t,x,\frac{x}{\epsilon}) & \text{on } [0,T]\times\Gamma_\epsilon, \\ \\ u_1^n(0,x) = U_1 & \text{in } \Omega_\epsilon, \end{cases} \tag{87}$$

and, if $i = 2,..,n$,

$$\begin{cases} \dfrac{\partial u_i^n}{\partial t} - d_i\,\Delta_x u_i^n = Q_i^n + F_i^n & \text{in } [0,T]\times\Omega_\epsilon, \\ \\ \dfrac{\partial u_i^n}{\partial \nu} := \nabla_x u_i^n \cdot n = 0 & \text{on } [0,T]\times\partial\Omega, \\ \\ \dfrac{\partial u_i^n}{\partial \nu} := \nabla_x u_i^n \cdot n = 0 & \text{on } [0,T]\times\Gamma_\epsilon, \\ \\ u_i^n(0,x) = 0 & \text{in } \Omega_\epsilon, \end{cases} \tag{88}$$

where the truncated coagulation and breakup kernels Q_i^n, F_i^n write

$$Q_i^n := \frac{1}{2}\sum_{j=1}^{i-1} a_{i-j,j}\,u_{i-j}^n\,u_j^n - \sum_{j=1}^{n} a_{i,j}\,u_i^n\,u_j^n, \tag{89}$$

$$F_i^n := \sum_{j=1}^{n-i} B_{i+j}\,\beta_{i+j,i}\,u_{i+j}^n - B_i\,u_i^n. \tag{90}$$

We then observe that the duality lemma (that is, Lemma 2.1 and Corollary 2.2) is still valid in this setting (with a proof that exactly follows the proof written above), so that we end up with the *a priori* estimate

$$\int_0^T \int_{\Omega_\varepsilon} \left| \sum_{i=1}^n i\,u_i^n(t,x) \right|^2 dtdx \le C, \tag{91}$$

where C is a constant which does not depend on n.

Using now a proof analogous to that of Lemmas 2.4 to 2.7, we can obtain the *a priori* estimate

$$||u_i^n||_{L^\infty([0,T]\times\Omega_\varepsilon)} \leq C_i, \tag{92}$$

where $C_i > 0$ is a constant which also does not depend on n (but may depend on i)

At this point, we use standard theorems for systems of reaction-diffusion equations in order to get the existence and uniqueness of a smooth solution to system (87) - (88) (for a given $n \in \mathbb{N} - \{0\}$). We refer to [10], Prop. 3.2 p. 97 and Thm. 3.3 p. 105 for a complete description of a case with a slightly different boundary condition (homogeneous Neumann instead of inhomogeneous Neumann) and a different right-hand side (but having the same crucial property, that is leading to an L^∞ *a priori* bound on the components of the unknown).

We now briefly explain how to pass to the limit when $n \to \infty$ in such a way that the limit of u_i^n satisfies the system (2) - (3). First, we notice that thanks to the duality estimate (91), each component sequence $(u_i^n)_{n\geq i}$ is bounded in $L^2([0,T]\times\Omega_\varepsilon)$. As a consequence, we can extract a subsequence from $(u_i^n)_{n\geq i}$ still denoted by $(u_i^n)_{n\geq i}$ (the extraction is done diagonally in such a way that it gives a subsequence which is common for all i) which converges in $L^2([0,T]\times\Omega_\varepsilon)$ weakly towards some function $u_i \in L^2([0,T]\times\Omega_\varepsilon)$. Using then the *a priori* estimates (92) and (91), we see that

$$||\frac{\partial u_i^n}{\partial t} - d_i\Delta_x u_i^n||_{L^2([0,T]\times\Omega_\varepsilon)} \leq C_i, \tag{93}$$

where C_i may depend on i but not on n, so that the convergence in fact holds for a.e $(t,x) \in [0,T]\times\Omega_\varepsilon$. This is sufficient to pass to the limit in system (87) - (88) and get a weak solution $(u_i)_{i\geq 1}$ to system (2) - (3). Moreover, thanks to estimates (92) and (93), this solution is strong, in the sense that for all $T > 0$, $u_i^\epsilon \in L^\infty([0,T]\times\Omega_\epsilon)$, $\frac{\partial u_i^\epsilon}{\partial t} \in L^2([0,T]\times\Omega_\epsilon)$, $\frac{\partial^2 u_i^\epsilon}{\partial x_k \partial x_l} \in L^2([0,T]\times\Omega_\epsilon)$ for all $k,l \in \{1,..,3\}$. □

3.2 Homogenization

We now present the end of the proof of our main Theorem 1.1, in which we use the solutions to system (2) - (3) for a given $\epsilon > 0$ obtained in Prop. 3.1,

and the (uniform w.r.t. ϵ) *a priori* estimates of Section 2, in order to perform the homogenization process corresponding to the limit $\epsilon \to 0$.

We recall that we use the notation $\widetilde{\ }$ for the extension by 0 to Ω of functions defined on Ω_ϵ, and the notation χ for the characteristic function of Y^*.

In view of Lemmas 2.7 and 2.8, the sequences $\widetilde{u_i^\epsilon}$, $\widetilde{\nabla_x u_i^\epsilon}$ and $\widetilde{\frac{\partial u_i^\epsilon}{\partial t}}$ $(i \geq 1)$ are bounded in $L^2([0,T]\times\Omega)$. Using Proposition B.2 and Proposition B.4, and following [1], Thm 2.9, p.1498 which is specially designed for perforated domains (in the elliptic case, but the transfer to the parabolic case is easy) they two-scale converge, up to a subsequence, respectively, to functions of the form: $[(t,x,y) \mapsto \chi(y)\, u_i(t,x)]$, $[(t,x,y) \mapsto \chi(y)\,(\nabla_x u_i(t,x) + \nabla_y u_i^1(t,x,y))]$, and $\left[(t,x,y) \to \chi(y)\,\frac{\partial u_i}{\partial t}(t,x)\right]$, for $i \geq 1$.

In the formulas above, $u_i \in L^2(0,T;H^1(\Omega))$ and $u_i^1 \in L^2([0,T]\times\Omega; H^1_\#(Y)/\mathbb{R})$.

In the case when $i = 1$, let us multiply the first equation of (2) by the test function $(t,x) \mapsto \phi_\epsilon(t,x,\frac{x}{\epsilon})$, where

$$\phi_\epsilon(t,x,y) := \phi(t,x) + \epsilon\,\phi_1(t,x,y), \tag{94}$$

with $\phi \in C^1([0,T]\times\overline{\Omega})$, and $\phi_1 \in C^1([0,T]\times\overline{\Omega}; C^\infty_\#(Y))$. Integrating, the divergence theorem yields

$$\begin{aligned}
&\int_0^T \int_{\Omega_\epsilon} \frac{\partial u_1^\epsilon}{\partial t}\,\phi_\epsilon(t,x,\frac{x}{\epsilon})\,dt\,dx + d_1 \int_0^T \int_{\Omega_\epsilon} \nabla_x u_1^\epsilon \cdot \nabla_x \left[(t,x) \mapsto \phi_\epsilon(t,x,\frac{x}{\epsilon})\right] dt\,dx \\
&+ \int_0^T \int_{\Omega_\epsilon} u_1^\epsilon \sum_{j=1}^\infty a_{1,j}\, u_j^\epsilon\, \phi_\epsilon(t,x,\frac{x}{\epsilon})\,dt\,dx = \epsilon\, d_1 \int_0^T \int_{\Gamma_\epsilon} \psi\left(t,x,\frac{x}{\epsilon}\right) \phi_\epsilon(t,x,\frac{x}{\epsilon})\,dt\,d\sigma_\epsilon(x) \\
&+ \int_0^T \int_{\Omega_\epsilon} \sum_{j=1}^\infty B_{1+j}\,\beta_{1+j,1}\, u_{1+j}^\epsilon\, \phi_\epsilon(t,x,\frac{x}{\epsilon})\,dt\,dx.
\end{aligned} \tag{95}$$

Using the two-scales convergences described above, we can directly pass to the limit in the two first terms of this weak formulation. It is also easy to pass to the limit in the fourth one thanks to Prop. B.6.

The passage to the limit in the last infinite sum can be performed thanks to Assumption C, the duality Lemma 2.1 (estimate (15)), and Cauchy-Schwarz inequality (used in the last inequality below), indeed

$$\left| \int_0^T \int_{\Omega_\epsilon} \sum_{j=K}^\infty B_{1+j}\,\beta_{1+j,1}\, u_{1+j}^\epsilon\, \phi_\epsilon\,dt\,dx \right|$$

$$\leq \int_0^T \int_{\Omega_\epsilon} \sum_{j=K}^{\infty} \gamma_1 \, a_{1,1+j} \; u_{1+j}^\epsilon \; dt \, dx \; ||\phi_\epsilon||_\infty$$

$$\leq C \int_0^T \int_{\Omega_\epsilon} \sum_{j=K}^{\infty} (1+j)^{1-\zeta} \, u_{1+j}^\epsilon \; dt \, dx$$

$$\leq C \, K^{-\zeta},$$

where C does not depend on ϵ.

The infinite sum in the third term of identity (95) can be treated in the same way, using moreover Lemma 2.7, indeed

$$\left| \int_0^T \int_{\Omega_\epsilon} u_1^\epsilon \sum_{j=K}^{\infty} a_{1,j} \, u_j^\epsilon \, \phi_\epsilon \, dt \, dx \right|$$

$$\leq C \int_0^T \int_{\Omega_\epsilon} \sum_{j=K}^{\infty} a_{1,j} \; u_j^\epsilon \; dt \, dx$$

$$\leq C \, K^{-\zeta},$$

where C does not depend on ϵ.

Note that the passage to the limit in quadratic terms like $u_1^\epsilon \, u_j^\epsilon$ can be performed thanks to Prop. B.3 (and the remark after this proposition), as done in [11].

Finally, the passage to the limit leads to the variational formulation:

$$\begin{aligned}
&\int_0^T \int_\Omega \int_{Y^*} \frac{\partial u_1}{\partial t}(t,x) \, \phi(t,x) \, dt \, dx \, dy \\
&+ d_1 \int_0^T \int_\Omega \int_{Y^*} [\nabla_x u_1(t,x) + \nabla_y u_1^1(t,x,y)] \cdot [\nabla_x \phi(t,x) + \nabla_y \phi_1(t,x,y)] \, dt \, dx \, dy \\
&+ \int_0^T \int_\Omega \int_{Y^*} u_1(t,x) \sum_{j=1}^{\infty} a_{1,j} \, u_j(t,x) \, \phi(t,x) \, dt \, dx \, dy \\
&= d_1 \int_0^T \int_\Omega \int_\Gamma \psi(t,x,y) \, \phi(t,x) \, dt \, dx \, d\sigma(y) \\
&+ \int_0^T \int_\Omega \int_{Y^*} \sum_{j=1}^{\infty} B_{1+j} \, \beta_{1+j,1} \, u_{1+j}(t,x) \, \phi(t,x) \, dt \, dx \, dy.
\end{aligned} \tag{96}$$

Thanks to an integration by parts, we see that (96) can be put in the strong form (associated to the following homogenized system):

$$-\nabla_y \cdot [d_1(\nabla_x u_1(t,x) + \nabla_y u_1^1(t,x,y))] = 0 \qquad \text{in } [0,T] \times \Omega \times Y^*, \tag{97}$$

$$[\nabla_x u_1(t,x) + \nabla_y u_1^1(t,x,y)] \cdot n = 0 \qquad \text{on } [0,T] \times \Omega \times \Gamma, \tag{98}$$

$$\begin{aligned} &\theta \, \frac{\partial u_1}{\partial t}(t,x) - \nabla_x \cdot \left[d_1 \int_{Y^*} (\nabla_x u_1(t,x) + \nabla_y u_1^1(t,x,y)) dy \right] \\ &+ \theta \, u_1(t,x) \sum_{j=1}^{\infty} a_{1,j} \, u_j(t,x) = d_1 \int_{\Gamma} \psi(t,x,y) \, d\sigma(y) \\ &+ \theta \sum_{j=1}^{\infty} B_{1+j} \, \beta_{1+j,1} \, u_{1+j}(t,x) \qquad \text{in } [0,T] \times \Omega, \end{aligned} \tag{99}$$

$$\left[\int_{Y^*} (\nabla_x u_1(t,x) + \nabla_y u_1^1(t,x,y)) \, dy \right] \cdot n = 0 \qquad \text{on } [0,T] \times \partial\Omega, \tag{100}$$

where

$$\theta = \int_Y \chi(y) dy = |Y^*|$$

is the volume fraction of material. Furthermore, a direct passage to the limit shows that

$$u_1(0,x) = U_1 \qquad \text{in } \Omega.$$

Eqs. (97) and (98) can be reexpressed as follows:

$$\triangle_y u_1^1(t,x,y) = 0 \qquad \text{in } [0,T] \times \Omega \times Y^*, \tag{101}$$

$$\nabla_y u_1^1(t,x,y) \cdot n = -\nabla_x u_1(t,x) \cdot n \qquad \text{on } [0,T] \times \Omega \times \Gamma. \tag{102}$$

Then, u_1^1 satisfying (101)-(102) can be written as

$$u_1^1(t,x,y) = \sum_{j=1}^{3} w_j(y) \, \frac{\partial u_1}{\partial x_j}(t,x), \tag{103}$$

where $(w_j)_{1 \le j \le 3}$ is the family of solutions of the cell problem:

$$\begin{cases} -\nabla_y \cdot [\nabla_y w_j + \hat{e}_j] = 0 & \text{in } Y^*, \\ (\nabla_y w_j + \hat{e}_j) \cdot n = 0 & \text{on } \Gamma, \\ y \mapsto w_j(y) \quad Y - \text{periodic}, \end{cases} \tag{104}$$

and $\hat{e}_j$ is the j-th unit vector of the canonical basis of $\mathbb{R}^3$.

By using the relation (103) in Eqs. (99) and (100), the system (12) can be immediately derived (cf. [1]).

We now consider $i \geq 2$, and multiply the first equation of (3) by the same test function $(t,x) \mapsto \phi_\epsilon(t,x,\frac{x}{\epsilon})$ as previously (with ϕ_ϵ defined by (94)). We get

$$\begin{aligned}
&\int_0^T \int_{\Omega_\epsilon} \frac{\partial u_i^\epsilon}{\partial t}\, \phi_\epsilon(t,x,\frac{x}{\epsilon})\, dt\, dx + d_i \int_0^T \int_{\Omega_\epsilon} \nabla_x u_i^\epsilon \cdot \nabla_x \left[(t,x) \mapsto \phi_\epsilon(t,x,\frac{x}{\epsilon})\right]\, dt\, dx \\
&= -\int_0^T \int_{\Omega_\epsilon} u_i^\epsilon \sum_{j=1}^{\infty} a_{i,j}\, u_j^\epsilon\, \phi_\epsilon(t,x,\frac{x}{\epsilon})\, dt\, dx + \frac{1}{2}\int_0^T \int_{\Omega_\epsilon} \sum_{j=1}^{i-1} a_{j,i-j}\, u_j^\epsilon\, u_{i-j}^\epsilon\, \phi_\epsilon(t,x,\frac{x}{\epsilon})\, dt\, dx \\
&+ \int_0^T \int_{\Omega_\epsilon} \sum_{j=1}^{\infty} B_{i+j}\, \beta_{i+j,i}\, u_{i+j}^\epsilon\, \phi_\epsilon(t,x,\frac{x}{\epsilon})\, dt\, dx - \int_0^T \int_{\Omega_\epsilon} B_i\, u_i^\epsilon\, \phi_\epsilon(t,x,\frac{x}{\epsilon})\, dt\, dx.
\end{aligned} \tag{105}$$

The passage to the two-scale limit can be done exactly as in the case when u_1^ϵ was concerned, and leads to

$$\begin{aligned}
&\int_0^T \int_\Omega \int_{Y^*} \frac{\partial u_i}{\partial t}(t,x)\, \phi(t,x)\, dt\, dx\, dy \\
&+ d_i \int_0^T \int_\Omega \int_{Y^*} [\nabla_x u_i(t,x) + \nabla_y u_i^1(t,x,y)] \cdot [\nabla_x \phi(t,x) + \nabla_y \phi_1(t,x,y)]\, dt\, dx\, dy \\
&= -\int_0^T \int_\Omega \int_{Y^*} u_i(t,x) \sum_{j=1}^{\infty} a_{i,j}\, u_j(t,x)\, \phi(t,x)\, dt\, dx\, dy \\
&+ \frac{1}{2}\int_0^T \int_\Omega \int_{Y^*} \sum_{j=1}^{i-1} a_{j,i-j}\, u_j(t,x)\, u_{i-j}(t,x)\, \phi(t,x)\, dt\, dx\, dy \\
&+ \int_0^T \int_\Omega \int_{Y^*} \sum_{j=1}^{\infty} B_{i+j}\, \beta_{i+j,i}\, u_{i+j}(t,x)\, \phi(t,x)\, dt\, dx\, dy \\
&- \int_0^T \int_\Omega \int_{Y^*} B_i\, u_i(t,x)\, \phi(t,x)\, dt\, dx\, dy.
\end{aligned} \tag{106}$$

An integration by parts shows that (106) is a variational formulation associated to the following homogenized system:

$$-\nabla_y \cdot [d_i(\nabla_x u_i(t,x) + \nabla_y u_i^1(t,x,y))] = 0 \qquad \text{in } [0,T] \times \Omega \times Y^*, \tag{107}$$

$$[\nabla_x u_i(t,x) + \nabla_y u_i^1(t,x,y)] \cdot n = 0 \qquad \text{on } [0,T] \times \Omega \times \Gamma, \tag{108}$$

$$
\begin{aligned}
&\theta\,\frac{\partial u_i}{\partial t}(t,x) - \nabla_x \cdot \left[d_i \int_{Y^*} (\nabla_x u_i(t,x) + \nabla_y u_i^1(t,x,y))dy \right] \\
&= -\theta\, u_i(t,x) \sum_{j=1}^{\infty} a_{i,j}\, u_j(t,x) + \frac{\theta}{2} \sum_{j=1}^{i-1} a_{j,i-j} u_j(t,x)\, u_{i-j}(t,x) \\
&+ \theta \sum_{j=1}^{\infty} B_{i+j}\, \beta_{i+j,i}\, u_{i+j}(t,x) - \theta\, B_i\, u_i(t,x) \qquad \text{in } [0,T] \times \Omega,
\end{aligned} \tag{109}
$$

$$
\left[\int_{Y^*} (\nabla_x u_i(t,x) + \nabla_y u_i^1(t,x,y))\, dy \right] \cdot n = 0 \qquad \text{on } [0,T] \times \partial\Omega, \tag{110}
$$

where θ still is the volume fraction of material. Once again, a direct passage to the limit shows that

$$
u_i(0,x) = 0 \qquad \text{in } \Omega.
$$

Eqs. (107) and (108) can be reexpressed as follows:

$$
\triangle_y u_i^1(t,x,y) = 0 \qquad \text{in } [0,T] \times \Omega \times Y^*, \tag{111}
$$

$$
\nabla_y u_i^1(t,x,y) \cdot n = -\nabla_x u_i(t,x) \cdot n \qquad \text{on } [0,T] \times \Omega \times \Gamma. \tag{112}
$$

Then, u_i^1 satisfying (111) - (112) can be written as

$$
u_i^1(t,x,y) = \sum_{j=1}^{3} w_j(y)\, \frac{\partial u_i}{\partial x_j}(t,x), \tag{113}
$$

where $(w_j)_{1 \le j \le 3}$ is the family of solutions of the cell problem (104).

By using the relation (113) in Eqs. (109) and (110), the system (13) can be immediately derived (cf. [1]).

This concludes the proof of our main Theorem (Thm. 1.1).

A Appendix A

We introduce in this Appendix some results related to the theory of perforated domains, proven in previous works. In the three Lemmas stated below, Ω_ϵ is a perforated domain satisfying Assumption 0.

Lemma A.1. *There exists a constant $C_1 > 0$ which does not depend on ϵ, such that when $v \in Lip\,(\Omega_\epsilon)$, then*

$$
\|v\|^2_{L^2(\Gamma_\epsilon)} \le C_1 \left[\epsilon^{-1} \int_{\Omega_\epsilon} |v|^2\, dx + \epsilon \int_{\Omega_\epsilon} |\nabla_x v|^2\, dx \right]. \tag{114}
$$

Proof. The inequality (114) can be easily obtained from the standard trace theorem by means of a scaling argument, cf. [2]. □

Lemma A.2. *There exists a family of linear continuous extension operators*

$$P_\epsilon : W^{1,p}(\Omega_\epsilon) \to W^{1,p}(\Omega)$$

and a constant $C > 0$ which does not depend on ϵ, such that

$$P_\epsilon v = v \quad \text{in } \Omega_\epsilon,$$

and

$$\int_\Omega |P_\epsilon v|^p dx \le C \int_{\Omega_\epsilon} |v|^p dx \ , \tag{115}$$

$$\int_\Omega |\nabla (P_\epsilon v)|^p dx \le C \int_{\Omega_\epsilon} |\nabla v|^p dx, \tag{116}$$

for each $v \in W^{1,p}(\Omega_\epsilon)$ and for any $p \in (1, +\infty)$.

Proof. For the proof of this Lemma, see for instance [5]. □

As a consequence of the existence of those extension operators, one can obtain Sobolev inequalities in $W^{1,p}(\Omega_\epsilon)$ with constants which do not depend on ϵ.

Lemma A.3 (Anisotropic Sobolev inequalities in perforated domains)**.**
(i) For q_1 and r_1 satisfying the conditions

$$\begin{cases} \dfrac{1}{r_1} + \dfrac{3}{2q_1} = \dfrac{3}{4}, \\ r_1 \in [2, \infty], \ q_1 \in [2, 6] \ , \end{cases} \tag{117}$$

the following estimate holds (for $v \in H^1(0, T; L^2(\Omega_\epsilon)) \cap L^2(0, T; H^1(\Omega_\epsilon))$):

$$\|v\|_{L^{r_1}(0,T;L^{q_1}(\Omega_\epsilon))} \le c \, \|v\|_{Q_\epsilon(T)}, \tag{118}$$

where $c > 0$ does not depend on ϵ, and (we recall that)

$$\|v\|^2_{Q_\epsilon(T)} := \sup_{0 \le t \le T} \int_{\Omega_\epsilon} |v(t)|^2 \, dx + \int_0^T dt \int_{\Omega_\epsilon} |\nabla v(t)|^2 \, dx; \tag{119}$$

(ii) For q_2 and r_2 satisfying the conditions

$$\begin{cases} \dfrac{1}{r_2} + \dfrac{1}{q_2} = \dfrac{3}{4}, \\ r_2 \in [2, \infty], \ q_2 \in [2, 4] \ , \end{cases} \tag{120}$$

the following estimate holds (for $v \in H^1(0,T;L^2(\Omega_\epsilon)) \cap L^2(0,T;H^1(\Omega_\epsilon))$):

$$\|v\|_{L^{r_2}(0,T;L^{q_2}(\Gamma_\epsilon))} \leq c\,\epsilon^{-\frac{3}{2}+\frac{2}{q_2}}\,\|v\|_{Q_\epsilon(T)}, \tag{121}$$

where $c > 0$ does not depend on ϵ.

Proof. For the proof of this Lemma, see [11]. □

B Appendix B

We present in this Appendix some results on two-scale convergence (cf. [1], [2], [17], [7], [12], [16]). Up to Prop. B.5, Ω is a bounded open set of $\mathbb{R}^3$ with smooth boundary, and $Y = [0,1[^3$. Then, for Prop. B.6 and Lemma B.7, Ω_ϵ is a perforated domain satisfying Assumption 0.

We start with the:

Definition B.1. *A sequence of functions v^ϵ in $L^2([0,T]\times\Omega)$ two-scale converges to $v_0 \in L^2([0,T]\times\Omega\times Y)$ if*

$$\lim_{\epsilon\to 0}\int_0^T\int_\Omega v^\epsilon(t,x)\,\phi\left(t,x,\frac{x}{\epsilon}\right)dt\,dx = \int_0^T\int_\Omega\int_Y v_0(t,x,y)\,\phi(t,x,y)\,dt\,dx\,dy, \tag{122}$$

for all $\phi \in C^1([0,T]\times\overline{\Omega};C^\infty_\#(Y))$.

We recall then the following classical Proposition:

Proposition B.2. *If v^ϵ is a bounded sequence in $L^2([0,T]\times\Omega)$, then there exists a function $v_0 := v_0(t,x,y)$ in $L^2([0,T]\times\Omega\times Y)$ such that, up to a subsequence, v^ϵ two-scale converges to v_0.*

Then, the following Proposition is useful for obtaining the limit of the product of two two-scale convergent sequences.

Proposition B.3. *Let v^ϵ be a sequence of functions in $L^2([0,T]\times\Omega)$ which two-scale converges to a limit $v_0 \in L^2([0,T]\times\Omega\times Y)$. Suppose furthermore that*

$$\lim_{\epsilon\to 0}\int_0^T\int_\Omega |v^\epsilon(t,x)|^2\,dt\,dx = \int_0^T\int_\Omega\int_Y |v_0(t,x,y)|^2\,dt\,dx\,dy. \tag{123}$$

Then, for any sequence w^ϵ in $L^2([0,T]\times\Omega)$ that two-scale converges to a limit $w_0 \in L^2([0,T]\times\Omega\times Y)$, we get the limit

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^T\int_\Omega v^\epsilon(t,x)\, w^\epsilon(t,x)\,\phi\left(t,x,\frac{x}{\epsilon}\right)dt\,dx \\ = \int_0^T\int_\Omega\int_Y v_0(t,x,y)\, w_0(t,x,y)\,\phi(t,x,y)\,dt\,dx\,dy,\end{aligned} \tag{124}$$

for all $\phi \in C^1([0,T]\times\overline{\Omega}; C^\infty_\#(Y))$.

Remark: Note that, in the setting of this paper, identity (123) can be obtained by standard computations, used in problems with perforated domains, thanks to the existence of the extension operators P_ϵ (stated in Lemma A.2).

The next Propositions yield a characterization of the two-scale limits of gradients of bounded sequences v^ϵ. This result is crucial for applications to homogenization problems.

Proposition B.4. *Let v^ϵ be a bounded sequence in $L^2(0,T;H^1(\Omega))$ that converges weakly to a limit $v := v(t,x)$ in $L^2(0,T;H^1(\Omega))$. Then, v^ϵ also two-scale converges to v, and there exists a function $v_1 := v_1(t,x,y)$ in $L^2([0,T]\times\Omega; H^1_\#(Y)/\mathbb{R})$ such that, up to extraction of a subsequence, ∇v^ϵ two-scale converges to $\nabla_x v(t,x) + \nabla_y v_1(t,x,y)$.*

Proposition B.5. *Let v^ϵ be a bounded sequence in $L^2([0,T]\times\Omega)$, such that $\epsilon\nabla_x v^\epsilon$ also is a bounded sequence in $L^2([0,T]\times\Omega)$. Then, there exists a function $v_1 := v_1(t,x,y)$ in $L^2([0,T]\times\Omega; H^1_\#(Y)/\mathbb{R})$ such that, up to extraction of a subsequence, v^ϵ and $\epsilon\nabla v^\epsilon$ respectively two-scale converge to v_1 and $\nabla_y v_1$.*

The main result of two-scale convergence can be generalized to the case of sequences defined in $L^2([0,T]\times\Gamma_\epsilon)$.

Proposition B.6. *Let v^ϵ be a sequence in $L^2([0,T]\times\Gamma_\epsilon)$ such that*

$$\epsilon\int_0^T\int_{\Gamma_\epsilon}|v^\epsilon(t,x)|^2\,dt\,d\sigma_\epsilon(x) \le C, \tag{125}$$

where C is a positive constant, independent of ϵ. There exist a subsequence (still denoted by ϵ) and a two-scale limit $v_0(t,x,y) \in L^2([0,T]\times\Omega; L^2(\Gamma))$ such that $v^\epsilon(t,x)$ two-scale converges to $v_0(t,x,y)$ in the sense that

$$\lim_{\epsilon\to 0} \epsilon \int_0^T \int_{\Gamma_\epsilon} v^\epsilon(t,x)\, \phi\left(t, x, \frac{x}{\epsilon}\right) dt\, d\sigma_\epsilon(x) = \int_0^T \int_\Omega \int_\Gamma v_0(t,x,y)\, \phi(t,x,y)\, dt\, dx\, d\sigma(y) \tag{126}$$

for any function $\phi \in C^1([0,T] \times \overline{\Omega}; C^\infty_\#(Y))$.

The proof of Prop. B.6 is very similar to the usual two-scale convergence theorem [1]. It relies on the following lemma [2]:

Lemma B.7. *Let* $B = C[\overline{\Omega}; C_\#(Y)]$ *be the space of continuous functions* $\phi(x,y)$ *on* $\overline{\Omega} \times Y$ *which are* Y*-periodic in* y*. Then,* B *is a separable Banach space which is dense in* $L^2(\Omega; L^2(\Gamma))$*, and such that any function* $\phi(x,y) \in B$ *satisfies*

$$\epsilon \int_{\Gamma_\epsilon} \left|\phi(x, \frac{x}{\epsilon})\right|^2 d\sigma_\epsilon(x) \leq C\, \|\phi\|_B^2, \tag{127}$$

and

$$\lim_{\epsilon\to 0} \epsilon \int_{\Gamma_\epsilon} \left|\phi\left(x, \frac{x}{\epsilon}\right)\right|^2 d\sigma_\epsilon(x) = \int_\Omega \int_\Gamma |\phi(x,y)|^2\, dx\, d\sigma(y). \tag{128}$$

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