THE LINEAR BOLTZMANN EQUATION FOR LONG-RANGE FORCES: A DERIVATION FROM PARTICLE SYSTEMS

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Abstract. In this paper we consider a particle moving in a random distribution of obstacles. Each obstacle generates an inverse power law potential $\frac{\varepsilon^s}{|x|^s}$, where ε is a small parameter and $s > 2$. Such a rescaled potential is truncated at distance $\varepsilon^{\gamma-1}$, where $\gamma \in]0,1[$ is suitably large. We assume also that the scatterer density is diverging as ε^{-d+1}, where d is the dimension of the physical space.

We prove that, as $\varepsilon \to 0$ (the Boltzmann-Grad limit), the probability density of a test particle converges to a solution of the linear (uncutoffed) Boltzmann equation with the cross section relative to the potential $V(x) = |x|^{-s}$.

1. Introduction

It is well known how interesting and challenging is the problem of obtaining a complete and rigorous derivation of the kinetic transport equations starting from the basic Hamiltonian particle dynamics.

The first result in this direction was obtained many years ago by G. Gallavotti who showed how to derive the linear Boltzmann equation (with hard–sphere cross section) starting from the dynamics of a single particle in a random distribution of fixed hard scatterers in the so–called Boltzmann–Grad limit. This paper (Cf. [G]), unfortunately unpublished and not widely known, is technically simple but has a deep content. In particular it is proved there for the first time that it is perfectly consistent to obtain an irreversible stochastic behavior as a limit of a sequence of deterministic Hamiltonian systems (in a random medium). Later on this result was improved (see [S1], [S2] and [BBuS]). More recently, the Boltzmann–Grad limit in the case when the distribution of scatterers is periodic (and not random) has also been considered in [BoGoW] (see also the references therein). Note that in this case, the result is totally different.

It is worthwile to mention also the well known Lanford's result for short times (see [L]) for the fully nonlinear Boltzmann equation, derived from a system of hard spheres. The reader will find in [CIP] (Ch. 4) additional results, references and further comments on the matter.

The Boltzmann equation for long-range potentials is more singular because of the presence of grazing collisions making meaningless the gain and loss terms of the collision operator taken separately. Indeed the collision term makes sense only by compensation (see e.g. [Gr], [A], [De], [Gou]).

In this paper we address the problem of a rigorous derivation of the linear Boltzmann equation for a long-range, inverse power law interaction along the following lines. We consider the behavior of a test particle under the action of a random distribution of obstacles. Given $\varepsilon > 0$ a small positive parameter, we assume that the density of distribution of scatterers is suitably diverging as well as the range of the interaction. More precisely a given scatterer localized in $c (\in \mathbb{R}^d)$ generates a potential of the form:

$$\check{V}_\varepsilon(x-c) = V_\varepsilon\Big(\frac{x-c}{\varepsilon}\Big), \tag{1.1}$$

where the unrescaled potential V_ε is given by:

$$V_\varepsilon(x) = \frac{1}{|x|^s} \quad \text{when} \quad |x| < \varepsilon^{-1+\gamma},$$

and

$$V_\varepsilon(x) = \varepsilon^{-s(\gamma-1)} \quad \text{when} \quad |x| \geq \varepsilon^{-1+\gamma}, \tag{1.2}$$

where $\gamma \in]0,1[$ is a parameter to be fixed. This is an inverse power law potential, cutoffed at large distances.

The distribution of scatterers is a Poisson law of intensity $\mu_\varepsilon = \varepsilon^{-d+1}\mu$, where $\mu > 0$ is fixed and d is the dimension of the physical space.

What we are considering here is nothing else than the usual Boltzmann-Grad limit for the Lorentz model (see e.g. [G], [BBS]..), with in addition a simultaneous divergence of the range of the potential allowing to recover the grazing collisions in the limit. In this framework we prove that the probability density associated to the test particle converges, in the limit $\varepsilon \to 0$, to a solution of the uncutoffed linear Boltzmann equation with a cross section given by the inverse power law potential $|x|^{-s}$.

We remark that one would really like to prove the same result directly for an uncutoffed potential $V(x) = |x|^{-s}$, giving, in this way, a complete derivation of the linear Boltzmann equation in terms of the basic Hamiltonian system. This problem however, presents deep additional difficulties which will be discussed in some details later on. Thus the present result can be viewed as a first step in this direction.

The proof we give here is very direct and is in the same spirit as that in [G]. Roughly speaking we basically show that a typical trajectory of the test particle is going to perform a random flight with infinitely many collisions. However, for a fixed angle $\alpha > 0$, only a finite number of collisions have a scattering angle larger that α. In other words, most of the collisions are grazing.

The plan of the paper is the following. In Section 2 we introduce the model, the scaling and establish the result. In Section 3 we give its proof. Comparing this proof with that of [G], we find an additional difficulty. Due to the fact that the range of the potential is infinite in the limit, the test particle interacts typically with infinitely many obstacles, so that the set of bad configurations of scatterers, preventing the Markov property of the limit (such as the set of configurations yielding recollisions) must be estimated explicitely, while for a short-range potential a simple dimensional argument is sufficient.

Finally, some useful estimates on the cross section are given in the Appendix.

2. Notation, results and comments

Consider a Poisson distribution of fixed particles (obstacles or scatterers) in $\mathbb{R}^d$ ($d = 2$ or 3 is the dimension of the physical space), of parameter $\mu_\varepsilon = \varepsilon^{-(d-1)}\mu$, where $\mu > 0$ is fixed and $\varepsilon \in]0,1]$. More explicitely, the probability distribution of finding exactly N obstacles in a bounded measurable set $\Lambda \subset \mathbb{R}^d$ is given by:

$$P(d\mathbf{c}_N) = e^{-\mu_\varepsilon|\Lambda|}\frac{\mu_\varepsilon^N}{N!}dc_1 \dots dc_N, \tag{2.1}$$

where $c_1 \dots c_N = \mathbf{c}_N$ are the positions of the scatterers and $|\Lambda|$ denotes the Lebesgue measure of Λ.

The expectation with respect to the Poisson repartition of parameter μ_ε will be denoted by $\mathbb{E}^\varepsilon$.

Consider a fixed $\gamma \in]0,1[$ and the cutoffed (rescaled) potential (1.1). Let $T^t_{\mathbf{c},\varepsilon,\gamma}$ be the Hamiltonian flow generated by the distribution of obstacles $\mathbf{c}$ associated with this potential. Namely, $T^t_{\mathbf{c},\varepsilon,\gamma}(x,v) = (x_\mathbf{c}(t), v_\mathbf{c}(t))$ is the solution of the problem:

$$\begin{aligned} \dot{x}_\mathbf{c}(t) &= v_\mathbf{c}(t), \\ \dot{v}_\mathbf{c}(t) &= \check{F}_\varepsilon(x_\mathbf{c}(t);\mathbf{c}), \\ x_\mathbf{c}(0) = x, &\qquad v_\mathbf{c}(0) = v, \end{aligned} \tag{2.2}$$

with

$$\check{F}_\varepsilon(x;\mathbf{c}) = -\sum_{c\in\mathbf{c}} \nabla \check{V}_\varepsilon(x-c). \tag{2.3}$$

The rescaled cutoffed potential $\check{V}_\varepsilon$ explicitely reads as

$$\check{V}_\varepsilon(x) = \frac{\varepsilon^s}{|x|^s} \quad \text{when} \quad |x| < \varepsilon^\gamma,$$

and

$$\check{V}_\varepsilon(x) = \varepsilon^{-s(\gamma-1)} \quad \text{when} \quad |x| \geq \varepsilon^\gamma. \tag{2.4}$$

We shall also denote this flow by $T^t_{\mathbf{c}}$ when no confusion can occur. Notice that the sum (2.3) is almost surely finite since the Poisson distribution gives probability one to the locally finite sets. Due to the discontinuity of $\check{F}_\varepsilon(x;\mathbf{c})$, the solution of Eq. (2.2) might not be defined if the trajectory became tangent to the union of spheres $\cup_{c\in\mathbf{c}}\,\{x/\,|x-c| = \varepsilon^\gamma\}$. However it is easy to show that this event happens for a zero-Poisson measure set of obstacles, and it can therefore be disregarded. Finally, the quantity $T^t_{\mathbf{c},\varepsilon,\gamma}(x,v)$ is defined for all $t\in\mathbb{R}$.

From now on we shall consider in detail only the two–dimensional case ($d=2$).

For a given initial datum $f_0 \in L^1 \cap L^\infty \cap C(\mathbb{R}^d \times \mathbb{R}^d)$, we can define the quantity

$$f_\varepsilon(t,x,v) = \mathbb{E}^\varepsilon\,[f_0(T^{-t}_{\mathbf{c},\varepsilon}(x,v))]. \tag{2.5}$$

In this paper we are interested in the asymptotic behavior of f_ε when $\varepsilon \to 0$ (and $|v|=1$ for the sake of simplicity). In this analysis, we are led to consider the following initial value problem, associated to the linear cutoffed Boltzmann kernel,

$$(\partial_t + v\cdot\nabla_x) h_{\varepsilon,\gamma}(t,x,v) = \mu \int_{\theta=-\pi}^{\pi} B_{\varepsilon,\gamma}(\theta) \left\{ h_{\varepsilon,\gamma}(t,x,R_\theta(v)) - h_{\varepsilon,\gamma}(t,x,v) \right\} d\theta,$$

$$h_{\varepsilon,\gamma}(0,x,v) = f_0(x,v). \tag{2.6}$$

Here, R_θ denotes the rotation of angle θ and $B_{\varepsilon,\gamma}$ is the cross section associated to a relative velocity of modulus one and to the unrescaled cutoffed potential V_ε given by (1.2).

Our main result is the following:

Theorem 2.1: *Assume that $s > 2$ and $\gamma \in]\frac{15}{17}, 1[$. Let the initial datum f_0 belong to $L^1 \cap W^{1,\infty}(\mathbb{R}^2 \times \mathbb{R}^2)$. Then, for any $T>0$, the quantity f_ε defined in (2.5) satisfies*

$$\lim_{\varepsilon\to 0} || f_\varepsilon|_{[0,T]\times\mathbb{R}^2\times S^1} - h_{\varepsilon,\gamma} ||_{L^\infty([0,T];L^1(\mathbb{R}^2\times S^1))} = 0. \tag{2.7}$$

The proof of this theorem is presented in section 3.

The proof of the transition from the particle system we are considering to the uncutoffed Boltzmann equation is thus reduced to a partial differential equation problem, namely that of the convergence when $\varepsilon \to 0$ of the solution of the cutoffed linear Boltzmann equation (2.6) towards the solution of the uncutoffed linear Boltzmann equation. Indeed, we prove in Appendix A (proposition A.2) that $h_{\varepsilon,\gamma} \to f$ (in $L^\infty([0,T] \times \mathbb{R}^2 \times S^1)$ weak $*$, up to extraction of subsequences), where f is a solution (in the weak sense precised in proposition A.2) of

$$(\partial_t + v \cdot \nabla_x) f(t,x,v) = \mu \int_{\theta=-\pi}^{\pi} B(\theta) \Big\{ f(t,x,R_\theta(v)) - f(t,x,v) \Big\} d\theta,$$
$$f(0,x,v) = f_0(x,v), \tag{2.8}$$

and B is the (singular) cross section corresponding to a relative velocity of modulus one and to the potential $V(x) = |x|^{-s}$.

Remark 2.1: The limit we are considering here can be seen in a different way, namely in terms of microscopic variables. Consider a Poisson distribution of scatterers of parameter ν_ε in $\mathbb{R}^d$ and a light particle under the action of the unrescaled potential $\sum_{c \in \mathbf{c}} V_\varepsilon(x-c)$. Consider $g^{\mathbf{c}}(t,x,v) = f_0(S^{-t}_{\mathbf{c},\varepsilon}(x,v))$, where $S^{-t}_{\mathbf{c},\varepsilon}$ is the flow generated by the obstacles $\mathbf{c}$. Scale hyperbolically space and time as for the hydrodynamical limit:

$$g^{\mathbf{c}}_\varepsilon(t,x,v) = \varepsilon^{-d} g^{\mathbf{c}}(\varepsilon^{-1}t, \varepsilon^{-1}x, v). \tag{2.9}$$

Considering also the density $\nu_\varepsilon = \varepsilon^{d-1}\mu$ (for a given fixed positive μ), and taking the expectation (denoted by $\mathbb{E}_{\nu_\varepsilon}$), we get

$$g_\varepsilon \equiv \mathbb{E}_{\nu_\varepsilon} g^{\mathbf{c}}_\varepsilon = f_\varepsilon,$$

so that g_ε also converges to f.

Remark 2.2: It would be more appropriate, from a physical point of view, to consider more general distributions of obstacles than the Poisson distribution, for instance the Gibbs distribution at a given temperature. We note however that this distribution is asymptotically equivalent to the Poisson distribution in the limit we are considering and that our approach works for other non–equilibrium distributions, not singular with respect to the free gas case we have considered explicitely.

Remark 2.3: On the basis of the present result one could hope to give a complete derivation of the linear Boltzmann equation for long-range forces by proving that the motion of the test particle under the action of (a random distribution of) obstacles generating uncutoffed long-range forces is asymptotic to that investigated here. Unfortunately, even though the long-range tails add a very small contribution to the total force for each typical scatterer distribution, the non-grazing collisions generate an exponential instability making the two trajectories very different. Thus the completion of the proof requires new ideas and techniques.

Remark 2.4: The assumption $s > 2$ is used in appendix A (more precisely just after formula (A.13)). We think it is probably possible to relax this assumption, but we shall not try to do so.

3. Proofs

This section is devoted to the proof of Theorem 2.1. In the following we shall denote by $B(x,R) = \{y \in \mathbb{R}^2 / \, |x-y| < R\}$ the disk of radius R. We fix an arbitrary time $T > 0$ and consider our dynamical problem for times t such that $|t| < T$. We shall also use the simplified notation $B(x) = B(x,T)$. Finally we shall denote by C any positive constant (possibly depending on the fixed parameters, but independent of ε), anf by $\varphi = \varphi(\varepsilon)$ any positive function vanishing with ε.

We start by giving a straightforward probability estimate:

Lemma 3.1: *Assume that* $\gamma \in]\frac{1}{2}, 1[$, *and for a given fixed* $x \in \mathbb{R}^2$, *consider the indicator*

$$\chi_1(\mathbf{c}_N) = \chi(\left\{ \mathbf{c}_N \in B(x)^N, \quad \forall i = 1 \dots N, \quad |c_i - x| > \varepsilon^\gamma \right\}). \tag{3.1}$$

Then,

$$\mathbb{E}^\varepsilon(\chi_1) \geq 1 - \varphi(\varepsilon). \tag{3.2}$$

Proof of lemma 3.1: We compute

$$\mathbb{E}^\varepsilon(1 - \chi_1(\mathbf{c}_N)) = \sum_{N \geq 0} e^{-\mu_\varepsilon |B(x)|} \frac{\mu_\varepsilon^N}{N!}$$

$$\int_{B(x)^N} \chi(\left\{ \exists i \in [1,N], \quad c_i \in B(x, \varepsilon^\gamma) \right\}) \, d\mathbf{c}_N$$

$$\leq \sum_{N \geq 0} e^{-\mu_\varepsilon |B(x)|} \frac{\mu_\varepsilon^N}{N!} N \int_{c_1 \in B(x,\varepsilon^\gamma)} \int_{c_2,..,c_N \in B(x)} d\mathbf{c}_N$$

$$\leq \sum_{N \geq 1} e^{-\mu_\varepsilon |B(x)|} \frac{\mu_\varepsilon^N}{(N-1)!} \pi \varepsilon^{2\gamma} |B(x)|^{N-1}$$

$$\leq \mu_\varepsilon \, \pi \, \varepsilon^{2\gamma}. \tag{3.3}$$

□We now come back to the proof of theorem 2.1. Given a configuration $\mathbf{c}$ of scatterers such that $\chi_1(\mathbf{c}) = 1$, the energy of a light particle of coordinates (x, v) in the phase space satisfies the following identity (recall that $|v| = 1$),

$$H(x, v, \mathbf{c}) \equiv \frac{1}{2}|v|^2 + \sum_{c \in \mathbf{c}} \tilde{V}_\varepsilon(x - c) = \frac{1}{2}, \tag{3.4}$$

where $\tilde{V}_\varepsilon(x - c) = \check{V}_\varepsilon(x - c) - \varepsilon^{-s(\gamma-1)}$.

Therefore for a configuration $\mathbf{c}$ such that $\chi_1(\mathbf{c}) = 1$, and any time $|t| \leq T$, we know that $|v_{\mathbf{c}}(t)| \leq 1$ and $x_{\mathbf{c}}(t) \in B(x)$, so that the only obstacles acting on the flow are those in $B(x)$ (at least when $\varepsilon^\gamma < T$).

Then, one can give for f_ε the following explicit formula,

$$f_\varepsilon(t, x, v) = e^{-\mu_\varepsilon |B(x)|} \sum_{N \geq 0} \frac{\mu_\varepsilon^N}{N!} \int_{B(x)^N} d\mathbf{c}_N \chi_1(\mathbf{c}_N) f_0(T^{-t}_{\mathbf{c}_N}(x, v)) + \varphi(\varepsilon). \tag{3.5}$$

From now on, we shall replace the flow $T^{-t}_{\mathbf{c}_N}$ by the flow $T^{t}_{\mathbf{c}_N}$. The result will be the same thanks to the reversibility of this (Hamiltonian) flow.

The rescaled cutoffed potential $\check{V}_\varepsilon$ has ε^γ as range (more precisely it is constant on $B(0, \varepsilon^\gamma)^c$ and therefore the corresponding force is 0 on this set). It means that the scatterer c_i has no influence on the flow whenever the light particle is outside its protection disk $B(c_i, \varepsilon^\gamma)$. Therefore, among the obstacles $c \in \mathbf{c} \cap B(x)$, we distinguish between those influencing the motion of the light particle and the others. Indeed we call "external"(up to time t) the obstacles $c \in \mathbf{c} \cap B(x)$ such that

$$\inf_{0 \leq s \leq t} |x_{\mathbf{c}}(s) - c| > \varepsilon^\gamma, \tag{3.6}$$

and "internal" all the others. Then we decompose a given configuration $\mathbf{c}_N$ of $B(x)^N$ in the following way,

$$\mathbf{c}_N = \mathbf{a}_P \cup \mathbf{b}_Q, \tag{3.7}$$

where $\mathbf{a}_P$ is the set of all external obstacles and $\mathbf{b}_Q$ is the set of all internal ones.

Realizing then that

$$T^t_{\mathbf{c}_N} = T^t_{\mathbf{b}_Q}, \qquad \chi_1(\mathbf{c}_N) = \chi_1(\mathbf{b}_Q) \tag{3.8}$$

(in fact χ_1 is the characteristic function of those configurations for which no obstacle is internal at time 0), we get

$$f_\varepsilon(t,x,v) = e^{-\mu_\varepsilon |B(x)|} \sum_{Q\geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{B(x)^Q} d\mathbf{b}_Q$$

$$\times \sum_{P\geq 0} \frac{\mu_\varepsilon^P}{P!} \int_{B(x)^P} d\mathbf{a}_P\, \chi(\Big\{ \text{ the } \mathbf{a_P} \text{ are external and the } \mathbf{b}_Q \text{ are internal } \Big\})$$

$$\times \chi_1(\mathbf{a}_P \cup \mathbf{b}_Q)\, f_0(T^t_{\mathbf{a}_P\cup\mathbf{b}_Q}(x,v)) + \varphi(\varepsilon)$$

$$= \sum_{Q\geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{B(x)^Q} d\mathbf{b}_Q e^{-\mu_\varepsilon |\mathcal{T}(\mathbf{b}_Q)|} \chi_1(\mathbf{b}_Q) \chi(\Big\{ \text{ the } \mathbf{b}_Q \text{ are internal} \Big\})$$

$$\times f_0(T^t_{\mathbf{b}_Q}(x,v)) + \varphi(\varepsilon). \tag{3.9}$$

The factor $e^{-\mu_\varepsilon |\mathcal{T}(\mathbf{b}_Q)|}$, where $\mathcal{T}(\mathbf{b}_Q)$ is the tube (at time t) defined by

$$\mathcal{T}(\mathbf{b}_Q) = \Big\{ y \in B(x), \quad \exists s \in [0,t], \quad |y - x_{\mathbf{b}_Q}(s)| < \varepsilon^\gamma \Big\}, \tag{3.10}$$

arises from the integration over $d\mathbf{a}_P$ which has been performed explicitely.

Note that

$$\chi(\Big\{ \text{ the } \mathbf{b}_Q \text{ are internal} \Big\}) = \chi(\Big\{ \mathbf{b}_Q \subset \mathcal{T}(\mathbf{b}_Q) \Big\}). \tag{3.11}$$

Note also that when $\chi_1(\mathbf{b}_Q) = 1$, the length of the curve $(x_{\mathbf{b}_Q}(s))_{s\in[0,t]}$ is not larger than t (since the velocity of the particle is bounded by 1), and therefore one has

$$|\mathcal{T}(\mathbf{b}_Q)| \leq 2\, t\, \varepsilon^\gamma. \tag{3.12}$$

We now set

$$\chi_2(\mathbf{b}_Q) = \chi(\Big\{ \mathbf{b_Q} \in B(x)^Q, \quad \forall\, 1 \leq i < j \leq Q, \quad |b_i - b_j| > 2\,\varepsilon^\gamma \Big\}). \tag{3.13}$$

Note that χ_2 is the characteristic function of the set of configurations $\mathbf{c}$ for which there is no overlapping of the protection disks of any pair of internal scatterers of $B(x)$.

Then, we can prove the

Lemma 3.2: *If* $\gamma \in]\frac{2}{3}, 1[$, *one has*

$$\sum_{Q\geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{B(x)^Q} e^{-\mu_\varepsilon |\mathcal{T}(\mathbf{b}_Q)|} \chi(\Big\{ \mathbf{b}_Q \subset \mathcal{T}(\mathbf{b}_Q) \Big\})\, \chi_1 \chi_2(\mathbf{b}_Q)\, d\mathbf{b}_Q \geq 1 - \varphi(\varepsilon). \tag{3.14}$$

Note however that $\chi_2(\mathbf{c}_N) = 0$ with a large probability (when we consider all the scatterers in the ball $B(x)$ and not only the internal ones).

Proof of Lemma 3.2: We consider $\gamma' \le \gamma$ and compute

$$I_\varepsilon = \sum_{Q\ge 0} e^{-\mu_\varepsilon |B(x)|} \frac{\mu_\varepsilon^Q}{Q!} \int_{B(x)^Q} \chi(\Big\{ \exists i,j \in [1,Q], \quad b_i, b_j \in \mathcal{T}(\mathbf{b}_Q) \quad \text{and} \quad |b_i - b_j| \le 2\,\varepsilon^{\gamma'} \Big\} d\mathbf{b}_Q. \tag{3.15}$$

Then:

$$I_\varepsilon \le \sum_{Q\ge 2} e^{-\mu_\varepsilon |B(x)|} \frac{\mu_\varepsilon^Q}{Q!} \frac{Q\,(Q-1)}{2} \times \int_{B(x)^Q} \chi(\Big\{ b_1, b_2 \in \mathcal{T}(\mathbf{b}_Q) \text{ and } |b_1 - b_2| \le 2\,\varepsilon^{\gamma'} \Big\}) d\mathbf{b}_Q. \tag{3.16}$$

Noting that

$$\chi(b_1, b_2 \in \mathcal{T}(\mathbf{b}_Q)) \le \chi(b_1 \in \mathcal{T}(b_3 \dots b_Q)) + \chi(b_2 \in \mathcal{T}(b_3 \dots b_Q)) \tag{3.17}$$

since

$$b_1 \notin \mathcal{T}(b_3 \dots b_Q), b_2 \notin \mathcal{T}(b_3 \dots b_Q) \quad \Rightarrow \quad \mathcal{T}(\mathbf{b}_Q) = \mathcal{T}(b_3 \dots b_Q),$$

we can write

$$I_\varepsilon \le \frac{1}{2} \sum_{Q\ge 2} e^{-\mu_\varepsilon |B(x)|} \frac{\mu_\varepsilon^Q}{(Q-2)!} \int_{B(x)^{Q-2}} \int_{c_1 \in B(x)} \int_{c_2 \in B(x)}$$

$$\Big(\chi(b_1 \in \mathcal{T}(b_3 \dots b_Q)) + \chi(b_2 \in \mathcal{T}(b_3 \dots b_Q)) \Big) \chi(\Big\{ |b_1 - b_2| \le 2\,\varepsilon^{\gamma'} \Big\}) d\mathbf{b}_Q$$

$$\le \sum_{Q\ge 2} e^{-\mu_\varepsilon |B(x)|} \frac{\mu_\varepsilon^Q}{(Q-2)!} \int_{B(x)^{Q-2}} |\mathcal{T}(b_3 \dots b_N)| db_3 \dots db_Q \; |B(0, 2\,\varepsilon^{\gamma'})|.$$

Then, if we restrict the integration over the set for which $\chi_1(\mathbf{b}_Q) = 1$, we bound the above integral by:

$$I_\varepsilon \le C(T)\,\varepsilon^{\gamma + 2\,\gamma' - 2}. \tag{3.18}$$

The lemma is then a consequence of this estimate when $\gamma' = \gamma$. □

For a given configuration $\mathbf{b}_Q \in B(x)^Q$ such that $\chi_1 \chi_2(\mathbf{b}_Q) = 1$ and such that the b_i's are internal for $i = 1 \dots Q$, we define

$$\chi_3(\mathbf{b}_Q) = \chi(\Big\{ \mathbf{b}_Q, \quad \forall i = 1 \dots Q, \quad x_{\mathbf{b}_Q}^{-1}(B(b_i, \varepsilon^\gamma)) \text{ is connected in } [0,t] \Big\}). \tag{3.19}$$

In other words, χ_3 is the characteristic function of the set of configurations for which there is no recollisions (up to time t) of the light particle with a given obstacle.

According to the previous analysis (and in particular Lemma 3.1 and 3.2), we can replace in (2.7) the quantity f_ε by $\hat{f}_\varepsilon$, defined in the following way,

$$\hat{f}_\varepsilon(t,x,v) = \sum_{Q\geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{B(x)^Q} e^{-\mu_\varepsilon |\mathcal{T}(\mathbf{b}_Q)|} \chi(\left\{ \mathbf{b}_Q \subset \mathcal{T}(\mathbf{b}_Q) \right\})$$
$$\times \chi_1\chi_2(\mathbf{b}_Q)\, f_0(T^t_{\mathbf{b}_Q})(x,v) d\mathbf{b}_Q. \tag{3.20}$$

However, instead of considering $\hat{f}_\varepsilon$ we shall analyze, for the moment, the behavior of $\tilde{f}_\varepsilon$ defined by

$$\tilde{f}_\varepsilon(t,x,v) = e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{B(x)^Q} \chi(\left\{ \mathbf{b}_Q \subset \mathcal{T}(\mathbf{b}_Q) \right\})$$
$$\times \chi_1\chi_2\chi_3(\mathbf{b}_Q)\, f_0(T^t_{\mathbf{b}_Q})(x,v) d\mathbf{b}_Q. \tag{3.21}$$

Note that

$$\tilde{f}_\varepsilon \leq \hat{f}_\varepsilon. \tag{3.22}$$

A typical trajectory for a configuration of scatterers which is such that $\chi_1\chi_2\chi_3 = 1$ (and such that the b_i are internal for $i = 1\dots Q$) can be visualized in fig. 1.

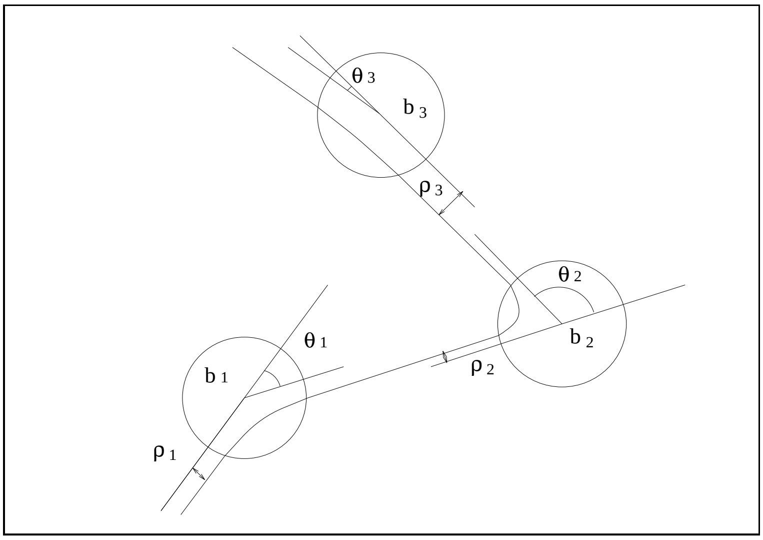

Fig. 1

We say that the light particle performs a collision with the scatterer b_i when it enters into its protection disk $B(b_i, \varepsilon^\gamma)$. Note that for a well behaving configuration described here, the light particle moves freely between two separated collisions. During the collision with the obstacle b_i (i.-e. for the times t such that $|x_{\mathbf{b}_Q}(t) - b_i| \leq \varepsilon^\gamma$), the dynamics is that of a particle moving in the potential $\check{V}_\varepsilon(\cdot - b_i)$ and can be computed "almost" exactly (see for instance [C]).

For such a trajectory, one can define, for each obstacle $b_i \in \mathbf{b}_Q$ ($i = 1 \dots Q$), the time t_i of the first (and unique because $\chi_3 = 1$) entrance in the protection disk $B(b_i, \varepsilon^\gamma)$, and the (unique) time $t_i' > t_i$ when the light particle gets out of this protection disk. We also define the impact parameter ρ_i, which is the (algebraic) distance between b_i and the straight line containing the straight trajectory followed by the light particle immediately before t_i (see fig. 1).

We now are in a position to perform the change of variables which is the crucial part of this section. We first note that, because of the symmetry with respect to $b_1 \dots b_Q$ of the expression inside the integral (3.21), one has

$$\tilde{f}_\varepsilon(t, x, v) = e^{-2\, t\, \mu_\varepsilon\, \varepsilon^\gamma} \sum_{Q \geq 0} \mu_\varepsilon^Q \int_{B(x)^Q} d\mathbf{b}_Q\, \chi_1 \chi_2 \chi_3(\mathbf{b}_Q)$$

$$\times \chi(\left\{ \mathbf{b}_Q \subset \mathcal{T}(\mathbf{b}_Q) \right\})\, \chi(\left\{ t_1 < t_2 < \dots < t_Q \right\})\, f_0(T_{\mathbf{b}_Q}^{-t}(x, v)). \quad (3.23)$$

We now use the change of variables (which depends of t, x, v, ε and γ)

$$\mathcal{Z} : \mathbf{b}_Q \to \{\rho_i, t_i\}_{i=1}^Q(\mathbf{b}_Q). \quad (3.24)$$

This mapping is indeed well–defined on the set $\Gamma \subset B(x)^Q$ of "well–ordered" configurations $\mathbf{b}_Q$ constituted of (internal) scatterers satisfying the property $\chi_1 \chi_2 \chi_3(\mathbf{b}_Q) = 1$.

The variables $\{\rho_i, t_i\}_{i=1}^Q$ satisfy then the constraints

$$0 \leq t_1 < t_2 < \dots < t_Q \leq t, \quad (3.25)$$

and

$$\forall i = 1, .., Q, \qquad |\rho_i| < \varepsilon^\gamma. \quad (3.26)$$

We now give the explicit way of finding the inverse mapping $\mathcal{Z}^{-1}$. Let a sequence $\{\rho_i, t_i\}_{i=1}^Q$ satisfying (3.25) and (3.26) be given. We build a corresponding sequence of obstacles $\beta_{\mathbf{Q}} = \beta_1 \mathinner{..} \beta_Q$ and a trajectory $(\xi(s), v(s))$ inductively. Suppose that one has been able to define the obstacles $\beta_1 \mathinner{..} \beta_{i-1}$ and a trajectory $(\xi(s), v(s))$ up to the time t_{i-1}.

We then define the trajectory between times t_{i-1} and t_i as that of the evolution of a particle moving in the potential $\check{V}_\varepsilon(\cdot - \beta_{i-1})$ with initial datum at time t_{i-1} given by $(\xi(t_{i-1}), v(t_{i-1}))$. Then, $\tau'_{i-1} > t_{i-1}$ is defined to be the first time of exit of the trajectory from the protection disk of β_{i-1}. Finally β_i is defined to be the only point at distance ε^γ of $\xi(t_i)$ and (algebraic) distance ρ_i from the straight line which is tengent to the trajectory at the point $\xi(t_i)$.

Note that for a given sequence $\{\rho_i, t_i\}_{i=1}^{Q}$, the sequence of obstacles $\beta_{\mathbf{Q}}$ and the trajectory $(\xi(s), v(s))$ can always be constructed, but the result of this construction sometimes gives rise to an unphysical trajectory, which means that the sequence $\beta_{\mathbf{Q}}$ is not in the range Γ of the mapping $\mathcal{Z}$. For instance the trajectory described in fig. 2 delivers various inconsistencies leading to such a sequence $\beta_{\mathbf{Q}}$, namely $\xi(s)$ enters into the protection disk of β_1 for $\tau'_3 < s < t_4$ (i.–e. there is recollision), β_2 overlaps β_3 ($\tau'_2 > t_3$), and β_6 belongs to the tube spanned by $\xi(s)$ for $s \in [\tau'_1, t_2]$ (we call that interferences).

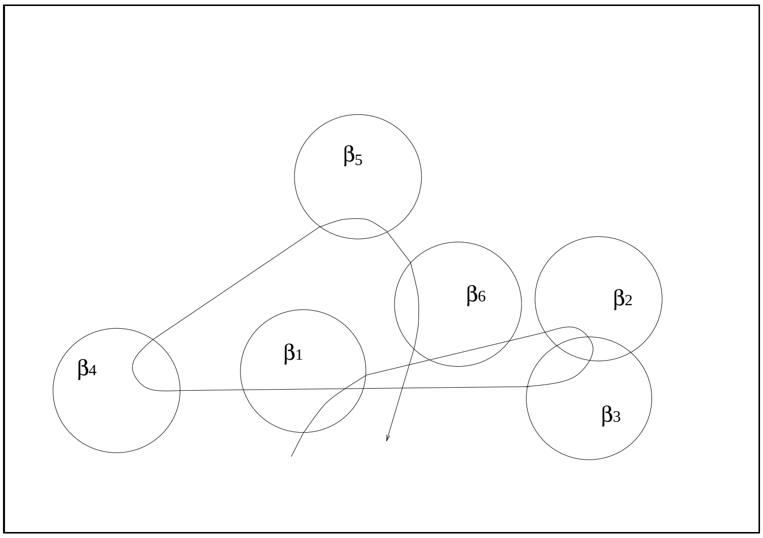

Fig. 2

It is however clear that $\mathcal{Z}$ is a diffeomorphism between Γ and the sequences $\{\rho_i, t_i\}_{i=1}^{Q}$ satisfying (3.25), (3.26), and such that for all $i = 2 \dots Q$, (with the convention $t_0 = 0$),

$$\beta_i \notin B(x, \varepsilon^\gamma) \cup B(\beta_1, 2\,\varepsilon^\gamma) \cup \dots \cup B(\beta_{i-1}, 2\,\varepsilon^\gamma), \tag{3.27}$$

(note that the condition $\tau'_{i-1} \le t_i$ is consequence of (3.27))

$$\min_{i=1\dots Q}\ \min_{j=i+2,\dots,Q}\ \inf_{\tau'_j\le s\le t_{j+1}} |\xi(s)-\beta_i| \ge \varepsilon^\gamma, \tag{3.28}$$

$$\min_{i=0\dots Q-1}\ \min_{j=i+2,\dots,Q}\ \inf_{\tau'_i\le s\le t_{i+1}} |\xi(s)-\beta_j| \ge \varepsilon^\gamma. \tag{3.29}$$

Note that condition (3.29) expresses the fact that the $i-th$ obstacle cannot be inserted in the tube generated by the light particle before the time of the first (and unique) entrance in the protection disk of β_i (what we call interference), while condition (3.28) eliminates the possibility of recollisions. Conditions (3.27–29) are indeed satisfied by the image of $\mathcal{Z}$ and ensure the admissibilty of the configurations $\beta_{\mathbf{Q}}$ and the trajectory $(\xi(s), v(s))$.

Reminding that the modulus of the initial velocity of our light particle is 1, the Jacobian of the previous change of variables is also 1.

We now can write

$$\tilde f_\varepsilon(t,x,v) = e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\ge 0} \mu_\varepsilon^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q$$
$$\int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_1 \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_2 \cdots \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_Q$$
$$\chi(\left\{\forall i=1\dots Q, \beta_i \notin B(x,\varepsilon^\gamma)\right\})\,\chi(\left\{\forall i,j=1\dots Q, i\ne j, \quad |\beta_i-\beta_j| > 2\,\varepsilon^\gamma\right\})$$
$$\times\,\chi(\left\{\min_{i=1\dots Q}\ \min_{j=i+2,\dots,Q}\ \inf_{\tau'_j\le s\le t_{j+1}} |\xi(s)-\beta_i| \ge \varepsilon^\gamma\right\})$$
$$\times\,\chi(\left\{\min_{i=0\dots Q-1}\ \min_{j=i+2,\dots,Q}\ \inf_{\tau'_i\le s\le t_{i+1}} |\xi(s)-\beta_j| \ge \varepsilon^\gamma\right\})\, f_0(\xi(t), v(t)). \tag{3.30}$$

The main point in the above representation is that the unphysical trajectories eliminated by the characteristic functions in the right hand side of eq. (3.30) are indeed negligible in the limit $\varepsilon \to 0$ (at least for suitable values of γ). More precisely, we can prove the

Proposition 3.1: *For* $\gamma \in]\frac{15}{17}, 1[$, *one has*

$$\tilde f_\varepsilon(t,x,v) = e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\ge 0} \mu_\varepsilon^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q$$
$$\int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_1 \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_2 \cdots \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_Q\ f_0(\xi(t), v(t)) + \varphi(\varepsilon). \tag{3.31}$$

Before proving Proposition 3.1, which is the central part of our proof, we conclude the proof of Theorem 2.1.

We make the change of variables

$$\{\rho_i\}_{i=1,\dots,Q} \to \{\theta_i\}_{i=1,\dots,Q}, \tag{3.32}$$

where θ_i is the angle of the scattering produced by the i–th obstacle (see fig. 1). The Jacobian determinant of this change of variables is given by $\prod_{i=1}^{Q} \check{B}_{\varepsilon,\gamma}(\theta_i) = \prod_{i=1}^{Q} \frac{d\rho_i}{d\theta_i}$ respectively. Here $\check{B}_{\varepsilon,\gamma}$ is the cross section associated to the rescaled cutoffed potential $\check{V}_\varepsilon$. Introducing the cross section $B_{\varepsilon,\gamma}$ of the unrescaled cutoffed potential V_ε (see def. (1.2)), we have $\check{B}_{\varepsilon,\gamma} = \varepsilon\, B_{\varepsilon,\gamma}$ and therefore,

$$\tilde{f}_\varepsilon(t,x,v) = e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\geq 0} \mu^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q$$

$$\int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\theta_2 \cdots \int_{-\pi}^{\pi} d\theta_Q \prod_{i=1}^{Q} B_{\varepsilon,\gamma}(\theta_i)\, f_0(\xi(t), v(t)) + \varphi(\varepsilon). \tag{3.33}$$

We denote by ψ_j the angle $\sum_{i=1}^{j} \theta_i$ (with the convention $\psi_0 = 0$), and use the convention $t_0 = 0$, $t_{Q+1} = t$. Then, the following estimate holds:

$$|\xi(t) - (x + \sum_{i=0}^{Q} R_{\psi_i}(v)\,(t_{i+1} - t_i))| \leq Q\,\varepsilon^\gamma. \tag{3.34}$$

Note also that $R_{\psi_Q}(v) = v(t)$ except when $t \in]t_Q, \tau'_Q[$.

A tedious exercice of classical mechanics based on the energy conservation shows that $|t_i - \tau'_i| \leq C\,\varepsilon^\gamma$.

Using this estimate and the fact that f_0 lies in $W^{1,\infty}$, we obtain:

$$\Big| \tilde{f}_\varepsilon(t,x,v) - e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\geq 0} \mu^Q \int_0^t dt_1 \cdots \int_{t_{Q-1}}^t dt_Q \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_Q$$

$$\prod_{i=1}^{Q} B_{\varepsilon,\gamma}(\theta_i) f_0(x + \sum_{i=0}^{Q} R_{\psi_i}(v)\,(t_{i+1} - t_i), R_{\psi_Q}(v)) \Big|$$

$$\leq C\, e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\geq 0} Q\,\varepsilon^\gamma\, \frac{(2\,t\,\mu_\varepsilon\,\varepsilon^\gamma)^Q}{Q!}$$

$$+ e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\geq 0} 2\,Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{\sup(t-5\,\varepsilon^\gamma, t_{Q-1})}^t dt_Q (2\,\mu_\varepsilon\,\varepsilon^\gamma)^Q$$

$$\leq C(T)\,\varepsilon^{2\,\gamma-1} + C\,\varepsilon^{2\,\gamma-1}. \tag{3.35}$$

Finally, we get

$$\tilde{f}_\varepsilon(t,x,v) = e^{-2\,t\,\mu\,\varepsilon^{\gamma-1}} \sum_{Q\geq 0} \mu^Q \int_0^t dt_1 \cdots \int_{t_{Q-1}}^t dt_Q \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_Q$$

$$\prod_{i=1}^{Q} B_{\varepsilon,\gamma}(\theta_i) f_0(x + \sum_{i=0}^{Q} R_{\psi_i}(v)\,(t_{i+1}-t_i), R_{\psi_Q}(v)) + \varphi(\varepsilon). \tag{3.36}$$

Noting that

$$\varepsilon \int_{-\pi}^{\pi} d\theta B_{\varepsilon,\gamma}(\theta) = \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho = 2\,\varepsilon^\gamma, \tag{3.37}$$

we see that the series expansion in the right hand side of (3.36) (which is obviously converging) is nothing else than $h_{\varepsilon,\gamma}$ in the form of the serie solution (obtained by perturbing around the loss term) to eq.(2.6).

On the other hand $h_{\gamma,\varepsilon} \geq 0$ and according to the conservation of mass,

$$\int h_{\varepsilon,\gamma}\, dxdv = \int f_0\, dxdv.$$

Then, according to (3.22), $\tilde{f}_\varepsilon$, $\check{f}_\varepsilon$ and $h_{\varepsilon,\gamma}$ have the same asymptotic behavior in $L^\infty_t(L^1_{x,v})$ as $\varepsilon \to 0$, so that (2.7) is proven. □

Proof of Proposition 3.1: We put $t_0 = 0$, $\theta_0 = 0$, $x = \beta_0$ and begin by estimating the probability of overlapping of two successive scatterers β_i, β_{i+1} (including the beginning of the trajectory $i = 0$). We have

$$A_\varepsilon = e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\geq 1} \mu_\varepsilon^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q$$

$$\int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_1 \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_2 \cdots \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_Q \sum_{i=0}^{Q-1} \chi(\left\{ |\beta_i - \beta_{i+1}| \leq 2\,\varepsilon^\gamma \right\})$$

$$\leq e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\geq 1} \mu_\varepsilon^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q$$

$$\int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_1 \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_2 \cdots \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_Q \sum_{i=0}^{Q-1} \chi(\left\{ t_{i+1} - t_i \leq C\,\varepsilon^\gamma \right\})$$

$$\leq C\, e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\geq 1} \frac{2\,\mu_\varepsilon\,\varepsilon^{\gamma Q}}{(Q-1)!}\, t^{Q-1}\, Q\, \varepsilon^\gamma$$

$$\leq C\,\varepsilon^{5\,\gamma-2}. \tag{3.38}$$

Then , we prove that other types of overlappings as well as interferences have small probability. We estimate the quantity

$$D_\varepsilon = e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\geq 1} \mu_\varepsilon^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q$$

$$\int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_1 \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_2 \cdots \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_Q \sum_{i=0}^{Q-1} \sum_{j=i+2}^{Q} \chi(\Big\{\beta_j \in \cup_{s\in]t_i,t_{i+1}[} B(\xi(s), 2\,\varepsilon^\gamma)\Big\}). \tag{3.39}$$

It is clear that if we show that D_ε is vanishing with ε, we also have proven that conditions (3.27) and (3.29) can be removed in (3.30).

We have $D_\varepsilon = D_\varepsilon^1 + D_\varepsilon^2$, where

$$D_\varepsilon^1 = e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\geq 1} \mu_\varepsilon^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_1 \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_2 \cdots \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_Q$$

$$\sum_{i=0}^{Q-1} \sum_{j=i+2}^{Q} \chi(\Big\{\beta_j \in \cup_{s\in]t_i,t_{i+1}[} B(\xi(s), 2\,\varepsilon^\gamma)\Big\})\,\chi(\Big\{d(\theta_{i+1}+\cdots+\theta_{j-1}, \pi\mathbb{Z}^*) \geq \varepsilon^\delta\Big\}), \tag{3.40}$$

and

$$D_\varepsilon^2 = e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\geq 1} \mu_\varepsilon^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_1 \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_2 \cdots \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_Q$$

$$\sum_{i=0}^{Q-1} \sum_{j=i+2}^{Q} \chi(\Big\{\beta_j \in \cup_{s\in]t_i,t_{i+1}[} B(\xi(s), 2\,\varepsilon^\gamma)\Big\})\,\chi(\Big\{d(\theta_{i+1}+\cdots+\theta_{j-1}, \pi\mathbb{Z}^*) \leq \varepsilon^\delta\Big\}), \tag{3.41}$$

where

$$d(x, \mathbb{Z}^*) = \inf_{k\in\mathbb{Z}, k\neq 0} |x-k|, \tag{3.42}$$

and θ_i are the scattering angles introduced in (3.32). We first estimate D_ε^1.

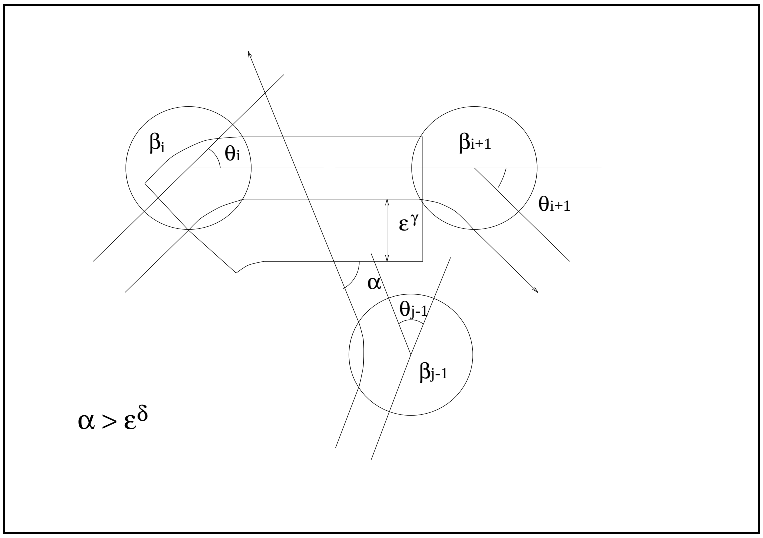

Fig. 3

According to fig. 3, we see that the integral on t_j can be restricted to a set Δ of measure at most $16\,\varepsilon^{\gamma-\delta}$ (for ε small enough).

Then, we can write

$$D_\varepsilon^1 \le e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\ge 1} (2\,\mu_\varepsilon\,\varepsilon^\gamma)^Q\, Q^2$$

$$\times 16\,\varepsilon^{\gamma-\delta} \int_0^t dt_1 \cdots \int_{t_{j-2}}^t dt_{j-1} \int_{t_{j+1}}^t dt_{j+1} \cdots \int_{t_{Q-1}}^t dt_Q$$

$$\le e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma}\, 16\,\varepsilon^{\gamma-\delta} \sum_{Q\ge 1} (2\,\mu_\varepsilon\,\varepsilon^\gamma)^Q\, Q^2\, \frac{t^{Q-1}}{(Q-1)!} \le C(T)\,\varepsilon^{4\gamma-3-\delta}. \quad (3.43)$$

Furthermore:

$$D_\varepsilon^2 \le\ e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\ge 1} \frac{(\mu_\varepsilon\,t)^Q}{Q!} \int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\theta_2 \cdots \int_{-\pi}^{\pi} d\theta_Q\, \check{B}_{\varepsilon,\gamma}(\theta_1)\dots \check{B}_{\varepsilon,\gamma}(\theta_Q)$$

$$\times \sum_{i=0}^{Q-1} \sum_{j=i+2}^{Q} \chi(\left\{ d(\theta_{i+1} + \cdots + \theta_{j-1}, \pi\mathbb{Z}^*) \le \varepsilon^\delta \right\})$$

$$\times \chi(\left\{ |\theta_k| \ge \frac{\pi}{2\,Q} \quad \text{for some} \quad k = i+1 \dots j-1 \right\}). \quad (3.44)$$

Note that the last characteristic function can be introduced because if for $k = i+1, .., j-1$ we had $|\theta_k| \le \frac{\pi}{2Q}$, then we would have $|\theta_{i+1} + \cdots + \theta_{j-1}| \le \frac{\pi}{2}$.

Introducing the set:

$$\mathbb{Z}_Q^* = \Big\{ -Q, -Q+1, \dots, -1, 1, 2, \dots, Q \Big\}, \tag{3.45}$$

we can write

$$D_\varepsilon^2 \le \; e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\ge 1} \frac{(2\,t\,\mu_\varepsilon\,\varepsilon^\gamma)^Q}{Q!} Q^3 \sup_{i=0,..,Q-1} \sup_{j=i+2,..,Q} \int_{-\pi}^{\pi} d\theta_{i+1} \cdots \int_{-\pi}^{\pi} d\theta_{j-1}$$

$$\frac{\check{B}_{\varepsilon,\gamma}(\theta_{i+1})}{2\,\varepsilon^\gamma} \cdots \frac{\check{B}_{\varepsilon,\gamma}(\theta_{j-1})}{2\,\varepsilon^\gamma}$$

$$\times \chi(\Big\{ d(\theta_{i+1} + \cdots + \theta_{j-1}, \pi \mathbb{Z}_Q^*) \le \varepsilon^\delta \Big\})\, \chi(\Big\{ |\theta_{i+1}| \ge \frac{\pi}{2\,Q} \Big\}). \tag{3.46}$$

Using estimate (A.1) of the appendix :

$$B_{\varepsilon,\gamma}(\theta) \le C\, |\theta|^{-(1+1/s)}, \tag{3.47}$$

we get

$$\frac{\check{B}_{\varepsilon,\gamma}(\theta)}{2\,\varepsilon^\gamma} \le C\, \varepsilon^{1-\gamma}\, |\theta|^{-(1+1/s)}. \tag{3.48}$$

Using the change of variable $\Theta = \theta_{i+1} + \cdots + \theta_{j-1}$, and keeping in mind that $s \ge 2$, we get

$$D_\varepsilon^2 \le \; e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\ge 1} \frac{(2\,t\,\mu_\varepsilon\,\varepsilon^\gamma)^Q}{Q!} Q^4 \sup_{i=0,..,Q-1} \sup_{j=i+2,..,Q}$$

$$\times \sup_{l\in\mathbb{Z}_Q^*} \int_{l\,\pi-\varepsilon^\delta}^{l\,\pi+\varepsilon^\delta} d\Theta\, C\, \varepsilon^{1-\gamma}\, (\frac{\pi}{2\,Q})^{-(1+1/s)}$$

$$\le 2\, e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\ge 1} \frac{(2\,t\,\mu_\varepsilon\,\varepsilon^\gamma)^Q}{Q!} Q^{11/2}\, \varepsilon^{\delta+1-\gamma}$$

$$\le C(T)\, \varepsilon^{9/2\,(\gamma-1)+\delta}. \tag{3.49}$$

Combining estimate (3.49) with (3.43), and optimizing δ, we finally obtain:

$$D_\varepsilon \le C(T)\, \varepsilon^{(17\gamma-15)/4}. \tag{3.50}$$

Thus it remains to prove that the set of configurations of scatterers yielding recollisions (namely satisfying (3.28)) is also negligible in the limit. To this purpose, we introduce the quantity $I_\varepsilon = I_\varepsilon^1 + I_\varepsilon^2$, where

$$I_\varepsilon^1 = e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\ge 0} \mu_\varepsilon^Q \sum_{i=1}^{Q} \sum_{j=i+2}^{Q} \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q$$

$$\int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_1 \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_2 \cdots \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_Q\, \chi(\Big\{ \inf_{\tau_j' \le s \le t_{j+1}} |\xi(s) - \beta_i| \le \varepsilon^\gamma \Big\}$$

$$\chi(\left\{ \sin \alpha_{jk} \leq \frac{\varepsilon^\nu}{4} \quad \forall k = i \dots j-1 \right\}, \tag{3.51}$$

$$I_\varepsilon^2 = e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q \geq 0} \mu_\varepsilon^Q \sum_{i=1}^{Q} \sum_{j=i+2}^{Q} \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q$$

$$\int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_1 \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_2 \cdots \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_Q \chi(\left\{ \inf_{\tau_j' \leq s \leq t_{j+1}} |\xi(s) - \beta_i| \leq \varepsilon^\gamma \right\}$$

$$\chi(\left\{ \sin \alpha_{jk} \geq \frac{\varepsilon^\nu}{4} \text{ for some } \quad k = i \dots j-1 \right\}, \tag{3.52}$$

and $\nu \in]0,1[$ will be chosen later on. Here α_{jk} denote the absolute values of the angles between the outgoing velocities $u_j = v(\tau_j')$ and $u_k = v(\tau_k')$ after the $j-$th and the $k-$th collisions respectively. Note that $\sin \alpha_{jk}$ is small when the collision is grazing (i.–e. when α_{jk} is close to 0), but also when the angle α_{jk} is close to π.

In order to evaluate I_ε^1 we first observe that if $\sin \alpha_{jk} \leq \frac{\varepsilon^\nu}{4}$ for all $k = i \dots j-1$ (and if there is a recollision), then $|\theta_r - \pi| < \varepsilon^\nu$ for some $r = 1 \dots j$. Indeed the light particle must escape from the cone $\mathcal{C}$ (see fig.4) to return (when looking bacwards in time) in the protection disk of β_i. This is clearly not possible if $\alpha_{jk} < \frac{\varepsilon^\nu}{2} \quad \forall k = i \dots j-1$ (see fig.4).

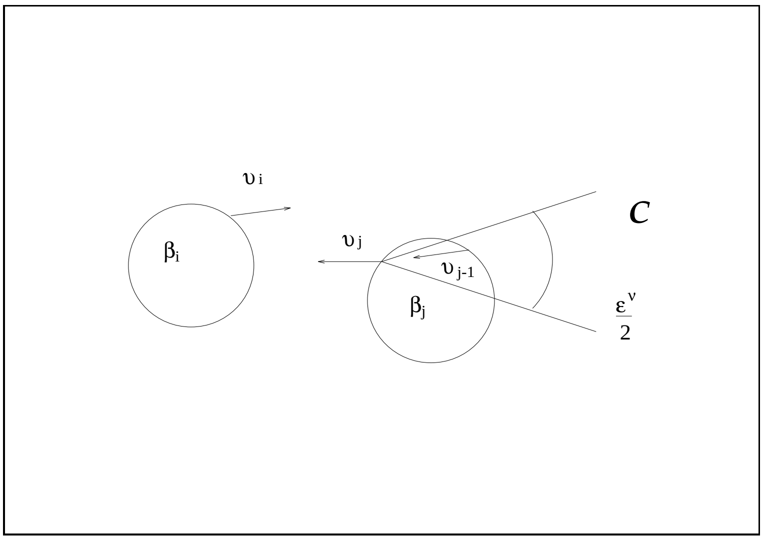

Fig. 4

Therefore,

$$I_\varepsilon^1 \le e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\ge 0} \mu_\varepsilon^Q \sum_{i=1}^{Q} \sum_{j=i+2}^{Q} \sum_{k=i}^{j-1} \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q$$

$$\int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_1 \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_2 \cdots \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_Q \, \chi(\{|\theta_k - \pi| < \varepsilon^\nu\}). \tag{3.53}$$

But

$$\int d\rho_k \, \chi(\{|\theta_k - \pi| < \varepsilon^\nu\}) = 2\,\varepsilon \int_{\pi-\varepsilon^\nu}^{\pi} d\theta \, B_{\varepsilon,\gamma}(\theta) \le C\,\varepsilon^{1+\nu}, \tag{3.54}$$

since away from $\theta = 0$, the cross section $B_{\varepsilon,\gamma}$ is bounded uniformly in ε (Cf. Appendix).

Then,

$$I_\varepsilon^1 \le e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \, C \sum_{Q\ge 1} \frac{(2\,t\,\mu_\varepsilon\,\varepsilon^\gamma)^Q}{Q!} Q^3 \varepsilon^{1+\nu-\gamma} \le C\,\varepsilon^{2\gamma+(1+\nu)-3}. \tag{3.55}$$

We now evaluate I_ε^2. We have

$$I_\varepsilon^2 \le e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\ge 0} \mu_\varepsilon^Q \sum_{i=1}^{Q} \sum_{j=i+2}^{Q} \sum_{k=i}^{j-1} \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q$$

$$\int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_1 \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_2 \cdots \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_Q \, \chi(\left\{ \inf_{\tau_j' \le s \le t_{j+1}} |\xi(s) - \beta_i| \le \varepsilon^\gamma \right\}$$

$$\times \chi(\left\{ \sin\alpha_{jk} \ge \frac{\varepsilon^\nu}{4} \right\}. \tag{3.56}$$

To estimate the above integrals, we fix all the variables $\{t_l\}_{l=1}^Q$ and $\{\rho_l\}_{l=1}^Q$, except t_{k+1}. Thus, everything is fixed but the length of the interval L described by the particle during its free flight between the k–th and $(k+1)$–th collision (see fig. 5).

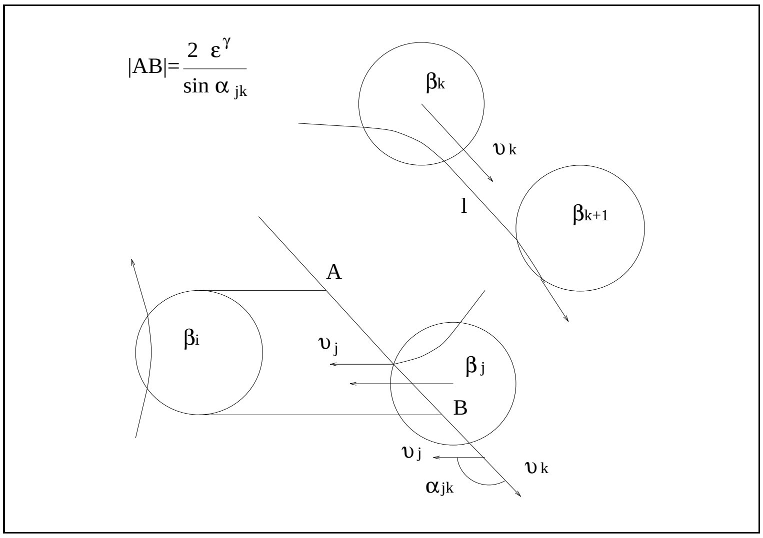

Fig. 5

By a simple geometrical argument, we argue that the integral over t_{k+1} is restricted over an interval of measure at most $8\,(\varepsilon^{\gamma-\nu})$. Then,

$$I_\varepsilon^2 \le e^{-2\,t\,\mu_\varepsilon\,\varepsilon^\gamma} \sum_{Q\ge 0} \frac{(2\,\varepsilon^\gamma \mu_\varepsilon)^Q}{(Q-1)!} Q^3\, t^{(Q-1)}\, 8\,\varepsilon^{\gamma-\nu} \le C(T)\,\varepsilon^{\gamma-\nu-4(1-\gamma)} \tag{3.57}$$

Optimizing ν in (3.55), (3.57) we finally obtain

$$I_\varepsilon \le C\,\varepsilon^{\frac{7\gamma-6}{2}}, \tag{3.58}$$

which concludes the proof. □

Remark 3.1: Note that if f_0 is only essentially bounded we can integrate against a test function and use Liouville's Theorem to get an estimate like eq. (2.7), but in the sense of L^1 weak only.

Remark 3.2: Note that only velocities of modulus 1 appear in eq. (2.6) so that $h_{\varepsilon,\gamma}$ a well as f_0 can be considered as an L^1 function on $\mathbb{R}^2 \times S^1$. On the contrary, for the approximated problems, f_ε and f_0 must be considered as functions defined on $\mathbb{R}^2 \times \mathbb{R}^2$.

Remark 3.3: We observe that eq. (2.8) is an evolution equation for the probability density associated to a particle performing random jumps in the velocity variable at random Markov times. On the contrary, the original system is Hamiltonian, the only stochasticity being that of the positions of the scatterers (and the initial state distributed according to f_0). The change of variables (3.24) outlines explicitely that the Poisson distribution of the scatterers induces a distribution of the instants and angles of collisions which, due to the recollisions, is neither independent nor Markov. The long tail memory is however lost in the limit. Notice, in addition, that the intensity of the process associated to eq. (2.6) (namely μ_ε), is diverging in the limit. This is indeed the effect of the grazing collisions. We control this from the point of view of the equation, asserting that the sequence $h_{\gamma,\varepsilon}$ converges to f in a suitable weak sense. It would be interesting to get a more detailed control from the point of view of the stochastic processes.

Remark 3.4: Comparing our proof with that in [G] we underline that, in our case, the range of the interaction is diverging in the limit. This forces us to give an explicit bound on the sets defined by (3.27), (3.28) and (3.29).

Remark 3.5: We underline that also in the linear case a notion of propagation of chaos can be established. Consider k independent test particles evolving in our random medium. Suppose also that the k particles are, at time zero, independently distributed according to the probability density $f_0^k(x_1, v_1 \dots x_k, v_k) = \prod_{i=1}^k f_0(x_i, v_i)$. Then the probability density at time t is defined by:

$$f_\varepsilon^k(t, x_1, v_1 \dots x_k, v_k) = \mathbb{E}^\varepsilon [f_0^k(T_{\mathbf{c},\varepsilon}^{-t}(x_1, v_1 \dots x_k, v_k))], \tag{3.59}$$

where $T_{\mathbf{c},\varepsilon}^{-t}$ is the Hamiltonian flow for the k-particle system. Even though the particles do not interact for a fixed ε, the quantity $f_\varepsilon^k(t)$ does not factorize any longer because two particles can interact with the same obstacle. However in the limit $\varepsilon \to 0$, such a factorization property (propagation of chaos) is recovered, thanks to our analysis. Indeed the same argument proving that recollisions are negligible also shows that the probability for a given particle to hit an obstacle visited by another particle is vanishing.

Appendix

Proposition A.1: *We denote by $B_{\varepsilon,\gamma}$ and B the cross sections relative to the potentials V_ε (see (1.2)) and $V(x) = |x|^{-s}$ respectively. Then, the following estimates hold for all $\gamma \in]0,1[$.*

$$\forall \theta' \in]-\pi, \pi[, \qquad \sup_{\delta<1} B_{\delta,\gamma}(\theta') \le C\,|\theta'|^{-1-1/s}, \tag{A.1}$$

$$\forall \theta \in]-\pi, \pi[, \qquad B(\theta) \le C\,|\theta|^{-1-1/s}. \tag{A.2}$$

Finally, B_ε converges towards B when $\delta \to 0$ for a.e. $\theta \in]-\pi, \pi[$.

Proof of proposition A.1: We recall the formula giving the scattering angle θ in term of the impact parameter ρ for a particle moving in the potential $V(r) = r^{-s}$,

$$\theta(\rho) = \pi - 2\int_{r_*}^{+\infty} \frac{\rho\, dr}{r^2\sqrt{1 - \frac{\rho^2}{r^2} - \frac{2}{r^s}}}, \qquad (A.3)$$

where

$$\frac{\rho^2}{r_*^2} + \frac{2}{r_*^s} = 1, \qquad (A.4)$$

and the corresponding formula when V is replaced by the cutoffed potential V_δ:

$$\theta'(\rho) = 2\int_{\rho}^{A} \frac{\rho\, dr}{r^2\sqrt{1 - \frac{\rho^2}{r^2}}} - 2\int_{r'_*}^{A} \frac{\rho\, dr}{r^2\sqrt{1 - \frac{\rho^2}{r^2} - (\frac{2}{r^s} - \frac{2}{A^s})}}, \qquad (A.5)$$

where

$$\frac{\rho^2}{r'^2_*} + (\frac{2}{r_*^s} - \frac{2}{A^s}) = 1, \qquad (A.6)$$

and $A = \delta^{\gamma-1} (A \geq 1)$.

Putting $u = \frac{\rho}{r}$ in (A.3) and $v = \frac{\rho}{r}$ in (A.5), we get

$$\theta = 2\int_0^{\frac{\pi}{2}} \left(1 - \frac{\sin\phi}{u + \frac{s}{\rho^s}\, u^{s-1}}\right) d\phi, \qquad (A.7)$$

after the following change of variables

$$u^2 + 2\,(\frac{u}{\rho})^s = \sin^2\phi. \qquad (A.8)$$

Analogously:

$$\theta' = 2\int_{\arcsin(\frac{\rho}{A})}^{\frac{\pi}{2}} \left(1 - \frac{\sin\phi}{v + \frac{s}{\rho^s}\, v^{s-1}}\right) d\phi, \qquad (A.9)$$

where:

$$v^2 + 2\left((\frac{v}{\rho})^s - (\frac{1}{A})^s\right) = \sin^2\phi. \qquad (A.10)$$

Then, we compute

$$\frac{d\theta}{d\rho} = \frac{2}{\rho^{s+1}}\int_0^{\frac{\pi}{2}} \frac{\sin\phi\, s\, u^{s-1}}{(u + \frac{s}{\rho^s}\, u^{s-1})^2}\left\{(s-1)\,\frac{\frac{1}{s-1} + s\,\frac{u^{s-2}}{\rho^s}}{1 + s\,\frac{u^{s-2}}{\rho^s}} - s\right\} d\phi, \quad (A.11)$$

and (for $|\rho| \leq A$),

$$\frac{d\theta'}{d\rho} = \frac{2}{\rho^{s+1}}\int_{\arcsin(\frac{\rho}{A})}^{\frac{\pi}{2}} \frac{\sin\phi\, s\, v^{s-1}}{(v + \frac{s}{\rho^s}\, v^{s-1})^2}\left\{(s-1)\,\frac{\frac{1}{s-1} + s\,\frac{v^{s-2}}{\rho^s}}{1 + s\,\frac{v^{s-2}}{\rho^s}} - s\right\} d\phi$$

$$-2\left\{1-\frac{1}{1+\frac{s}{\rho^s}\,(\frac{\rho}{A})^{s-2}}\right\}, \qquad (A.12)$$

so that

$$|\frac{d\theta'}{d\rho}| = \frac{2}{\rho^{s+1}}\int_{\arcsin(\frac{\rho}{A})}^{\frac{\pi}{2}} \frac{\sin\phi\, s\, v^{s-1}}{(v+\frac{s}{\rho^s}\,v^{s-1})^2}\left\{s-(s-1)\,\frac{\frac{1}{s-1}+s\,\frac{v^{s-2}}{\rho^s}}{1+s\,\frac{v^{s-2}}{\rho^s}}\right\}d\phi$$

$$+\frac{\frac{2\,s}{\rho^s}\,(\frac{\rho}{A})^{s-2}}{1+\frac{s}{\rho^s}\,(\frac{\rho}{A})^{s-2}}. \qquad (A.13)$$

It is clear (using for example Lebesgue's theorem of dominated convergence) that $\frac{d\theta'}{d\rho}$ converges (for any ρ) towards $\frac{d\theta}{d\rho}$ when $A\to+\infty$ (note that $\rho\le A$ when A is large enough).

Note now that if $\rho\ge\frac{1}{2}\,A$, then

$$|\frac{d\theta'}{d\rho}| \ge \frac{2\,s^{\,s}}{\rho}\,(\frac{1}{2})^{s-2}\,\frac{1}{1+2^s\,s\,2^{2-s}} \ge C\,\rho^{-s}. \qquad (A.14)$$

On the other hand, if $\rho\le\frac{1}{2}\,A$, then

$$|\frac{d\theta'}{d\rho}| \ge \frac{2}{\rho^{s+1}}\int_{\arcsin(\frac{1}{2})}^{\frac{\pi}{2}} \frac{\sin\phi\, s\, v^{s-1}}{(v+\frac{s}{\rho^s}\,v^{s-1})^2}\,d\phi. \qquad (A.15)$$

Using the estimates

$$v+\frac{s}{\rho^s}\,v^{s-1}\le\frac{2\,s}{v}, \qquad (A.16)$$

and

$$v\ge\inf(1/2, 8^{-1/s}\,\rho), \qquad (A.17)$$

we get

$$|\frac{d\theta'}{d\rho}| \ge \inf(\frac{1}{4\,s}(\frac{1}{2\,\rho})^{s+1}, \frac{1}{4\,s}\,8^{-1-1/s}). \qquad (A.18)$$

Finally, we have for all $\rho\le A$,

$$|\frac{d\theta'}{d\rho}| \ge \frac{C}{1+\rho^{s+1}}, \qquad (A.19)$$

and the same estimate holds for $\frac{d\theta}{d\rho}$ by passage to the limit.

Moreover, observing that $v\le\sin\phi$, we see that

$$\theta' \le \frac{2\,s}{\rho^s}\int_{\arcsin(\frac{\rho}{A})}^{\frac{\pi}{2}} \frac{v^{s-1}}{v+\frac{s}{\rho^s}\,v^{s-1}}\,d\phi$$

$$\le\frac{C}{1+\rho^s}. \qquad (A.20)$$

Combining (A.19) and (A.20), we get

$$B_{\delta,\gamma}(\theta') = \frac{d\rho}{d\theta'}(\theta') \le C\,|\theta'|^{-1-1/s}, \tag{A.21}$$

and the same estimate holds for $B(\theta)$.

We also know that $B_{\delta,\gamma}$ tends towards B everywhere because its inverse mapping converges everywhere. □

Proposition A.2: *Let $s > 2$, $\gamma \in]0,1[$, and $f_0 \in L^\infty(\mathbb{R}^2_x \times S^1_v)$. Then the solution $h_{\delta,\gamma}$ of eq. (2.9) converges (up to extraction of a subsequence) in L^∞ weak $*$ towards a solution f of eq. (2.12) in the following weak sense: for any $\phi \in \mathcal{D}([0,T[\times\mathbb{R}^2_x \times S^1_v)$,*

$$-\int_0^T \int_{\mathbb{R}^2}\int_{S^1} f(t,x,v)\left\{\partial_t\phi + v\cdot\nabla_x\phi\right\}dv\,dx\,dt - \int_{\mathbb{R}^2}\int_{S^1} f_0(x,v)\,\phi(0,x,v)\,dv\,dx$$

$$= \int_0^T \int_{\mathbb{R}^2}\int_{S^1} f(t,x,v)\int_{-\pi}^{\pi}(\phi(t,x,R_\theta(v)) - \phi(t,x,v))\,B(\theta)\,d\theta\,dv\,dx\,dt. \tag{A.22}$$

Proof of proposition A.2: According to the maximum principle, the sequence $h_{\delta,\gamma}$ converges (up to extraction of a subsequence) in $L^\infty([0,T]\times\mathbb{R}^2\times S^1)$ weak $*$ towards a function f, so that the left–hand side of eq. (2.6) (integrated against the test function ϕ) converges to the left–hand side of eq. (A.22).

Using estimates (A.1), (A.2) and the convergence a.e. of $B_{\delta,\gamma}$ towards B, we get the convergence in $L^1([0,T]\times\mathbb{R}^2\times S^1)$ (strong) of

$$a_\delta(t,x,v) = \int_{-\pi}^{\pi}(\phi(t,x,R_\theta(v)) - \phi(t,x,v))\,B_{\delta,\gamma}(\theta)\,d\theta \tag{A.23}$$

towards

$$a(t,x,v) = \int_{-\pi}^{\pi}(\phi(t,x,R_\theta(v)) - \phi(t,x,v))\,B(\theta)\,d\theta. \tag{A.24}$$

Finally, the right–hand side of eq. (2.6) (integrated against the test function ϕ) converges to the right–hand side of eq. (A.22). □

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