ABOUT THE USE OF THE FOURIER TRANSFORM FOR THE BOLTZMANN EQUATION

Laurent Desvillettes
Ecole Normale Supérieure de Cachan
Centre de Mathématiques et Leurs Applications
61, avenue du Président Wilson, 94235 Cachan Cedex
e-mail: desville@cmla.ens-cachan.fr

March 14, 2003

Contents

Abstract

We propose here a survey of the results for the Boltzmann equation which use the Fourier transform. In particular, we introduce various versions of the averaging lemmas, of the properties of smoothness of Boltzmann's kernel, and various other computations.

1 Introduction

1.1 Kinetic equations

We usually denote by "kinetic equations" those equations in which the unknown is the phase space density

$$f(t,x,v) \geq 0 \tag{1}$$

of particles which at time t and point x move with velocity v.

Such a modeling is in some situations an alternative to the study of equations (such as the Navier-Stokes system) in which the unknowns (such as the usual density ρ, the mean velocity u or the internal energy e) only depend on t and x.

The phase space density (1) typically verifies an equation of the form

$$\partial_t f + v \cdot \nabla_x f = R, \tag{2}$$

where R often depends on f. The reason for that is that when there is no interaction between the particles and their surrounding environment (including themselves), they will move at a constant velocity and along straight lines. In other words, for all times t and τ, point x and velocity v, a particle which at time t sits at point x and move with velocity v will sit at time $t+\tau$ at point $x + v\tau$ and will keep its velocity v. This entails that

$$\forall \tau, \qquad f(t+\tau, x+v\tau, v) = f(t,x,v), \tag{3}$$

or, after differentiation with respect to τ,

$$\partial_t f + v \cdot \nabla_x f = 0. \tag{4}$$

Then, the left-hand side R appears as the contribution of the environment on the motion of the particles.

Note that formulas like (2) are typical of classical mechanics. When relativistic or quantum effects must be taken into account, the variable v is replaced by the momentum p or the wave vector k, and equation (2) becomes

$$\partial_t f + v(p) \cdot \nabla_x f = R \tag{5}$$

or

$$\partial_t f + v(k) \cdot \nabla_x f = R \tag{6}$$

Those are still considered as kinetic equations, as long as the function v is not constant on some substantial part of the domain of variation of p or k (this is of course always true in the relativistic context, and in most of the other situations).

Equations like (5) are also typical of the kinetic formulations of conservation laws.

Note finally that in many situations (e.g. in the study of radiative transfer and in the study of realistic gases, or in the modeling of sprays), the density f also depends on extra variables (such as the frequency ν of the photons for the radiative transfer, the internal energy I of diatomic gases, the size r, the temperature θ and even sometimes the eccentricity y of the droplets for the sprays).

The behavior of the solutions of eq. (2) strongly depends on the form of the term R.

When a given force $F(t,x)$ acts on the particles (such a force can also depend on v in specific situations, for example when the particles are charged and feel the action of a magnetic field, or when the force is the drag force due to a surrounding gas), the particles will follow the trajectories of the following system of differential equations :

$$\dot{x}(t) = v(t), \tag{7}$$

$$\dot{v}(t) = F(t, x(t)), \tag{8}$$

and the corresponding partial differential equation satisfied by f (that is, the PDE whose characteristic curves are exactly the solutions of eq. (7), (8)), is the Vlasov equation

$$\partial_t f + v \cdot \nabla_x f + F(t,x) \cdot \nabla_v f = 0. \tag{9}$$

In many cases, the force F is itself related to f (through Poisson's or Maxwell's equations for example). That leads to the classical Vlasov-Poisson or Vlasov-Maxwell systems.

The equations we wish to investigate in this document are of a different type. We describe them in the sequel.

1.2 The Boltzmann equation

When the forces acting on the particles are mainly due to the collisions of the particles between themselves, one is led to write down the Boltzmann equation. This equation is valid when one is interested in a situation where the typical dimension of the physical objects under study are of the same order as the mean free path of the particles (that is, the length of the trajectory of a typical particle between two collisions). When this is the case, the gas is said to be rarefied. For gases which are not rarefied, one has to use the equations of fluid mechanics (such as the compressible or incompressible Euler or Navier-Stokes systems).

Many features of the Boltzmann equation are related to the hypothesis that the gas is rarefied. In particular, this assumption implies that the collisions are binary (that is, the ternary, etc., collisions are neglected), they are localized in time and space (that is, the size of the region in which the velocities of the particles vary is small in front of the size of the objects under study), and no correlations occur between the velocities of the particles (that is, roughly speaking, collisions do not occur very often, so that the probability for a particle to encounter a particle which has already interacted with it (through other particles) is negligeable).

Starting from the general form (2) of kinetic equations, we see thanks to the property of locality in space and time that

$$R(t, x, v) = R(f(t, x, \cdot))(v).$$

It is therefore sufficient to define the effect of R on a function f depending on v only.

We denote by $f_2(v_1, v_2)$ the joint density of two particles with respective velocities v_1 and v_2. We see (thanks to the assumption that the collisions are binary) that we must take into account only two distincts phenomena which modify the number density of particles with velocity v.

First, because of a possible collision with a particle of velocity v_*, a particle which had v for velocity will end up with a velocity v' (its partner in the collision will end up with velocity v'_*).

Secondly, some particle with a velocity w will encounter a particle with velocity w_* and will end up with a velocity v after the collision (its partner in the collision will end up with velocity w'_*).

We now denote by $p(v_1, v_2 \to v_3, v_4)$ the (density of) probability that for two particles sitting at the same point x at a given time t, a collision occurs and transforms the ingoing velocities v_1 and v_2 in the outgoing velocities v_3,

v_4 (we shall see that in the so-called non cutoff case, this quantity is in fact far from being a probability density, since it is not integrable).

We see that $R(f)$ is the sum of two terms $-R^-(f)$ and $R^+(f)$ which respectively correspond to the two phenomena described above.

According to their definition, R^- and R^+ write

$$R^-(f)(v) = \int_{v_*}\int_{v'}\int_{v'_*} f_2(v, v_*)\, p(v, v_* \to v', v'_*)\, dv'_* dv' dv_*,$$

and

$$R^+(f)(v) = \int_w \int_{w_*}\int_{w'_*} f_2(w, w_*)\, p(w, w_* \to v, w'_*)\, dw'_* dw_* dw.$$

According to the hypothesis that no correlations occur, we can replace in the previous formula $f_2(v, v_*)$ by $f(v)\, f(v_*)$ and $f_2(w, w_*)$ by $f(w)\, f(w_*)$. Then, R is clearly quadratic as a function of f. As a consequence, we shall from now on denote it by $Q(f, f)$, and we obtain the formulas

$$Q(f, f) = Q^+(f, f) - Q^-(f, f),$$

with

$$Q^-(f, f)(v) = \int_{v_*}\int_{v'}\int_{v'_*} f(v)\, f(v_*)\, p(v, v_* \to v', v'_*)\, dv'_* dv' dv_*,$$

and

$$Q^+(f, f)(v) = \int_w \int_{w_*}\int_{w'_*} f(w)\, f(w_*)\, p(w, w_* \to v, w'_*)\, dw'_* dw_* dw.$$

We now introduce the microreversibility assumption

$$\forall v_1, v_2, v_3, v_4, \qquad p(v_1, v_2 \to v_3, v_4) = p(v_3, v_4 \to v_1, v_2).$$

This assumption is justified by the fact that the motion of two interacting particles is modeled by ordinary differential equations which are reversible.

We get the formula

$$Q(f, f)(v) = \int_{v_*}\int_{v'}\int_{v'_*} \Big(f(v')\, f(v'_*) - f(v)\, f(v_*) \Big)\, p(v, v_* \to v', v'_*)\, dv'_* dv' dv_*.$$

Then, we use the conservation of momentum and kinetic energy in a collision :

$$v + v_* = v' + v'_*, \tag{10}$$

$$\frac{|v|^2}{2} + \frac{|v_*|^2}{2} = \frac{|v'|^2}{2} + \frac{|v'_*|^2}{2}. \tag{11}$$

Note that the conservation of kinetic energy holds only in the case of monoatomic gases. For gases of the real atmosphere such as diatomic nitrogene N_2 and diatomic oxygene O_2, only the conservation of the total energy holds : one has to introduce various kinds of internal energy (vibration, rotation) in order to get a realistic modeling.

As a consequence, the measure p is concentrated on the set defined by identities (10) and (11). At this point, it is useful to parametrize those equations.

When we are interested in a two-dimensional situation, the best way to parametrize seems to use the center of mass reference frame, that is, the frame moving with velocity $\frac{v+v_*}{2}$. Then, the conservation of energy simply becomes

$$|v - v_*|^2 = |v' - v'_*|^2.$$

Finally, v' and v'_* are defined by

$$v' = \frac{v+v_*}{2} + R_\theta\Big(\frac{v-v_*}{2}\Big),$$

$$v'_* = \frac{v+v_*}{2} - R_\theta\Big(\frac{v-v_*}{2}\Big),$$

where R_θ is the rotation of angle θ.

The situation is not so good in dimension N equal or bigger than three. Then, two different parametrizations are traditionally used. The first one uses symmetries, and has the advantage of being with respect to v, v_*. It writes

$$v' = v + ((v_* - v)\cdot\omega)\,\omega,$$

$$v'_* = v - ((v_* - v)\cdot\omega)\,\omega,$$

with ω varying in the sphere (or half sphere) S^{N-1}.

We shall however rather use the parametrization which uses the center of mass reference frame, and which writes

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\,\sigma, \tag{12}$$

$$v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\,\sigma, \tag{13}$$

with σ varying in the sphere S^{N-1}.

Note that σ and ω are related by a simple change of variables (Cf. [19] for example to get a precise formula for the corresponding Jacobian).

The Galilean invariance which holds in the context of binary collisions entails that the measure $p(v, v_* \to v', v'_*)$ can only depend on $|v - v_*|$ and $\frac{v-v_*}{|v-v_*|} \cdot \sigma$ (or $\left|\frac{v-v_*}{|v-v_*|} \cdot \omega\right|$, or even $|\theta|$ in dimension 2).

We now can write down the "final" form of Boltzmann's collision operator :

$$Q(f,f)(v) = \int_{v_* \in \mathbb{R}^N} \int_{\sigma \in S^{N-1}} \Big(f(v')\, f(v'_*) - f(v)\, f(v_*) \Big)$$
$$\times B\bigg(|v - v_*|, \frac{v - v_*}{|v - v_*|} \cdot \sigma \bigg)\, d\sigma dv_*, \tag{14}$$

where B is called the cross section (sometimes a slightly different definition of the cross section is presented, namely $B/|v - v_*|$), and v', v'_* are given by formulas (12), (13).

We shall also use the bilinear form $Q(g,f)$ related to the quadratic form $Q(f,f)$, and defined by

$$Q(g,f)(v) = \int_{v_* \in \mathbb{R}^N} \int_{\sigma \in S^{N-1}} \Big(f(v')\, g(v'_*) - f(v)\, g(v_*) \Big)$$
$$\times B\bigg(|v - v_*|, \frac{v - v_*}{|v - v_*|} \cdot \sigma \bigg)\, d\sigma dv_*. \tag{15}$$

Finally, we write down the standard form of the Boltzmann equation :

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \tag{16}$$

where Q is given by (14).

For a general exposition of the theory of the Boltzmann equation, we refer to [23], [25] and [70].

The rigorous derivation of the Boltzmann equation starting from the dynamics of N particles in interaction is performed in [53] and [22] in the context of local (in time) solutions or of global (in time) solutions close to vacuum.

1.3 Cross sections

It is possible to (almost) explicitly compute the cross section B when the interparticle force is proportional to r^{-s} (with r denoting the interparticle distance and $s > 2$). In such a case (and in dimension 3), B writes (with $\cos\theta = \frac{v-v_*}{|v-v_*|} \cdot \sigma$) :

$$B\left(|v-v_*|, \cos\theta\right) = |v-v_*|^{\frac{s-5}{s-1}}\, b(\cos\theta),$$

with b a smooth function except at point 1 and

$$\sin\theta\, b(\cos\theta) \sim_{\theta\to 0} \frac{K}{|\theta|^{\frac{s+1}{s-1}}}, \tag{17}$$

with $K > 0$.

Since $\frac{s+1}{s-1} > 1$, the singularity in the angular variable θ is always non integrable. Because of the difficulties entailed by this singularity, Grad has proposed to introduce an angular cutoff near $\theta = 0$ (Cf. [42]). It means that we replace B by a new cross section

$$\tilde{B}\left(|v-v_*|, \cos\theta\right) = |v-v_*|^{\frac{s-5}{s-1}}\, \tilde{b}(\cos\theta),$$

with b smooth, or at least such that $\theta \mapsto \sin\theta\, \tilde{b}(\cos\theta)$ is integrable near $\theta = 0$.

In the sequel, we shall speak of cutoff cross sections (or cutoff potentials) when B is locally integrable, and of non cutoff cross sections (or non cutoff potentials) when B has a singularity like in (17).

Note that the decomposition $Q = Q^+ - Q^-$, with

$$Q^+(f,f)(v) = \int_{v_*\in\mathbb{R}^N} \int_{\sigma\in S^{N-1}} f(v')\, f(v'_*) B\Big(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\, d\sigma dv_*, \tag{18}$$

$$Q^-(f,f)(v) = \int_{v_*\in\mathbb{R}^N} \int_{\sigma\in S^{N-1}} f(v)\, f(v_*)\, B\Big(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\, d\sigma dv_*, \tag{19}$$

holds only when the cross section B is integrable (that is, cutoff).

We shall also speak of hard potentials when $B \to +\infty$ as its first variable tends to infinity, of soft potentials when $B \to 0$ as its first variable tends to infinity, and of Maxwellian molecules when B does not depend on the first variable (what we shall call cutoff Maxwellian molecules in the sequel is sometimes called pseudo Maxwellian molecules).

Finally, note that the case when $s = 2$ (that is, the Coulomb potential), leads to a different equation, namely the Fokker-Planck-Landau equation.

1.4 Basic properties of Boltzmann's kernel

We shall systematically use in the sequel the so-called pre/post collisional change of variables $(v, v_*, \sigma) \mapsto (v', v'_*, \sigma)$ which ensures that for all functions $f \equiv f(v, v_*, v', v'_*, \sigma)$, one has (at the formal level) :

$$\int_{I\!R^N}\int_{I\!R^N}\int_{S^{N-1}} f(v, v_*, v', v'_*, \sigma)\, d\sigma dv_* dv$$
$$= \int_{I\!R^N}\int_{I\!R^N}\int_{S^{N-1}} f(v', v'_*, v, v_*, \sigma)\, d\sigma dv_* dv.$$

This formula is obvious when one uses the parametrization with θ in dimension two (or, in fact, the parametrization with ω in higher dimension). The proof can be found for example in [19].

We shall also use the change of variables $(v, v_*, \sigma) \mapsto (v_*, v, \sigma)$, which ensures that for all function $f \equiv f(v, v_*, v', v'_*, \sigma)$, one has (still at the formal level)

$$\int_{I\!R^N}\int_{I\!R^N}\int_{S^{N-1}} f(v, v_*, v', v'_*, \sigma)\, d\sigma dv_* dv$$
$$= \int_{I\!R^N}\int_{I\!R^N}\int_{S^{N-1}} f(v_*, v, v'_*, v', \sigma)\, d\sigma dv_* dv.$$

As an immediate consequence of those formulas, we get the following various weak formulations for Boltzmann's kernel Q :

$$\int_{I\!R^N} Q(f,f)(v)\,\phi(v)\,dv = \int_{I\!R^N}\int_{I\!R^N}\int_{S^{N-1}} \Big(\phi(v') - \phi(v)\Big) \times f(v)\,f(v_*)\,B\,d\sigma dv_* dv, \tag{20}$$

$$\int_{I\!R^N} Q(f,f)(v)\,\phi(v)\,dv = \frac{1}{2}\int_{I\!R^N}\int_{I\!R^N}\int_{S^{N-1}} \Big(\phi(v'_*) + \phi(v') - \phi(v') - \phi(v)\Big) \times f(v)\,f(v_*)\,B\,d\sigma dv_* dv, \tag{21}$$

$$\int_{I\!R^N} Q(f,f)(v)\,\phi(v)\,dv = -\frac{1}{4}\int_{I\!R^N}\int_{I\!R^N}\int_{S^{N-1}} \Big(\phi(v'_*) + \phi(v') - \phi(v') - \phi(v)\Big) \times \Big(f(v')\,f(v'_*) - f(v)\,f(v_*)\Big)\,B\,d\sigma dv_* dv. \tag{22}$$

Plugging $\phi(v) = 1, v_i, \frac{|v|^2}{2}$ in formula (21), we get the conservation of mass, momentum and energy at the level of the Boltzmann operator :

$$\int_{I\!\!R^N} Q(f,f)(v) \begin{pmatrix} 1 \\ v_i \\ \frac{|v|^2}{2} \end{pmatrix} dv = 0. \tag{23}$$

Boltzmann's H-theorem is obtained by plugging $\phi = \log f$ in (22).

Defining the entropy dissipation by

$$D(f) = -\int_{I\!\!R^N} Q(f,f)(v) \log f(v)\, dv$$

we get

$$D(f) = \frac{1}{4} \int_{I\!\!R^N} \int_{I\!\!R^N} \int_{S^{N-1}} \Big(f(v')\, f(v'_*) - f(v)\, f(v_*) \Big) \tag{24}$$
$$\times \log \left(\frac{f(v')\, f(v'_*)}{f(v)\, f(v_*)} \right) B\, d\sigma dv_* dv,$$

we observe (this is the first part of Boltzmann's H-theorem) that $D(f) \geq 0$.

Then, it is possible to prove (under suitable, but rather weak assumptions on B and f) that

$$D(f) = 0 \qquad \Longleftrightarrow \qquad \forall v \in I\!\!R^N, \quad Q(f,f)(v) = 0$$

$$\Longleftrightarrow \qquad \exists \rho \geq 0, T > 0, u \in I\!\!R^N, \quad f(v) = \frac{\rho}{(2\pi T)^N} \exp\left(-\frac{|v-u|^2}{2T} \right).$$

This is the second part of Boltzmann's H-theorem (Cf. [70]).

1.5 A priori estimates

Since this work is more concerned with the qualitative properties of the solutions of Boltzmann's equation than with the existence theory, we shall only state some basic a priori estimates related to the conservation properties of the previous section, and only one theorem of existence.

We first introduce the Cauchy problem for the spatially homogneous Boltzmann equation. That consists in looking for solutions to the full Boltzmann equation (16) which only depend on the variables t and v, a compatible initial datum being given.

In other words, it writes

$$\partial_t f(t,v) = Q(f,f)(t,v), \tag{25}$$

$$f(0,\cdot) = f_{in}. \tag{26}$$

According to the results of the previous subsection, the solutions of this equation (at least formally) satisfy the conservation of mass, momentum and energy

$$\forall t \geq 0, \qquad \int_{v\in\mathbb{R}^N} f(t,v) \begin{pmatrix} 1 \\ v_i \\ \frac{|v|^2}{2} \end{pmatrix} dv = \int_{v\in\mathbb{R}^N} f_{in}(v) \begin{pmatrix} 1 \\ v_i \\ \frac{|v|^2}{2} \end{pmatrix} dv, \tag{27}$$

and the decay of the entropy (defined by $\int f \log f \, dv$ and not by $-\int f \log f \, dv$ as in physics)

$$\forall t \geq 0, \qquad \int_{v\in\mathbb{R}^N} f(t,v) \log f(t,v)\, dv + \int_0^t D(f)(s)\, ds$$
$$\leq \int_{v\in\mathbb{R}^N} f_{in}(v) \log f_{in}(v) dv. \tag{28}$$

Then, it is easy to show (still at the formal level) that as soon as the initial datum has finite mass, energy and entropy (in the two next formulas, $f \log f$ is replaced by $f\,|\log f|$, so that only nonnegative quantities are considered : this does not lead to any difficulties), that is when

$$K_{in} = \int_{v\in\mathbb{R}^N} f_{in}(v)\,(1+|v|^2+|\log f_{in}(v)|)\, dv < +\infty, \tag{29}$$

there exists for all $T>0$ a constant $C_T>0$ (only depending on K_{in}) such that

$$\sup_{t\in[0,T]} \int_{v\in\mathbb{R}^N} f(t,v)\,(1+|v|^2+|\log f(t,v)|)\, dv + \int_0^T D(f)(s)\, ds \leq C_T. \tag{30}$$

In the sequel, we shall use the following (now classical) theorem of existence, proven in [8], [9] and [40]:

Theorem 1 *Let B be a (nonnegative) cross section satisfying (for $x \in \mathbb{R}$ and $\theta \in [0,\pi]$),*

$$\sin\theta\, B(x,\cos\theta) \leq K\,(1+|x|)\,|\theta|^{-1-\gamma},$$

for some $K > 0$ and $\gamma < 2$ (that is, cutoff or non cutoff hard potentials or Maxwellian molecules).

Let also f_{in} be a (nonnegative) measurable function from $\mathbb{R}^N$ to $\mathbb{R}$ such that $K_{in} < +\infty$ (K_{in} is defined by (29)).

Then, there exists a solution $f \equiv f(t,v)$ lying in $L^\infty(\mathbb{R}_+; L^1_2(\mathbb{R}^N))$ and $C(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}^N))$ to eq. (25) written in the weak form (Cf. eq. (21)) for all test functions $\phi \in \mathcal{D}(\mathbb{R}^N)$,

$$\partial_t \int_{\mathbb{R}^N} f(t,v)\, \phi(v) dv = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} \Big(\phi(v'_*) + \phi(v')$$

$$-\phi(v_*) - \phi(v)\Big)\, f(t,v)\, f(t,v_*)\, B\, d\sigma dv_* dv.$$

This solution can be constructed in such a way that the conservations of mass, momentum and energy and the decrease of the entropy hold.

1.6 Simplified models

In the sequel, we shall be led to consider various simplifications of Boltzmann's kernel, which we now describe.

The first one is the so-called Kac's operator (Cf. [47]). It acts on functions of a one-dimensional variable ($v \in \mathbb{R}$) and writes

$$Q(f,f)(v) = \int_{\mathbb{R}} \int_0^{2\pi} \Big(f(v\, \cos\theta - v_*\, \sin\theta)\, f(v\, \sin\theta + v_*\, \cos\theta)$$

$$-f(v)\, f(v_*)\Big) \beta(|\theta|)\, d\theta dv_* \tag{31}$$

for some nonnegative cross section β. We shall conserve for this model the terminology on cross sections that we adopted for the Boltzmann equation. That is, it is said to be cutoff if β is integrable, and non cutoff if $\beta(\theta) \sim_{\theta\to 0} |\theta|^{-1-\gamma}$, for $\gamma \in]0,2[$.

Mass and energy, but not momentum, are conserved for this operator. The H theorem is also valid except that in the second part of the theorem, the set of all Maxwellians must be replaced by the set of centered Maxwellians. As we shall see in the sequel, this operator is very close to the Boltzmann operator for Maxwellian molecules when it is restricted to the radially symmetric functions.

The second model that we shall introduce is even simpler. It acts on functions of a periodic variable ($v \in T^1$), and writes

$$Q(f,f)(v) = \int_{-1/2}^{1/2} \int_{\mathbb{T}^1} [f(v+\theta)\, f(v'-\theta) - f(v)\, f(v')]\, \beta(|\theta|)\, d\theta dv'. \quad (32)$$

This operator is close to a linear operator in the sense that (at the formal level)

$$Q(f,f)(v) = \int_{\mathbb{T}^1} f(w)\, dw \int_{-1/2}^{1/2} \Big(f(v+\theta) - f(v) \Big) \beta(|\theta|)\, d\theta. \quad (33)$$

It is associated to a spatially inhomogeneous equation which writes

$$\partial_t f(t,x,v) + \cos(2\pi v)\, \partial_x f(t,x,v) = Q(f,f)(t,x,v), \quad (34)$$

where the unknown is the number density $f \equiv f(t,x,v)$. Here, $t \geq 0$ is the time variable, the position variable is $x \in \mathbb{T}^1$, and $v \in \mathbb{T}^1$ parametrizes the velocity $\cos(2\pi v)$ of the particles. This model was introduced in [30].

Finally we introduce the classical linear Fokker-Planck operator

$$Q(f)(v) = \nabla \cdot (\nabla f + v\, f),$$

and the corresponding (confined) linear Vlasov-Fokker-Planck equation (sometimes also called kinetic Fokker-Planck equation)

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = \nabla_v \cdot (\nabla_v f + v\, f), \quad (35)$$

where V is the confining potential. Here x and v vary in $\mathbb{R}^N$, and the equation models the motion of a particle in a thermal bath.

1.7 The Fourier transform in the context of the Boltzmann equation

For a given function $f : \mathbb{R}^N \to \mathbb{R}$, we define its Fourier transform $\hat{f}$ (sometimes also denoted by $\mathcal{F}f$) by the formula

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi}\, f(x)\, dx.$$

With this definition, the inversion formula writes

$$f(x) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{ix\cdot\xi}\, \hat{f}(\xi)\, d\xi,$$

and Plancherel's formula becomes

$$\int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 \, d\xi = (2\pi)^N \int_{\mathbb{R}^N} |f(x)|^2 \, dx.$$

We shall also use the relationship between derivatives and moments. Denoting by α a multiindex of $\mathbb{N}^N$, we have

$$\widehat{\partial_\alpha f}(\xi) = (i\xi)^\alpha \, \hat{f}(\xi),$$

and

$$(\widehat{-ix})^\alpha f(\xi) = \partial_\alpha \hat{f}(\xi).$$

In the sequel, we shall use the Fourier transform with respect to various variables (t and x, x and v, v only, etc.). We shall therefore systematically recall which variables are concerned and what are the name of the corresponding Fourier variables.

Like for other PDEs, the Fourier transform is useful in many ways in the context of the Boltzmann equation. For example, it enables to obtain explicit solutions in some situations (typically, in the case of Maxwellian molecules, which somehow plays a role in the theory of the Boltzmann equation analogous to that played in the theory of PDEs by the linear equations with constant coefficients, Cf. [15] and [16]). It is also extremely useful for the study of the smoothness of the solutions, as we shall see repeatedly in the sequel.

We recall that the large $|\xi|$ behavior of $\hat{f}(\xi)$ is related to the smoothness of f. This link is best seen in the context of Sobolev spaces based on L^2. Precisely, for all $s \in \mathbb{N}$, the norms

$$\left(\sum_{|\alpha| \le s} \int_{\mathbb{R}^N} |\partial_\alpha f(x)|^2 \, dx \right)^{1/2}$$

and

$$\left(\int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 \, (1 + |\xi|^2)^s \, d\xi \right)^{1/2}$$

are equivalent and define the same space $H^s(\mathbb{R}^N)$.

So are the norms

$$\left(\int_{\mathbb{R}^N} |f(x)|^2 \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^2}{|x-y|^{N+2s}} \, dy dx \right)^{1/2}$$

and

$$\left(\int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 \, (1 + |\xi|^2)^s \, d\xi \right)^{1/2},$$

for the space $H^s(\mathbb{R}^N)$ with $s \in]0, 1[$.

1.8 Some notations for spaces of functions

In addition to the norms of H^s defined above, that is

$$||f||_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 \, (1+|\xi|^2)^s \, d\xi \right)^{1/2},$$

we introduce for $0 < s < N/2$ the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^N)$ of functions f of $L^{2N/(N-2s)}(\mathbb{R}^N)$ such that $\widehat{f} \in L^1_{loc}(\mathbb{R}^N)$ and $|\xi|^s \widehat{f}(\xi) \in L^2(\mathbb{R}^N_\xi)$. Its norm is given by

$$||f||_{\dot{H}^s(\mathbb{R}^N)} = \left(\int_{\xi \in \mathbb{R}^N} |\widehat{f}(\xi)|^2 |\xi|^{2s} \, d\xi \right)^{1/2}. \tag{36}$$

We shall also use for $p \geq 1$, $q \geq 0$, the weighted space $L^p_q(\mathbb{R}^N)$ embedded with the norm

$$||f||_{L^p_q(\mathbb{R}^N)} = \left(\int_{v \in \mathbb{R}^N} |f(v)|^p (1+|v|)^{pq} \, dv \right)^{1/p}, \tag{37}$$

and for $k \in \mathbb{N}$ the Sobolev spaces

$$W^{k,\infty}(\mathbb{R}^N) = \left\{ f \in L^\infty(\mathbb{R}^N), \quad \forall \alpha \in \mathbb{N}^N, |\alpha| \leq k, ||\partial_\alpha f||_{L^\infty(\mathbb{R}^N)} < +\infty \right\},$$

embedded with the norm

$$||f||_{W^{k,\infty}(\mathbb{R}^N)} = \sum_{|\alpha| \leq k} ||\partial_\alpha f||_{L^\infty(\mathbb{R}^N)}.$$

2 Averaging Lemmas

2.1 Introduction

Averaging lemmas are designed for the study of the regularity of the solutions of kinetic (transport) equations of type

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = g(t,x,v) \tag{38}$$

or of the (space independant) type

$$v \cdot \nabla_x f(x,v) = g(x,v). \tag{39}$$

Because of the hyperbolicity of the operators $v \cdot \nabla_x$ and $\partial_t + v \cdot \nabla_x$ (their respective symbols are (with obvious notations) $i\, v \cdot \xi$ and $i\, \tau + i\, v \cdot \xi$), there is no hope that the solution f of eq. (38) (or eq. (39)) be smoother than the right-hand side g. In fact, for any f (that is, as singular as one wants), $f(x - vt, v)$ is a (weak) solution of eq. (38) with $g = 0$.

However, the set of ξ (different from 0) such that $v \cdot \frac{\xi}{|\xi|} = 0$ varies when v varies, so that when one takes averages in v of f (weak) solution of eq. (38) (or eq. (39)), there is some hope of getting a function (of t, x) smoother than g.

Unfortunately, though eq. (38) has a very simple explicit solution, namely

$$f(t,x,v) = f(0, x - vt, v) + \int_0^t g(s, x - v\,(t-s), v)\, ds,$$

it seems very difficult to prove such a gain of smoothness by using this formula without Fourier transform.

The use of the Fourier transform, on the other hand, enables to obtain this gain of smoothness. This was first observed in [39], [38] and [1].

In the next two subsections, we give two proofs using the Fourier transform, but in a very different way. In the first one, better adapted to a steady equation, or to a situation in which one needs smoothness in the time variable, the Fourier transform is taken with respect to t and x. In the second one, better adapted to situations where smoothness in the time variable is not required, the Fourier transform is taken with respect to x and v.

2.2 Use of the Fourier Transform in x or t, x

We begin here by recalling the proof of [38] in the case of the steady equation, when the averaging function ϕ is L^∞ (and compactly supported). We give estimates which are fully explicit, but not necessarily optimal in some respects. In particular, sums of norms instead of products appear in the right-hand sides of our estimates.

Theorem 2 *Let $f \equiv f(x,v)$ be a function of $L^2(\mathbb{R}^N \times \mathbb{R}^N)$ such that $g = v \cdot \nabla_x f$ also lies in $L^2(\mathbb{R}^N \times \mathbb{R}^N)$. Then, for all function ϕ in $L^\infty(\mathbb{R}^N)$ with its support included in $[-R, R]^N$, the following estimate holds :*

$$\Big|\Big| \int_{\mathbb{R}^N} f(\cdot, v)\, \phi(v)\, dv \Big|\Big|^2_{H^{1/2}(\mathbb{R}^N)} \le 4\,(2R)^{N-1}\, ||\phi||^2_{L^\infty(\mathbb{R}^N)} \Big(||f||^2_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}$$

$$+\,||g||^2_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}\Big). \tag{40}$$

Proof : We denote by $\hat{f}$ the Fourier transform of f in the x variable only, and by ξ the corresponding Fourier variable. Then, $\hat{g}(\xi,v) = i\,(v\cdot\xi)\,\hat{f}(\xi,v)$.

The idea is to consider separately those $v\in\mathbb{R}^N$ such that $|v\cdot\frac{\xi}{|\xi|}|$ is large and those such that $|v\cdot\frac{\xi}{|\xi|}|$ is small.

The computation gives (for some function $\delta\equiv\delta(\xi)$ which will be chosen later on)

$$\Big|\int_{\mathbb{R}^N}\hat{f}(\xi,v)\,\phi(v)\,dv\Big|^2 \le 2\,\Big|\int_{|v\cdot\frac{\xi}{|\xi|}|\ge\delta}\hat{f}(\xi,v)\,\phi(v)\,dv\Big|^2$$

$$+\,2\,\Big|\int_{|v\cdot\frac{\xi}{|\xi|}|\le\delta}\hat{f}(\xi,v)\,\phi(v)\,dv\Big|^2$$

$$\le 2\,\Big|\int_{|v\cdot\frac{\xi}{|\xi|}|\ge\delta}\frac{1}{|\xi|}\frac{|v\cdot\xi|}{|v\cdot\frac{\xi}{|\xi|}|}\,\hat{f}(\xi,v)\,\phi(v)\,dv\Big|^2 +\,2\,\Big|\int_{|v\cdot\frac{\xi}{|\xi|}|\le\delta}\hat{f}(\xi,v)\,\phi(v)\,dv\Big|^2$$

$$\le \frac{2}{|\xi|^2}\int_{|v\cdot\frac{\xi}{|\xi|}|\ge\delta}\frac{|\phi(v)|^2}{|v\cdot\frac{\xi}{|\xi|}|^2}\,dv\int_{\mathbb{R}^N}|\hat{g}(\xi,v)|^2\,dv$$

$$+\,2\int_{|v\cdot\frac{\xi}{|\xi|}|\le\delta}|\phi(v)|^2\,dv\int_{\mathbb{R}^N}|\hat{f}(\xi,v)|^2\,dv$$

$$\le \frac{2}{|\xi|^2}\,||\phi||^2_{L^\infty(\mathbb{R}^N)}\,(2R)^{N-1}\,2\int_\delta^{+\infty}\frac{dv_1}{v_1^2}\int_{\mathbb{R}^N}|\hat{g}(\xi,v)|^2\,dv$$

$$+\,2\,||\phi||^2_{L^\infty(\mathbb{R}^N)}\,(2\delta)\,(2R)^{N-1}\int_{\mathbb{R}^N}|\hat{f}(\xi,v)|^2\,dv$$

$$\le 4\,(2R)^{N-1}\,||\phi||^2_{L^\infty(\mathbb{R}^N)}\Big(\frac{1}{\delta|\xi|^2}\int_{\mathbb{R}^N}|\hat{g}(\xi,v)|^2\,dv+\delta\int_{\mathbb{R}^N}|\hat{f}(\xi,v)|^2\,dv\Big).$$

We conclude by taking $\delta(|\xi|)=\frac{1}{|\xi|}$.

Note that a different choice of δ would enable to obtain at the end a product of norms of f and g instead of a sum of such norms. $\square$

This result can be extended in many ways. We give here the proof of two useful such extensions.

The first one enables to treat the case of kinetic equations with right-hand sides including derivatives in the v variable (first-order derivatives as in Vlasov, or second-order derivatives as in Landau, but also fractional derivatives such as in the non cutoff Boltzmann equation). The second one enables to treat space-dependant equations. Of course those two extensions can be combined in a single theorem, but we shall not write down such a theorem in this work, since we wish to present only typical proofs, not optimal results.

We begin with the theorem adapted to the Vlasov equation. The estimate given here is almost explicit (that is, explicit up to a numerical constant which can be estimated). With respect to the previous theorem, It needs more derivatives of the averaging functions ϕ. The proof is very close to that of [34].

Theorem 3 *Let $f \equiv f(x,v)$ be a function of $L^2(\mathbb{R}^N \times \mathbb{R}^N)$ such that $g = v\cdot\nabla_x f$ is of the form $g = \partial_v^K h$, where $h \in L^2(\mathbb{R}^N \times \mathbb{R}^N)$ and ∂_v^K denotes any derivative in the v variable of order K. Then, for all function ϕ in $W^{K,\infty}(\mathbb{R}^N)$ with its support included in $[-R,R]^N$, the following estimate holds (for some constant $C_K > 0$) :*

$$\Big|\Big|\int_{\mathbb{R}^N} f(\cdot,v)\,\phi(v)\,dv\Big|\Big|^2_{H^{1/2(K+1)}(\mathbb{R}^N)} \le C_K\,R^{N-1}\,||\phi||^2_{W^{K,\infty}(\mathbb{R}^N)}$$

$$\times\Big(||f||^2_{L^2(\mathbb{R}^N\times\mathbb{R}^N)} + ||h||^2_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}\Big). \tag{41}$$

Proof: We still denote by $\hat{f}$ the Fourier transform of f in the x variable only, and by ξ the corresponding Fourier variable.

Moreover, we introduce (for $\delta \equiv \delta(\xi)$ to be chosen later) a cutoff function χ_δ of $\mathcal{D}(\mathbb{R})$ which takes its values in $[0,1]$, has its support in $[-2\delta, 2\delta]$, and satisfies $\chi_\delta(x) \equiv 1$ for $x \in [-\delta,\delta]$. We still use the identity $\hat{g}(\xi,v) = i\,(v\cdot\xi)\,\hat{f}(\xi,v)$.

We compute :

$$\Big|\int_{\mathbb{R}^N} \hat{f}(\xi,v)\,\phi(v)\,dv\Big|^2 \le 2\Big|\int_{\mathbb{R}^N} \hat{f}(\xi,v)\,\chi_\delta(v\cdot\frac{\xi}{|\xi|})\,\phi(v)\,dv\Big|^2$$

$$+\,2\Big|\int_{\mathbb{R}^N} \hat{f}(\xi,v)\,(1-\chi_\delta(v\cdot\frac{\xi}{|\xi|}))\,\phi(v)\,dv\Big|^2$$

$$\leq 2 \left| \int_{\mathbb{R}^N} \hat f(\xi, v)\, \chi_\delta(v\cdot \frac{\xi}{|\xi|})\, \phi(v)\, dv \right|^2 + 2 \left| \int_{\mathbb{R}^N} \frac{\partial_v^K \hat h(\xi, v)}{v\cdot \xi}\, (1-\chi_\delta(v\cdot \frac{\xi}{|\xi|}))\, \phi(v)\, dv \right|^2$$

$$\leq 2 \left| \int_{\mathbb{R}^N} \hat f(\xi, v)\, \chi_\delta(v \cdot \frac{\xi}{|\xi|})\, \phi(v)\, dv \right|^2$$

$$+ \sum_{P+Q+R=K} C_{P,Q} \left| \int_{\mathbb{R}^N} \hat h(\xi, v) \Big(\frac{\xi}{|\xi|}\Big)^P \partial^P (1-\chi_\delta)(v\cdot \frac{\xi}{|\xi|})\, \partial^Q \phi(v)\, \frac{\xi^R}{(v\cdot\xi)^{R+1}}\, dv \right|^2$$

(with obvious notations)

$$\leq 2 \int_{|v\cdot \frac{\xi}{|\xi|}| \leq 2\delta} |\phi(v)|^2\, dv \int_{\mathbb{R}^N} |\hat f(\xi, v)|^2\, dv$$

$$+ \sum_{P+Q+R=K} C_{P,Q} \int_{|v\cdot \frac{\xi}{|\xi|}| \geq 2\delta} \delta^{-2P}\, |\partial^Q \phi(v)|^2\, \frac{|\xi|^{2R}}{|v\cdot\xi|^{2R+2}}\, dv \int_{\mathbb{R}^N} |\hat h(\xi, v)|^2\, dv$$

$$\leq 8\,\delta\,(2R)^{N-1}\, ||\phi||^2_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\hat f(\xi, v)|^2\, dv$$

$$+ \sum_{P+Q+R=K} C_{P,Q}\, (2R)^{N-1} \frac{\delta^{-2P}}{|\xi|^2}\, ||\partial^Q \phi||^2_{L^\infty(\mathbb{R}^N)}$$

$$\times \int_{|v_1| \geq \delta} \frac{dv_1}{|v_1|^{2R+2}} \int_{\mathbb{R}^N} |\hat h(\xi, v)|^2\, dv$$

$$\leq C_K\, R^{N-1} \Big(\delta\, ||\phi||^2_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\hat f(\xi, v)|^2\, dv$$

$$+ \sum_{P+Q+R=K} \frac{\delta^{-2P-2R-1}}{|\xi|^2}\, ||\partial^Q \phi||^2_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\hat h(\xi, v)|^2\, dv \Big).$$

Choosing

$$\delta = |\xi|^{-\frac{1}{K+1}},$$

the previous computation yields the estimate

$$\left| \int_{\mathbb{R}^N} \hat f(\xi, v)\, \phi(v)\, dv \right|^2 \leq C_K\, R^{N-1}\, ||\phi||^2_{W^{K,\infty}(\mathbb{R}^N)}$$

$$\times \Big(|\xi|^{-\frac{1}{K+1}} \int_{\mathbb{R}^N} |\hat f(\xi, v)|^2\, dv + \sum_{S\leq K} |\xi|^{-2+\frac{2S+1}{K+1}} \int_{\mathbb{R}^N} |\hat h(\xi, v)|^2\, dv \Big).$$

This in turn enables us to write down the estimate

$$\int_{|\xi|\geq 1} |\xi|^{\frac{1}{K+1}} \left| \int_{\mathbb{R}^N} \hat f(\xi, v)\, \phi(v)\, dv \right|^2 d\xi \leq C_K\, R^{N-1}\, ||\phi||^2_{W^{K,\infty}(\mathbb{R}^N)}$$

$$\times \Big(\int \int_{|\xi| \geq 1} |\hat{f}(\xi,v)|^2 \, dv d\xi + \sum_{S \leq K} \int \int_{|\xi| \geq 1} |\xi|^{2 \, \frac{S-K}{K+1}} \, |\hat{h}(\xi,v)|^2 \, dv d\xi \Big).$$

Since on the other hand, it is easy to estimate

$$\int_{|\xi| \leq 1} |\xi|^{\frac{1}{K+1}} \Big| \int_{\mathbb{R}^N} \hat{f}(\xi,v) \, \phi(v) \, dv \Big|^2 d\xi$$

by the L^2 norm of f, we conclude the proof. $\square$

We now treat the second extension of theorem 2. This is the case when f, which also depends on t, satisfies eq. (38) on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$. It enables to get smoothness of the averages in v of f in both variables t and x. The proof is close to that of [38]. In order to use such a result in the context of the study of the Cauchy problem for a partial differential equation, one has in general to use techniques of truncation, etc., in the time variable. Those technicalities can sometimes be avoided when one uses the results of next chapter.

Theorem 4 *Let $f \equiv f(t,x,v)$ be a function of $L^2(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)$ such that $g = \partial_t f + v \cdot \nabla_x f$ lies in $L^2(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)$. Then, for all function ϕ in $L^\infty(\mathbb{R}^N)$ with support included in $[-R,R]^N$, the following estimate holds :*

$$\Big|\Big| \int_{\mathbb{R}^N} f(\cdot,\cdot,v) \, \phi(v) \, dv \Big|\Big|^2_{H^{1/2}(\mathbb{R} \times \mathbb{R}^N)} \leq 4 \, ||\phi||^2_{L^\infty(\mathbb{R}^N)} \, (2R)^{N-2} \, (5 + 14 \, R$$

$$+ 12 \, R^2 + 8 \, R^3) \times \Big(||f||^2_{L^2(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)} + ||g||^2_{L^2(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)} \Big). \tag{42}$$

Proof: We now denote by $\hat{f}$ the Fourier transform of f in the t and x variable only, and by τ and ξ the corresponding Fourier variables. The relation between $\hat{f}$ and $\hat{g}$ is now $\hat{g}(\tau,\xi,v) = i \, (\tau + v \cdot \xi) \, \hat{f}(\tau,\xi,v)$.

We compute (for any δ)

$$\Big| \int_{\mathbb{R}^N} \hat{f}(\tau,\xi,v) \, \phi(v) \, dv \Big|^2 \leq 4 \, \Big| 1_{|\xi| \leq 1, |\tau| \geq 2R} \int_{\mathbb{R}^N} \frac{\hat{g}(\tau,\xi,v)}{i(\tau + v \cdot \xi)} \, \phi(v) \, dv \Big|^2$$

$$+ 4 \, \Big| 1_{|\xi| \leq 1, |\tau| \leq 2R} \int_{\mathbb{R}^N} \hat{f}(\tau,\xi,v) \, \phi(v) \, dv \Big|^2$$

$$+ 4 \, \Big| 1_{|\xi| \geq 1} \int_{|\tau + v \cdot \xi| \leq \delta} \hat{f}(\tau,\xi,v) \, \phi(v) \, dv \Big|^2$$

$$+\,4\left|1_{|\xi|\geq 1}\int_{|\tau+v\cdot\xi|\geq\delta}\frac{\hat{g}(\tau,\xi,v)}{i(\tau+v\cdot\xi)}\,\phi(v)\,dv\right|^2$$

$$\leq 4\,1_{|\xi|\leq 1,|\tau|\geq 2R}\,||\phi||^2_{L^\infty(\mathbb{R}^N)}\left(\int_{|v|\leq R}\frac{dv}{|\tau+v\cdot\xi|^2}\right)\left(\int_{\mathbb{R}^N}|\hat{g}(\tau,\xi,v)|^2\,dv\right)$$

$$+\,4\,1_{|\xi|\leq 1,|\tau|\leq 2R}\,||\phi||^2_{L^\infty(\mathbb{R}^N)}\left(\int_{|v|\leq R}dv\right)\left(\int_{\mathbb{R}^N}|\hat{f}(\tau,\xi,v)|^2\,dv\right)$$

$$+\,4\,1_{|\xi|\geq 1}\,||\phi||^2_{L^\infty(\mathbb{R}^N)}\left(\int_{|\tau+v\cdot\xi|\leq\delta,|v|\leq R}dv\right)\left(\int_{\mathbb{R}^N}|\hat{f}(\tau,\xi,v)|^2\,dv\right)$$

$$+\,4\,1_{|\xi|\geq 1}\,||\phi||^2_{L^\infty(\mathbb{R}^N)}\left(\int_{|\tau+v\cdot\xi|\geq\delta,|v|\leq R}\frac{dv}{|\tau+v\cdot\xi|^2}\right)\left(\int_{\mathbb{R}^N}|\hat{g}(\tau,\xi,v)|^2\,dv\right).$$

We now observe that

$$1_{|\xi|\leq 1,|\tau|\geq 2R}\int_{|v|\leq R}\frac{dv}{|\tau+v\cdot\xi|^2}\leq\frac{(2R)^N}{\frac{1}{2}|\tau|^2-R^2}1_{|\xi|\leq 1,|\tau|\geq 2R},$$

$$1_{|\xi|\leq 1,|\tau|\leq 2R}\int_{|v|\leq R}dv\leq(2R)^N\,1_{|\xi|\leq 1,|\tau|\leq 2R},$$

$$1_{|\xi|\geq 1}\int_{|\tau+v\cdot\xi|\leq\delta,|v|\leq R}dv\leq(2R)^{N-1}\,\frac{\delta}{|\xi|}\,1_{|\xi|\geq 1,|\tau|\leq\delta+R\,|\xi|},$$

$$1_{|\xi|\geq 1}\int_{|\tau+v\cdot\xi|\geq\delta,|v|\leq R}\frac{dv}{|\tau+v\cdot\xi|^2}\leq\frac{(2R)^{N-1}}{|\xi|^2}\int_{|w|\geq\frac{\delta}{|\xi|},|w-\frac{\tau}{|\xi|}|\leq R}\frac{dw}{|w|^2}\,1_{|\xi|\geq 1}$$

$$\leq(2R)^{N-1}\left(1_{|\tau|\leq 2R\,|\xi|}\frac{1}{\delta\,|\xi|}+1_{|\tau|\geq 2R\,|\xi|}\frac{1}{\tau\,|\xi|}\right)1_{|\xi|\geq 1}.$$

Then,

$$\int\int_{\mathbb{R}\times\mathbb{R}^N}\Big(|\tau|+|\xi|\Big)\left|\int_{\mathbb{R}^N}\hat{f}(\tau,\xi,v)\,\phi(v)\,dv\right|^2d\xi\,d\tau$$

$$\leq\int\int_{|\xi|\leq 1}\Big(|\tau|+|\xi|\Big)\left|\int_{\mathbb{R}^N}\hat{f}(\tau,\xi,v)\,\phi(v)\,dv\right|^2d\xi\,d\tau$$

$$+\int\int_{|\xi|\geq 1,|\tau|\leq 2R\,|\xi|}\Big(|\tau|+|\xi|\Big)\left|\int_{\mathbb{R}^N}\hat{f}(\tau,\xi,v)\,\phi(v)\,dv\right|^2d\xi\,d\tau$$

$$+\int\int_{|\xi|\geq 1,|\tau|\geq 2R\,|\xi|}\Big(|\tau|+|\xi|\Big)\left|\int_{\mathbb{R}^N}\hat{f}(\tau,\xi,v)\,\phi(v)\,dv\right|^2d\xi\,d\tau$$

$$\leq\int\int_{|\xi|\leq 1}\Big(|\tau|+1\Big)\left|\int_{\mathbb{R}^N}\hat{f}(\tau,\xi,v)\,\phi(v)\,dv\right|^2d\xi\,d\tau$$

$$+\int\int_{|\xi|\geq 1, |\tau|\leq 2R|\xi|} (1+2R)\,|\xi| \left| \int_{\mathbb{R}^N} \hat{f}(\tau,\xi,v)\,\phi(v)\,dv \right|^2 d\xi\,d\tau$$

$$+\int\int_{|\xi|\geq 1, |\tau|\geq 2R|\xi|} \Big(|\tau|+|\xi|\Big) \left| \int_{\mathbb{R}^N} \hat{f}(\tau,\xi,v)\,\phi(v)\,dv \right|^2 d\xi\,d\tau$$

$$\leq 4\,||\phi||^2_{L^\infty(\mathbb{R}^N)}\,(2R)^{N-1} \int\int_{\mathbb{R}\times\mathbb{R}^N} \Big(|\tau|+1\Big) \left\{ \frac{8R}{|\tau|^2}\,1_{|\tau|\geq 2R} + 2R\,1_{|\tau|\leq 2R} \right\}$$

$$\times \left\{ \int_{\mathbb{R}^N} |\hat{f}(\tau,\xi,v)|^2\,dv + \int_{\mathbb{R}^N} |\hat{g}(\tau,\xi,v)|^2\,dv \right\} d\xi\,d\tau$$

$$+4\,||\phi||^2_{L^\infty(\mathbb{R}^N)}\,(2R)^{N-1} \int\int_{|\xi|\geq 1} (1+2R)\,|\xi| \left\{ \frac{\delta}{|\xi|} + \frac{1}{\delta\,|\xi|} \right\}$$

$$\times \left\{ \int_{\mathbb{R}^N} |\hat{f}(\tau,\xi,v)|^2\,dv + \int_{\mathbb{R}^N} |\hat{g}(\tau,\xi,v)|^2\,dv \right\} d\xi\,d\tau$$

$$+4\,||\phi||^2_{L^\infty(\mathbb{R}^N)}\,(2R)^{N-1} \int\int_{|\tau|\geq 2R, |\xi|\geq 1} \frac{|\tau|+|\xi|}{|\tau|\,|\xi|}$$

$$\times \left\{ \int_{\mathbb{R}^N} |\hat{f}(\tau,\xi,v)|^2\,dv + \int_{\mathbb{R}^N} |\hat{g}(\tau,\xi,v)|^2\,dv \right\} d\xi\,d\tau.$$

Note finally that $\delta = 1$ yields the theorem. □

Many more extensions of the previous results can be found in the works of [36], [37], [35], [12], [58], [64] and [52]. Among those extensions, one can write down results in L^p instead of L^2 (those are obtained by interpolation techniques), one can replace v by $a(v)$, where a is any function satisfying a non degeneracy condition, and finally one can introduce in the right-hand side of the equation derivatives in t, x of order strictly less than one.

2.3 Use of the Fourier Transform in x, v

We now introduce a different way of looking at averaging lemmas. We are interested in this section only in the time-dependant case, but we don't try to get regularity in the t variable. As a consequence, the results we shall get are more adapted to solutions of the transport equation which are defined on a time interval $[0, T]$, for which the initial datum is given.

Though the results are weaker than those of the previous section, the proofs turn out to be more easily extendable in the case of discretized in time equations.

The idea used here consists in writing down the Fourier transform in x and v of the free transport operator, instead of its Fourier transform in t and x. This procedure was used in particular by Golse (Cf. [37]) and by P.-L. Lions and Perthame (Cf. [59]).

As we noticed previously, the interest of this method lies in the fact that it yields results when some discretization in time is in order. Such a situation is described in [31]. In this work, the operator splitting technique between the free transport part and the collisional part of the Boltzmann equation is studied, in the framework of renormalized solutions. We give in next subsection another example of dicretization in time.

The proof given here is inspired of [20]. We denote by $L^2 - w$ the weak topology of L^2.

Theorem 5 *Let $f \in C([0,T], L^2(\mathbb{R}^N_x \times \mathbb{R}^N_v) - w)$ solve eq. (38) for some $g \in L^2(]0,T[\times \mathbb{R}^N_x \times \mathbb{R}^N_v)$. We denote $f_0 = f(0,\cdot)$. Then, for any $\psi \in C^\infty_c(\mathbb{R}^N)$, the average quantity*

$$\rho_\psi(t,x) = \int_{\mathbb{R}^N} f(t,x,v)\, \psi(v)\, dv \tag{43}$$

lies in $L^2(]0,T[, H^{1/2}(\mathbb{R}^N_x))$, and for all $s > (N-1)/2$,

$$\|\rho_\psi\|_{L^2(]0,T[,H^{1/2}(\mathbb{R}^N_x))} \leq C_{N,s} \bigg(\int\int_{\mathbb{R}^N \times \mathbb{R}^N} |f_0(x,v)|^2\, |\psi(v)|^2\, (1+|v|^2)^s\, dvdx$$

$$+ \int\int\int_{[0,T]\times\mathbb{R}^N\times\mathbb{R}^N} (|f(t,x,v)|^2 + |g(t,x,v)|^2)\, |\psi(v)|^2\, (1+|v|^2)^s\, dvdxdt \bigg). \tag{44}$$

Proof: Let us denote $\widehat{f}(t,\xi,v)$ the Fourier transform of f in the x variable, and $\mathcal{F}f(t,\xi,\eta)$ the Fourier transform of f in the x,v variables. Then, (38) yields

$$\partial_t \widehat{f} + i\, v\cdot\xi\, \widehat{f} = \widehat{g}. \tag{45}$$

Solving this equation in the sense of distributions, we get

$$\widehat{f}(t,\xi,v) = e^{-i\, v\cdot\xi\, t}\widehat{f}_0(\xi,v) + \int_0^t e^{-i\, v\cdot\xi\, s}\widehat{g}(t-s,\xi,v)\, ds. \tag{46}$$

Multiplying (46) by $\psi(v)$, we obtain

$$\widehat{f}(t,\xi,v)\psi(v) = e^{-i\,v\cdot\xi\,t}\widehat{f_0\psi}(\xi,v) + \int_0^t e^{-i\,v\cdot\xi\,s}\widehat{g\psi}(t-s,\xi,v)\,ds, \tag{47}$$

and after integration in v,

$$\widehat{\rho}_\psi(t,\xi) = \mathcal{F}(f_0\psi)(\xi,t\xi) + \int_0^t \mathcal{F}(g\psi)(t-s,\xi,s\xi)ds. \tag{48}$$

This type of formula with double Fourier transform evaluated at $(\xi,t\xi)$ was used in [37]. For *a.e.* $\xi \in I\!R^N$, we estimate this quantity thanks to Cauchy–Schwarz inequality, and get

$$|\widehat{\rho}_\psi(t,\xi)|^2 \leq 2\,|\mathcal{F}(f_0\psi)(\xi,t\xi)|^2 + 2\,t\int_0^t |\mathcal{F}(g\psi)(t-s,\xi,s\xi)|^2\,ds. \tag{49}$$

Integrating this estimate on $]0,T[$, and using the variable $\tau = t-s$, we obtain

$$\int_0^T |\widehat{\rho}_\psi(t,\xi)|^2\,dt \leq 2\int_0^T |\mathcal{F}(f_0\psi)(\xi,t\xi)|^2\,dt$$

$$+2\,T\int_0^T\int_0^T |\mathcal{F}(g\psi)(\tau,\xi,s\xi)|^2 d\tau ds$$

$$\leq \frac{2}{|\xi|}\int_0^{T\,|\xi|} |\mathcal{F}(f_0\psi)(\xi,\theta\frac{\xi}{|\xi|})|^2\,d\theta + \frac{2T}{|\xi|}\int_0^T\int_0^{T\,|\xi|} |\mathcal{F}(g\psi)(\tau,\xi,\theta\frac{\xi}{|\xi|})|^2 d\tau ds. \tag{50}$$

Let us now state a very classical trace lemma.

Lemma 1 *Let $\phi \in H^s(I\!R^N_\eta)$ with $s > (N-1)/2$. Then, for any $\sigma \in I\!R^N$ such that $|\sigma| = 1$,*

$$\|\phi(z\sigma)\|_{L^2(z\in I\!R)} \leq C_{N,s}\,\|(Id-\Delta_\eta)^{s/2}\phi\|_{L^2(I\!R^N_\eta)}. \tag{51}$$

For each integral in z, we use this lemma and Plancherel's identity. We get for *a.e.* ξ,

$$\int_0^T |\widehat{\rho}_\psi(t,\xi)|^2\,dt \leq \frac{C_{N,s}}{|\xi|}\int_{v\in I\!R^N} \left|\widehat{f_0\psi}(\xi,v)\right|^2 (1+|v|^2)^s\,dv$$

$$+\frac{C_{N,s}}{|\xi|}\int_{t=0}^T\int_{v\in I\!R^N} \left|\widehat{g\psi}(t,\xi,v)\right|^2\,(1+|v|^2)^s\,dvd\tau.$$

Then,

$$\int_0^T \int_{I\!R^N} |\xi| \, |\widehat{\rho}_\psi(t,\xi)|^2 \, d\xi dt \leq C_{N,s} \Big(\int\int_{I\!R^N \times I\!R^N} |f_0(x,v)|^2 \, |\psi(v)|^2$$

$$\times \, (1+|v|^2)^s \, dvdx + \int\int\int_{[0,T]\times I\!R^N \times I\!R^N} |f(t,x,v)|^2 \, |\psi(v)|^2 \, (1+|v|^2)^s \, dvdxdt. \Big)$$

□

2.4 Time discretization

We use here the techniques of the previous subsection to get averaging lemmas adapted to a time discretization of eq. (38). More precisely, we present the Euler implicit scheme and the second-order Crank-Nicolson scheme corresponding to the free transport equation (that is, eq. (38) with $g = 0$). The results of this subsection are extracted from [20].

Note that another example of time discretization is presented in [20]. It concerns the convergence of the operator splitting method for the Boltzmann equation in the renormalized framework (Cf. [31]). Let us also mention that there exists another method to prove the convergence of the splitting algorithm, which does not use averaging lemmas, see [72].

Finally, we underline the fact that the results of this subsection belong to the general class of the so-called "averaging lemmas at the limit". Those are designed to prove the convergence of the numerical schemes towards the solutions of the kinetic equations. They can concern other variables than t.

We introduce implicit methods for solving the free transport equation $\partial_t f + v \cdot \nabla_x f = 0$. The distribution function f is approximated by f^n at time $n\Delta t$ ($\Delta t > 0$, $n \in I\!N$). We treat the cases of the Euler implicit scheme

$$\frac{f^{n+1} - f^n}{\Delta t} + v \cdot \nabla_x f^{n+1} = 0, \tag{52}$$

and of the second-order Crank-Nicolson scheme

$$\frac{f^{n+1} - f^n}{\Delta t} + v \cdot \nabla_x \frac{f^n + f^{n+1}}{2} = 0. \tag{53}$$

The initial datum $f_{in} = f^0$ is assumed to belong to $L^2(I\!R_x^N \times I\!R_v^N)$. Then f^n is uniformly bounded in L^2, $\|f^n\|_{L^2(I\!R_x^N \times I\!R_v^N)} \leq \|f^0\|_{L^2(I\!R_x^N \times I\!R_v^N)}$. For any test function $\psi \in C_c^\infty(I\!R_v^N)$, we define the averages

$$\rho_\psi^n(x) = \int_{I\!R^N} f^n(x,v)\psi(v)\,dv \quad \in L^2(I\!R_x^N). \tag{54}$$

We begin with an easy computation for the Euler implicit scheme.

Theorem 6 *For the Euler implicit scheme (52), $\rho_\psi^n \in H^{1/2}(I\!R^N)$ for any $n \geq 1$, and for any $s > (N-1)/2$,*

$$\Delta t \sum_{n=1}^{\infty} \|\rho_\psi^n\|^2_{\dot H^{1/2}(I\!R^N)} \leq C_{N,s} \|\psi(v)(1+|v|^2)^{s/2}\|^2_{L^\infty(I\!R_v^N)} \|f^0\|^2_{L^2(I\!R_x^N \times I\!R_v^N)}. \tag{55}$$

Proof : We denote by $\hat f$ or by $\mathcal{F} f$ the Fourier transform of f with respect to the x variable, and by ξ the corresponding Fourier variable.

The solution f^{n+1} of (52) is given in terms of f^n by

$$f^{n+1}(x,v) = \int_0^\infty e^{-s} f^n(x - \Delta t s v, v)\,ds, \tag{56}$$

and we easily deduce by induction that for any $n \geq 1$,

$$\begin{aligned} f^n(x,v) &= \textstyle\int_0^\infty e^{-s} \frac{s^{n-1}}{(n-1)!} f^0(x - \Delta t s v, v)\,ds, \\ \widehat{f}^n(\xi,v) &= \textstyle\int_0^\infty e^{-s} \frac{s^{n-1}}{(n-1)!} e^{-i\Delta t\, s\, v\cdot\xi}\, \widehat{f}^0(\xi,v)\,ds. \end{aligned} \tag{57}$$

Then, for *a.e.* $\xi \in I\!R^N$, the Fourier transform ρ_ψ of the average in v of f,

$$\begin{aligned} \widehat{\rho}_\psi^n(\xi) &= \textstyle\int_{I\!R^N} \widehat{f}^n(\xi,v)\psi(v)\,dv \\ &= \textstyle\int_0^\infty e^{-s} \frac{s^{n-1}}{(n-1)!} \mathcal{F}(f^0\psi)(\xi, \Delta t s \xi)\,ds. \end{aligned} \tag{58}$$

According to the Cauchy-Schwarz inequality,

$$|\widehat{\rho}_\psi^n(\xi)|^2 \leq \int_0^\infty e^{-s} \frac{s^{n-1}}{(n-1)!}\,ds \int_0^\infty e^{-s} \frac{s^{n-1}}{(n-1)!} |\mathcal{F}(f^0\psi)(\xi, \Delta t s \xi)|^2\,ds, \tag{59}$$

and since the first integral has value 1,

$$\begin{array}{rl}\Delta t\sum_{n=1}^{\infty}|\widehat{\rho}_{\psi}^{n}(\xi)|^2 & \leq \Delta t\int_0^\infty\left(\sum_{n=1}^\infty e^{-s}\frac{s^{n-1}}{(n-1)!}\right)|\mathcal{F}(f^0\psi)(\xi,\Delta ts\xi)|^2\,ds\\ & =\frac{1}{|\xi|}\int_0^\infty|\mathcal{F}(f^0\psi)(\xi,z\frac{\xi}{|\xi|})|^2\,dz\\ & \leq\frac{C_{N,s}}{|\xi|}\int_{v\in\mathbb{R}^N}|\widehat{f}^0(\xi,v)|^2\,dv\ \|\psi(v)(1+|v|^2)^{s/2}\|^2_{L^\infty(\mathbb{R}^N_v)}\end{array} \tag{60}$$

by the same estimate as in Theorem 5. The result (55) follows by integration with respect to the variable ξ. $\square$

We now turn to the Crank-Nicolson scheme, and propose a very different type of estimate.

Theorem 7 *For the Crank-Nicolson scheme (53), the following compactness estimate for averages in time holds. For any $R>0$,*

$$\int_{|\xi|>R}\left|\Delta t\sum_{n=0}^m\chi_n\widehat{\rho}_\psi^n(\xi)\right|^2 d\xi\leq C_\psi\left(\Delta t^2A^2+\frac{AB}{R}\right)\|f^0\|^2_{L^2(\mathbb{R}_x^N\times\mathbb{R}_v^N)}, \tag{61}$$

where $m\in\mathbb{N}$, $(\chi_n)_{0\leq n\leq m}$ are arbitrary complex numbers, and

$$A=\sum_{n=0}^{m-1}|\chi_n-\chi_{n+1}|+|\chi_m|,\qquad B=\Delta t\sum_{n=0}^m|\chi_n| \tag{62}$$

represent respectively the total variation and the L^1 norm of χ.

Proof : We use the same notations as in the proof of theorem 6.

The solution f^{n+1} of (53) is given in terms of f^n by

$$\begin{array}{r}f^{n+1}(x,v)=2\int_0^\infty e^{-s}f^n(x-\frac{\Delta t}{2}sv,v)\,ds-f^n(x,v),\\ \widehat{f}^{n+1}(\xi,v)=\frac{1-i\frac{\Delta t}{2}v\cdot\xi}{1+i\frac{\Delta t}{2}v\cdot\xi}\,\widehat{f}^n(\xi,v).\end{array} \tag{63}$$

Therefore, for any $n\geq 0$, we obtain by induction

$$\widehat{f}^n(\xi,v)=\left(\frac{1-i\frac{\Delta t}{2}v\cdot\xi}{1+i\frac{\Delta t}{2}v\cdot\xi}\right)^n\widehat{f}^0(\xi,v), \tag{64}$$

and

$$\begin{array}{rl}\widehat{\rho}_\psi^n(\xi) & =\int_{\mathbb{R}^N}\widehat{f}^n(\xi,v)\psi(v)\,dv\\ & =\int_{\mathbb{R}^N}\left(\frac{1-i\frac{\Delta t}{2}v\cdot\xi}{1+i\frac{\Delta t}{2}v\cdot\xi}\right)^n\widehat{f}^0(\xi,v)\psi(v)\,dv.\end{array} \tag{65}$$

Let us now introduce the angle $\theta \in]-\pi, \pi[$ defined by

$$\frac{1 - i\frac{\Delta t}{2}\, v \cdot \xi}{1 + i\frac{\Delta t}{2}\, v \cdot \xi} = e^{-i\theta}, \tag{66}$$

or equivalently $\theta = 2\ \mathrm{Arctg}(\frac{\Delta t}{2}\, v \cdot \xi)$. Then,

$$\Delta t \sum_{n=0}^{m} \chi_n \widehat{\rho}_\psi^n(\xi) = \int_{I\!R^N} \varphi(\theta) \widehat{f}^0(\xi, v)\psi(v)\, dv, \tag{67}$$

$$\varphi(\theta) = \Delta t \sum_{n=0}^{m} \chi_n e^{-in\theta}. \tag{68}$$

Using Abel's transform, we get

$$\begin{gathered} \varphi(\theta) = \Delta t \sum_{n=0}^{m-1} (\chi_n - \chi_{n+1}) \left(\sum_{l=0}^{n} e^{-il\theta} \right) + \Delta t \chi_m \sum_{l=0}^{m} e^{-il\theta}, \\ |\varphi(\theta)| \leq \frac{\Delta t A}{|\sin(\theta/2)|}. \end{gathered} \tag{69}$$

Now, since $\sin(\theta/2) = \frac{\Delta t}{2}\, v \cdot \xi \,/ \sqrt{1 + (\frac{\Delta t}{2}\, v \cdot \xi)^2}$, we obtain

$$|\varphi(\theta)| \leq \Delta t A + \frac{2A}{|v \cdot \xi|}. \tag{70}$$

But we can also use the trivial estimate $|\varphi(\theta)| \leq B$, and combined with (70) this yields

$$|\varphi(\theta)| \leq \Delta t\, A + \min\left(\frac{2A}{|v \cdot \xi|}, B \right). \tag{71}$$

Now, coming back to (67) we get for *a.e.* $\xi \in I\!R^N$

$$\begin{aligned} \left| \Delta t \sum_{n=0}^{m} \chi_n \widehat{\rho}_\psi^n(\xi) \right|^2 &\leq \int_{I\!R^N} |\widehat{f}^0(\xi, v)|^2\, dv \int_{I\!R^N} |\varphi(\theta)|^2 |\psi(v)|^2\, dv \\ &\leq 2 \int_{I\!R^N} |\widehat{f}^0(\xi, v)|^2\, dv \left[\Delta t^2 A^2 \int_{I\!R^N} |\psi(v)|^2\, dv \right. \\ &\quad \left. + \int_{I\!R^N} \min{}^2 \left(\frac{2A}{|v \cdot \xi|}, B \right) |\psi(v)|^2\, dv \right]. \end{aligned} \tag{72}$$

The last integral can be computed,

$$\int_{I\!R^N} \min{}^2 \left(\frac{2A}{|v \cdot \xi|}, B \right) |\psi(v)|^2\, dv \tag{73}$$

$$= \int_{u=-\infty}^{\infty} \min{}^2\left(\frac{2A}{|\xi||u|}, B\right)\left(\int_{v'\in\xi^\perp} |\psi(u\frac{\xi}{|\xi|}+v')|^2\,dv'\right)du$$
$$\leq C_{N,s}\,\|\psi(v)(1+|v|^2)^{s/2}\|^2_{L^\infty(\mathbb{R}^N_v)}\int_{-\infty}^{\infty}\min{}^2\left(\frac{2A}{|\xi||u|}, B\right)du$$
$$= C_{N,s}\,\|\psi(v)(1+|v|^2)^{s/2}\|^2_{L^\infty(\mathbb{R}^N_v)}\frac{8AB}{|\xi|}.$$

Finally, estimate (72) gives for any $s > (N-1)/2$

$$\left|\Delta t\sum_{n=0}^{m}\chi_n\widehat{\rho}^n_\psi(\xi)\right|^2 \leq 2\int_{\mathbb{R}^N}|\widehat{f}^0(\xi,v)|^2\,dv\,\Big(\|\psi\|^2_{L^2(\mathbb{R}^N)}\Delta t^2A^2$$
$$+C_{N,s}\,\|\psi(v)(1+|v|^2)^{s/2}\|^2_{L^\infty(\mathbb{R}^N_v)}\frac{AB}{|\xi|}\Big), \tag{74}$$

and (61) follows by integration in ξ. $\square$

Let us now emphasize the big difference between the two schemes described above. Using the implicit scheme (52), we immediately see that for $n \geq 1$, $f^n + \Delta t\, v\cdot\nabla_x f^n \in L^2_{x,v}$. Therefore, according to [38], $\rho^n_\psi \in H^{1/2}_x$. However, in general $\rho^0_\psi \notin H^{1/2}$ (for example, take for f^0 a tensor product). Then, in an estimate like (61), we only get a term in $1/R$ (a term in Δt appears if the sum starts at $n=0$).

For the Crank-Nicolson scheme (53), the situation is very different since there is time reversibility, as in the continuous case (the L^2 norm of f^n is constant). When f^0 varies in L^2, f^n also varies in L^2, and thus ρ^n_ψ only lies in L^2_x (for a given n). Compactness only occurs for averages in time, and we must have a term in Δt in (61). However, the situation here is worse than in the continuous case, since we can only estimate an average in time with respect to a smooth function $\chi(t)$ (of bounded variation), whereas in the continuous case, an L^2 function is enough. Note that this regularity of χ is really needed. There is no inequality like (61) with the L^2 norm in time instead of the average with respect to χ. This can be seen by writing (65) as

$$\widehat{\rho}^n_\psi(\xi) = \int_{\theta=-\pi}^{\pi} e^{-in\theta}\int_{v'\in\xi^\perp}\widehat{f^0\psi}\Big(\xi, \frac{2}{\Delta t|\xi|}\tan\frac{\theta}{2}\,\frac{\xi}{|\xi|}+v'\Big)dv'\frac{1+\tan^2\frac{\theta}{2}}{\Delta t|\xi|}\,d\theta. \tag{75}$$

Then by Parseval's formula

$$\Delta t\sum_{n\in\mathbb{Z}}|\widehat{\rho}^n_\psi(\xi)|^2$$

$$= 2\pi\Delta t \int_{\theta=-\pi}^{\pi} \left| \int_{v'\in\xi^\perp} \widehat{f^0\psi}\Big(\xi, \frac{2}{\Delta t|\xi|}\tan\frac{\theta}{2}\frac{\xi}{|\xi|} + v'\Big)\, dv' \, \frac{1+\tan^2\frac{\theta}{2}}{\Delta t|\xi|} \right|^2 d\theta$$

$$= 2\pi \int_{u=-\infty}^{\infty} \left| \int_{v'\in\xi^\perp} \widehat{f^0\psi}\Big(\xi, u\frac{\xi}{|\xi|} + v'\Big)\, dv' \right|^2 \frac{1+(\frac{\Delta t}{2}|\xi|u)^2}{|\xi|}\, du,$$

and it is impossible to control the term in $\Delta t^2|\xi|$.

3 Regularity of Q^+

3.1 Introduction

We recall the general form of the positive part Q^+ of the Boltzmann operator (18),

$$Q^+(f,f)(v) = \iint\limits_{\substack{v_*\in\mathbb{R}^N \\ \sigma\in S^{N-1}}} f\left(\frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma\right) f\left(\frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma\right) \tag{76}$$

$$\times B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\, d\sigma dv_*,$$

where B is the cross section.

The classical assumption of angular cutoff of Grad (Cf. [42]) that B is integrable will always be made in this section.

The properties of Q^+ with the assumption of angular cutoff of Grad (without this assumption, Q^+ is not defined even for very smooth functions f) have first been investigated by P.-L. Lions in [54], [55]. In this work, it is proven that if B is a very smooth function with support avoiding certain points, then

$$\|Q^+(f,f)\|_{\dot H^{(N-1)/2}(\mathbb{R}^N_v)} \le C\, \|f\|_{L^1(\mathbb{R}^N_v)}\, \|f\|_{L^2(\mathbb{R}^N_v)} \tag{77}$$

for any $f\in L^1\cap L^2(\mathbb{R}^N_v)$.

The proof of this estimate used the theory of Fourier integral operators. The very restricting conditions on B were not a serious inconvenience since in the application to the inhomogeneous Boltzmann equation, only the strong compactness in L^1 of $Q^+(f)$ was used, and not the estimate itself, so that these assumptions could be relaxed by a suitable approximation of B.

An extension of this work to the case of the relativistic Boltzmann kernel can be found in [6].

Then, a simplified proof of (77) was given by Wennberg (Cf. [78] and [79]) with the help of the regularizing properties of the (generalized) Radon transform. The hypothesis on B were considerably diminished, so that for example forces in r^{-s} with angular cutoff and $s \geq 9$ were included.

We intend here to give a yet simplified proof of (77)-like estimates, using only elementary properties of the Fourier transform. Moreover, we prove that the estimate holds for a large class of cross sections B, including all hard potentials with cutoff (that is when $s \geq 5$).

One of the drawbacks of the results here given is that instead of having a L^1 norm times a L^2 norm in the right-hand side of (77), we only get a L^2 norm to the square. The proofs of this section are extracted from [19]. They are also close of that of [60].

3.2 A simplified situation

We begin with the simplest possible cross section, that is $B \equiv 1$. We only treat here the three-dimensional case for the sake of simplicity (the two-dimensional case is in fact slightly more involved because some part of the computation cannot be written down explicitly).

Our theorem writes :

Theorem 8 *For any $\varepsilon > 0$, there exists a constant C_ε only depending on ε such that for any $f \in L^1_1(I\!R^3) \cap L^2_{(3+\varepsilon)/2}(I\!R^3)$, $Q^+(f) \in \dot{H}^1(I\!R^3)$ with*

$$\|Q^+(f,f)\|_{\dot{H}^1(I\!R^3)} \leq C_\varepsilon \, \|f\|^2_{L^2_{(1+\varepsilon)/2}}. \tag{78}$$

Proof: We note first that for all $f \in L^1_1(I\!R^3) \cap L^2_{(3+\varepsilon)/2}(I\!R^3)$, the kernel $Q^+(f,f)$ lies in $L^1(I\!R^3)$.

Therefore, we can compute the Fourier transform of $Q^+(f,f)$,

$$Q^+\widehat{(f,f)}(\xi) = \iiint_{\substack{v,v_* \in I\!R^3 \\ \sigma \in S^2}} e^{-iv\cdot\xi} f\left(\frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma\right)$$

$$\times f\left(\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma\right)\,d\sigma dv dv_* \tag{79}$$

$$= \iiint\limits_{\substack{v,v_*\in\mathbb{R}^3\\ \sigma\in S^2}} e^{-i\xi\cdot(v+v_*-|v-v_*|\sigma)/2} f(v)f(v_*)\,d\sigma dv dv_*,$$

according to the pre-post collisional change of variables.

We then note that

$$\int\limits_{\sigma\in S^2} e^{i|v-v_*|\sigma\cdot\xi/2}\,d\sigma = 2\pi\int_{u=-1}^{+1} e^{i|v-v_*||\xi|u/2}\,du$$

$$= 8\pi\frac{\sin(\frac{1}{2}|v-v_*|\,|\xi|)}{|v-v_*|\,|\xi|}.$$

Thus we obtain

$$\widehat{Q^+(f,f)}(\xi) = 8\pi \iint\limits_{v,v_*\in\mathbb{R}^3} e^{-i\xi\cdot(v+v_*)/2} f(v)f(v_*)\frac{\sin(\frac{1}{2}|v-v_*|\,|\xi|)}{|v-v_*|\,|\xi|}\,dv dv_*. \tag{80}$$

Using the variables

$$z=\frac{v+v_*}{2}, \qquad w=v-v_*, \tag{81}$$

we get

$$\widehat{Q^+(f,f)}(\xi) = 8\pi\int\int_{\mathbb{R}^3\times\mathbb{R}^3} e^{-i\,z\cdot\xi}\, f(z+w/2)f(z-w/2)$$

$$\times\frac{\sin(\frac{1}{2}|w|\,|\xi|)}{|w|\,|\xi|}\,dw dz$$

$$= \frac{8\pi}{|\xi|}\int_{\mathbb{R}^3} \widehat{f(\cdot+\frac{w}{2})f(\cdot-\frac{w}{2})}\,\frac{\sin(\frac{1}{2}|w|\,|\xi|)}{|w|}\,dw.$$

According to Cauchy-Schwarz's inequality and Plancherel's identity,

$$\int_{\mathbb{R}^3}|\xi|^2\,|\widehat{Q^+(f)}(\xi)|^2\,d\xi \le 64\,\pi^2\int_{\mathbb{R}^3}\frac{dw}{|w|^2\,(1+|w|)^{1+\varepsilon}}$$

$$\times\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\widehat{f(\cdot+\frac{w}{2})f(\cdot-\frac{w}{2})}(\xi)|^2 d\xi\,(1+|w|)^{1+\varepsilon}\,dw$$

$$\leq C_\varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f(z+\frac{w}{2})f(z-\frac{w}{2})|^2\,(1+|w|)^{1+\varepsilon}\,dwdz$$
$$\leq C_\varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)^2\,f(v_*)^2\,(1+|v-v_*|)^{1+\varepsilon}\,dvdv_*$$
$$\leq C_\varepsilon\,\|f\|^4_{L^2_{(1+\varepsilon)/2}}.$$

and the proof is complete. □

3.3 General cutoff cross sections

We now turn to the general case, that is when cutoff hard potentials (or Maxwellian molecules) are considered (note that assumption (82) below is satisfied only by potentials gently cutoff).

The proof, extracted from [19], follows the same lines as that of the previous section, but is slightly more involved. We still only consider dimension three.

Theorem 9 *Let B be a continuous cross section from $]0,\infty[\times[-1,1]$ to $\mathbb{R}$, admitting a continuous derivative in the second variable. We assume that B satisfies the estimate :*

$$\forall x>0,\quad \forall u\in[-1,1],\qquad \Big|B(x,u)\Big| + \Big|\frac{\partial B}{\partial u}(x,u)\Big| \leq K_B(1+x). \tag{82}$$

Then, for any $\varepsilon>0$, there exists a constant C_ε only depending on ε such that for any $f\in L^1_1(\mathbb{R}^3)\cap L^2_{(3+\varepsilon)/2}(\mathbb{R}^3)$, $Q^+(f,f)\in \dot{H}^1(\mathbb{R}^3)$ with

$$\|Q^+(f,f)\|_{\dot{H}^1(\mathbb{R}^3)} \leq C_\varepsilon\,K_B\,\|f\|^2_{L^2_{(3+\varepsilon)/2}}. \tag{83}$$

Proof : We first note that since $|B(x,u)|\leq K_B(1+x)$, the integral (76) defining $Q^+(f,f)$ is absolutely convergent for *a.e.* v. Moreover, $Q^+(f,f)\in L^1(\mathbb{R}^3)$, and

$$\|Q^+(f,f)\|_{L^1} \leq 4\pi\,K_B\,\|f\|^2_{L^1_1}. \tag{84}$$

Therefore, we can compute the Fourier transform of $Q^+(f,f)$,

$$\widehat{Q^+(f,f)}(\xi) = \iiint_{\substack{v,v_*\in\mathbb{R}^3\\ \sigma\in S^2}} e^{-iv\cdot\xi}\,f\left(\frac{v+v_*}{2}-\frac{|v-v_*|}{2}\sigma\right)\,f\left(\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma\right) \tag{85}$$

$$\times B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\, d\sigma dv dv_*$$

$$= \iiint_{\substack{v,v_*\in \mathbb{R}^3\\ \sigma\in S^2}} e^{-i\xi\cdot(v+v_*-|v-v_*|\sigma)/2} f(v)f(v_*)\, B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\, d\sigma dv dv_*,$$

according to the pre-post collisional change of variables. Thus we obtain

$$\widehat{Q^+(f,f)}(\xi) = \iint_{v,v_*\in\mathbb{R}^3} e^{-i\xi\cdot(v+v_*)/2}\, f(v)f(v_*)\, D(v-v_*,\xi)\, dv dv_*, \tag{86}$$

where for any $w, \xi \in \mathbb{R}^3\backslash\{0\}$

$$D(w,\xi) = \int_{\sigma\in S^2} e^{i|w|\sigma\cdot\xi/2} B\left(|w|, \frac{w}{|w|}\cdot\sigma\right)\, d\sigma \tag{87}$$

$$= \int_{u=-1}^{+1} e^{i|w||\xi|u/2} \int_{\varphi=0}^{2\pi}$$

$$B\left(|w|\,,\ u\,\frac{\xi}{|\xi|}\cdot\frac{w}{|w|} + \sqrt{1-u^2}\sqrt{1-\left(\frac{\xi}{|\xi|}\cdot\frac{w}{|w|}\right)^2}\cos\varphi\right)\, d\varphi\, du,$$

with spherical coordinates and

$$u = \sigma\cdot\frac{\xi}{|\xi|}. \tag{88}$$

Integrating by parts, we get

$$D(w,\xi) = -\int_{u=-1}^{+1} \frac{2e^{i|w||\xi|u/2}}{i|w||\xi|}$$

$$\times\int_{\varphi=0}^{2\pi}\left(\frac{\xi}{|\xi|}\cdot\frac{w}{|w|} - \frac{u}{\sqrt{1-u^2}}\sqrt{1-\left(\frac{\xi}{|\xi|}\cdot\frac{w}{|w|}\right)^2}\cos\varphi\right)$$

$$\times\frac{\partial B}{\partial u}\left(|w|\,,\ u\,\frac{\xi}{|\xi|}\cdot\frac{w}{|w|} + \sqrt{1-u^2}\sqrt{1-\left(\frac{\xi}{|\xi|}\cdot\frac{w}{|w|}\right)^2}\cos\varphi\right)\, d\varphi\, du$$

$$+\frac{2e^{i|w||\xi|/2}}{i|w||\xi|}\, 2\pi B\left(|w|, \frac{\xi}{|\xi|}\cdot\frac{w}{|w|}\right) - \frac{2e^{-i|w||\xi|/2}}{i|w||\xi|}\, 2\pi B\left(|w|, -\frac{\xi}{|\xi|}\cdot\frac{w}{|w|}\right), \tag{89}$$

and therefore

$$|D(w,\xi)| \le \frac{4\pi}{|w||\xi|} K_B (1+|w|) \int_{-1}^{+1} \left(1 + \frac{|u|}{\sqrt{1-u^2}}\right) du \tag{90}$$

$$+\frac{8\pi}{|w||\xi|} K_B (1+|w|)$$

$$\le \frac{24\pi}{|\xi|} K_B (1+1/|w|).$$

Coming back to (86) and using the variables

$$z = \frac{v+v_*}{2}, \qquad w = v - v_*, \tag{91}$$

we get

$$\widehat{Q^+(f,f)}(\xi) = \int_{w\in \mathbb{R}^3} W(f)(w,\xi) D(w,\xi)\, dw, \tag{92}$$

where

$$W(f)(w,\xi) = \int_{z\in \mathbb{R}^3} e^{-iz\cdot\xi} f(z+w/2) f(z-w/2)\, dz \tag{93}$$

is a Wigner-type transform of f. Then, according to Cauchy-Schwarz's inequality, we get for any $\varepsilon > 0$

$$\left|\widehat{Q^+(f,f)}(\xi)\right|^2 \le \int_{w\in \mathbb{R}^3} |W(f)(w,\xi)|^2 (1+|w|)^{3+\varepsilon}\, dw$$

$$\times \int_{w\in \mathbb{R}^3} |D(w,\xi)|^2 \frac{dw}{(1+|w|)^{3+\varepsilon}} \tag{94}$$

$$\le C_\varepsilon \frac{K_B^2}{|\xi|^2} \int_{w\in \mathbb{R}^3} |W(f)(w,\xi)|^2 (1+|w|)^{3+\varepsilon}\, dw.$$

Finally, using Plancherel's identity, we obtain

$$\int_{\xi\in \mathbb{R}^3} |\xi|^2 \left|\widehat{Q^+(f,f)}(\xi)\right|^2 d\xi \tag{95}$$

$$\le C_\varepsilon K_B^2 \int_{w\in \mathbb{R}^3} \left(\int_{\xi\in \mathbb{R}^3} |W(f)(w,\xi)|^2 \, d\xi \right) (1+|w|)^{3+\varepsilon}\, dw$$

$$= C_\varepsilon K_B^2 (2\pi)^3 \int\limits_{w\in\mathbb{R}^3} \left(\int\limits_{z\in\mathbb{R}^3} |f(z+w/2)f(z-w/2)|^2 \, dz \right) (1+|w|)^{3+\varepsilon} \, dw$$

$$= C_\varepsilon K_B^2 (2\pi)^3 \iint\limits_{v,v_*\in\mathbb{R}^3} |f(v)f(v_*)|^2 \, (1+|v-v_*|)^{3+\varepsilon} \, dv dv_*$$

$$\leq C_\varepsilon K_B^2 (2\pi)^3 \|f\|^4_{L^2_{(3+\varepsilon)/2}},$$

and the proof is complete. □

Note that assumption (82) can be relaxed (in order to treat (not too) soft potentials for example). The estimate is then not as good as in the previous theorem (Cf. [19] for more details).

3.4 Propagation of Singularities for the spatially homogeneous Boltzmann equation

The results obtained in the previous subsections can be directly applied to the study of the propagation of singularities for the spatially homogeneous Boltzmann equation.

This is due to the fact that as soon as the cross section is cutoff, the Boltzmann operator Q can be written under the form

$$Q(f,f) = Q^+(f,f) - f\,Lf,$$

where Q^+ is defined by (18) and

$$Lf(v) = (A * f)(v),$$

with

$$A(x) = \int_{\sigma\in S^{N-1}} B(x, \frac{x}{|x|}\cdot\sigma) \, d\sigma.$$

As a consequence, a solution of (25), (26) can be written under the "Duhamel" or mild form

$$f(t) = f_{in} \exp\left(- \int_0^t (A*f)(\tau) d\tau \right)$$
$$+ \int_0^t Q^+(f,f)(s) \exp\left(- \int_s^t (A*f)(\tau) d\tau \right) ds. \tag{96}$$

Let us look for example at cross sections like

$$B(x,u) = |x|^\alpha \, b(u),$$

where $\alpha \in]0,1[$ and b is of class C^1 on $[-1,1]$ (that is, typical cutoff hard potentials).

We consider solutions of (25), (26) which lie in $L^\infty(I\!R_+; L^2_s(I\!R^N))$ for some large s (such solutions are known to exist as soon as the initial datum also lie in the same space).

Then, for all $s, \tau \geq 0$, $(A * f)(\tau) \in H^{N/2+\alpha}_{loc}(I\!R^N)$ and $Q^+(f,f)(s) \in H^{(N-1)/2}(I\!R^N)$. According to formula (96), we see that for all $t \geq 0$, and $p \leq (N-1)/2$,

$$f_{in} \in H^p_{loc}(I\!R^N) \qquad \Longleftrightarrow \qquad f(t) \in H^p_{loc}(I\!R^N).$$

This can be seen as a theorem of propagation of singularities. As can be deduced from formula (96), the singularities of the initial datum are propagated (in a trivial way : they stay at the same position in the space of velocities) and decrease exponentially fast. Such a behavior is confirmed by numerical simulations.

4 Propagation of Singularities for the spatially inhomogeneous Boltzmann equation

4.1 Introduction

In this section, we investigate the smoothness (more precisely, the lack of smoothness, that is, the singularities) of the solution of the full cutoff Boltzmann equation (16).

In the sequel, we shall in fact limit ourselves to cross sections B which satisfy the following assumption:

Assumption 1. The nonnegative cross section B lies in $W^{1,\infty}(I\!R_+ \times [-1,1])$.

We denote as in the previous section

$$A(x) = \int_{\sigma \in S^{N-1}} B(x, \frac{x}{|x|} \cdot \sigma) \, d\sigma,$$

and

$$Q(f,f) = Q^+(f,f) - f \, Lf.$$

Note that the classical cross sections of (cutoff) Maxwellian molecules or (cutoff) regularized soft potentials satisfy this assumption. The case of (cutoff) hard potentials, which do not satisfy assumption 1 because of the large relative velocities, is briefly discussed in a remark at the end of section 2.

In this section, we shall deal with solutions of the full Boltzmann equation (16), for which many kinds of solutions exist.

Global renormalized solutions have been proven to exist for a large class of initial data by DiPerna and P.-L. Lions in [33] (Cf. also [54] and [55]). Global solutions (in the whole space) close to the equilibrium have been studied by Imai and Nishida in [46] and Ukai and Asano in [71]. Finally, global solutions for small initial data were introduced by Kaniel and Shinbrot (Cf. [48]) and studied by Bellomo and Toscani (Cf. [11]), Goudon (Cf. [41]), Hamdache (Cf. [43]), Illner and Shinbrot (Cf. [45]), Mischler and Perthame (Cf. [61]), Polewczak (Cf. [65]) and Toscani (Cf. [68]).

In our study of how the singularities of the initial datum are propagated by the Boltzmann equation, we need some smoothness (basically, we need that f be L^∞ with some decay in x, v), and we shall therefore concentrate on the framework of small initial data, where such estimates are available. We think that our work is likely to extend to solutions close to the equilibrium, but we shall not investigate this case.

We consider only the dimension three for the sake of simplicity.

We recall here one of the theorems of existence of such small solutions. We use a formulation adapted to our study, which is inspired from [61].

Theorem 10 *Let B be a cross section satisfying assumption 1 and f_{in} be an initial datum such that, for all $x, v \in \mathbb{R}^3 \times \mathbb{R}^3$,*

$$0 \le f_{in}(x,v) \le (81\, \|A\|_{L^\infty})^{-1} \exp\Big(-\frac{1}{2}(|x|^2 + |v|^2)\Big). \tag{97}$$

Then there exists a global distributional solution f to Boltzmann equation (16) with initial datum f_{in}, such that, for all $T > 0$, $t \in [0,T]$ and $x, v \in \mathbb{R}^3 \times \mathbb{R}^3$,

$$0 \le f(t,x,v) \le C_T \exp\Big(-\frac{1}{2}(|x - vt|^2 + |v|^2)\Big) := M_T(t,x,v), \tag{98}$$

where C_T is a constant only depending on T and $\|A\|_{L^\infty}$.

We now state the main result of this section. It concerns the form of the singularities of the solution of the Boltzmann equation (in our setting), and is extracted from [21]. An analogous result in a different setting can be found in [7].

Theorem 11 *Let B be a cross section satisfying assumption 1 and f_{in} be an initial datum such that (97) holds. Then we can write, for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$,*

$$f(t,x,v) = f_{in}(x-vt)\,\Gamma_1(t,x,v) + \Gamma_2(t,x,v),$$

where $\Gamma_1, \Gamma_2 \in H^{\alpha}_{loc}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ for all $\alpha \in]0, 1/25[$.

This theorem shows that the singularities of the initial datum (that is, for example, the points around which $f_{\rm in}$ is in L^2 but not in H^s for any $s > 0$) are propagated with the free flow, and decrease exponentially fast (since in fact Γ_1 has an exponential decay).

In particular, an x-dependant version of the result of subsection 3.4 holds. Namely, for all $t \geq 0$ and $s < 1/25$,

$$f(t) \in H^s(\mathbb{R}^3 \times \mathbb{R}^3) \qquad \Longleftrightarrow \qquad f_{\rm in} \in H^s(\mathbb{R}^3 \times \mathbb{R}^3).$$

The proof of theorem 11 uses the regularizing properties of the kernel Q^+ presented in the previous section. We recall that they were first studied by P.-L. Lions in [58], and extended by Wennberg in [78], [79], by Bouchut and Desvillettes in [19], and by Lu in [60]. We also recall that those properties are exactly what is needed to give the form of the singularities of the solutions to the *spatially homogeneous* cutoff Boltzmann equation (this is the result of subsection 3.4). In order to conclude in our inhomogeneous setting, we also have to use the averaging lemmas of Golse, P.-L. Lions, Perthame and Sentis (Cf. [38]), in the form of theorem 5.

Proof : We briefly sketch the proof of theorem 11 before detailing it. The main idea is the following : we write down the Duhamel form of the solution of the Boltzmann equation (as in the spatially homogeneous case), also called the mild exponential form. For $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$, we have

$$f(t,x,v) \quad = \quad f_{\rm in}(x-vt,v)\exp\left(-\int_0^t Lf(\sigma, x-v(t-\sigma), v)\,d\sigma\right)$$

$$+\int_0^t \Big[Q^+(f,f)(s,x-v(t-s),v) \tag{99}$$
$$\times \exp\Big(-\int_s^t Lf(\sigma,x-v(t-\sigma),v)\,d\sigma\Big)\Big]ds.$$

We are going to prove that both Lf and $Q^+(f,f)$ lie in $L^2_{\rm loc}(I\!R_+;H^\alpha_{\rm loc}(I\!R^3\times I\!R^3))$ for any $\alpha\in]0,1/25[$.

We now begin to give a detailed proof. Next subsection is devoted to the study of the regularity of Lf.

4.2 Regularity of Lf

Denoting by $\mathcal{B}_R$ the ball of radius R and center 0 in $I\!R^3$, we prove the following intermediate result :

Proposition 1 *Suppose that B satisfies assumption 1 and that f_{in} is such that (97) holds. Then, for any $T>0$ and $R>0$, there exists $K_{T,R}>0$ such that*

$$||Lf||_{L^2([0,T];H^{1/2}(\mathcal{B}_R\times\mathcal{B}_R))}\leq K_{T,R}\,||A||_{L^\infty(I\!R^3)}.$$

Proof : Let us choose $T>0$. Since Lf is the convolution with respect to v by A, we obviously have that, under assumption 1, $Lf\in L^2([0,T]_t\times I\!R^3_x;H^{1/2}_{\rm loc}(I\!R^3_v))$ (in fact, Lf lies in $L^2([0,T]_t\times I\!R^3_x;W^{1,\infty}_{\rm loc}(I\!R^3_v)))$ and satisfies

$$||Lf||_{L^2([0,T]\times\mathcal{B}_R;H^{1/2}(\mathcal{B}_R))}\leq K'_{T,R}\,||A||_{W^{1,\infty}(I\!R^3)}.$$

It remains to prove that $Lf\in L^2([0,T]_t\times I\!R^3_v;H^{1/2}_{\rm loc}(I\!R^3_x))$.

Let us define the function T_λ, $0<\lambda<1/2$, by $T_\lambda(v_*)=\mathrm{e}^{-\lambda v_*}$, and study the following quantity

$$||Lf||^2_{L^2([0,T]_t\times I\!R^3_v;H^{1/2}(I\!R^3_x))} \tag{100}$$
$$=\int_{t,v}\int_{x,h}\Big|\int_{v_*}A(v-v_*)(f(t,x+h,v_*)-f(t,x,v_*))dv_*\Big|^2dx\frac{dh}{|h|^4}dvdt.$$

We want to use theorem 5, which we here recall under the form :

Lemma 2 *Let $f \in C([0,T]_t; L^2_w(\mathbb{R}^3_x \times \mathbb{R}^3_{v_*}))$ solve the equation*

$$\partial_t f + v_* \cdot \nabla_x f = g \qquad in \qquad]0,T[\times \mathbb{R}^3 \times \mathbb{R}^3,$$

for some $g \in L^2([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$.

Then, for any $\psi \in \mathcal{D}(\mathbb{R}^3)$, the average quantity defined by

$$\rho_\psi(f)(t,x) = \int_{v_* \in \mathbb{R}^3} f(t,x,v_*)\, \psi(v_*)\, dv_*$$

belongs to $L^2([0,T]; H^{1/2}(\mathbb{R}^3))$ and satisfies, for any $s > 1$,

$$\|\rho_\psi(f)\|^2_{L^2([0,T];H^{1/2}(\mathbb{R}^3))} \le C_s \left[\int_{x,v} |f(0,x,v_*)|^2 |\psi(v_*)|^2 (1+|v_*|^2)^s dv_* dx \right.$$
$$\left. + \int_{t,x,v_*} |g(t,x,v_*)|^2 |\psi(v_*)|^2 (1+|v_*|^2)^s dv_* dx dt \right],$$

where C_s is a constant only depending on s.

Using lemma 2, eq. (100) becomes, for any $s > 1$ and any open ball $\mathcal{B}_R$ of $\mathbb{R}^3$,

$$\begin{aligned}
&\|Lf\|^2_{L^2([0,T]_t \times \mathcal{B}_{R_v}; H^{1/2}(\mathbb{R}^3_x))} \\
&\le \int_{v \in \mathcal{B}_R} \left\| \rho_{A(v-\cdot)T_\lambda}\left(\frac{f}{T_\lambda}\right) \right\|^2_{L^2([0,T];H^{1/2}(\mathbb{R}^3))} dv \\
&\le C_s \int_{v \in \mathcal{B}_R} \left[\int_{x,v_*} \left|\frac{f_{\text{in}}(x,v_*)}{T_\lambda(v_*)}\right|^2 |A(v-v_*)|^2 |T_\lambda(v_*)|^2 (1+|v_*|^2)^s dv_* dx \right. \\
&\qquad + \int_{t,x,v_*} \left|(\partial_t + v_* \cdot \nabla_x) \frac{f}{T_\lambda}\right|^2 \\
&\qquad\qquad \left. \times |A(v-v_*)|^2 |T_\lambda(v_*)|^2 (1+|v_*|^2)^s dv_* dx dt \right] dv \\
&\le C_{R,s} {M_{\lambda,s}}^2 \|A\|^2_{L^\infty(\mathbb{R}^3)} \qquad\qquad (101) \\
&\qquad \times \left(\left\| \frac{f_{\text{in}}}{T_\lambda} \right\|^2_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} + \left\| (\partial_t + v \cdot \nabla_x) \frac{f}{T_\lambda} \right\|^2_{L^2([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)} \right),
\end{aligned}$$

where $C_{R,s}$ is a constant and

$$M_{\lambda,s} = \sup_{v_* \in \mathbb{R}^3} |T_\lambda(v_*)(1+|v_*|^2)^{s/2}|. \qquad (102)$$

Note that, since we have (97), the following estimate holds :

$$0 \le \frac{f_{\text{in}}(x,v)}{T_\lambda(v)} \le \kappa \;\; e^{-|x|^2/2} \; e^{(\lambda-1/2)|v|^2},$$

where κ is an absolute constant, so that (recall that $0 < \lambda < 1/2$) we can find a constant $C_\lambda > 0$ such that

$$\left\| \frac{f_{\text{in}}}{T_\lambda} \right\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \le C_\lambda. \tag{103}$$

Moreover, we have

$$\left| (\partial_t + v \cdot \nabla_x) \frac{f}{T_\lambda} \right| \le \frac{|Q^+(f,f)|}{T_\lambda} + \frac{|fLf|}{T_\lambda}. \tag{104}$$

It is clear, by (98), that

$$\begin{aligned} \frac{|f(t,x,v)\; Lf(t,x,v)|}{T_\lambda(v)} &\le \frac{M_T(t,x,v)\; LM_T(t,x,v)}{T_\lambda(v)} \\ &\le C_T'^2 \, (2\pi)^{3/2} \, \|A\|_{L^\infty} \, e^{-\frac{1}{2}|x-vt|^2} \, e^{(\lambda-\frac{1}{2})|v|^2}. \end{aligned}$$

Hence there exists a constant C_λ such that

$$\left\| \frac{fLf}{T_\lambda} \right\|_{L^2([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)} \le C_\lambda. \tag{105}$$

It is also clear that, for $(t,x,v) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$,

$$\begin{aligned} \frac{|Q^+(f,f)(t,x,v)|}{T_\lambda(v)} &= \frac{1}{T_\lambda(v)} \left| \int_{v_*,\sigma} f(t,x,v') f(t,x,v'_*) \; B \, d\sigma dv_* \right| \\ &\le \frac{Q^+(M_T, M_T)(t,x,v)}{T_\lambda(v)} \\ &= \frac{M_T(t,x,v)\; LM_T(t,x,v)}{T_\lambda(v)}, \end{aligned}$$

so that

$$\left\| \frac{Q^+(f,f)}{T_\lambda} \right\|_{L^2([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)} \le C_\lambda. \tag{106}$$

Taking (105)–(106) into account, (104) implies that

$$\left\| (\partial_t + v \cdot \nabla_x) \frac{f}{T_\lambda} \right\|_{L^2([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)} \le C_\lambda. \tag{107}$$

Then, using (103) and (107) in (101), we get

$$\|Lf\|^2_{L^2([0,T]_t\times I\!R^3_v;H^{1/2}(I\!R^3_x))} \le C_s {C_\lambda}^2 {M_{\lambda,s}}^2 \|A\|^2_{L^\infty}.$$

Recalling that $Lf \in L^2([0,T]_t \times I\!R^3_x; H^{1/2}_{\text{loc}}(I\!R^3_v))$, we finally obtain that

$$Lf \in L^2([0,T]; H^{1/2}_{\text{loc}}(I\!R^3_x \times I\!R^3_v)), \tag{108}$$

which ends the proof of proposition 1. $\square$

We now turn to the more complicated term $Q^+(f,f)$.

4.3 Regularity of $Q^+(f,f)$

Studying $Q^+(f,f)$, a new difficulty arises when we try to prove that this term is (somewhat) smooth in x, v. Namely, $Q^+(f,f)$ itself cannot easily be expressed in terms of averages in v of f, whereas it was possible for Lf in the previous section.

However, its own averages in v (that is, for ζ smooth, quantities like $\int_v Q^+(f,f)(t,x,v)\,\zeta(v)\,dv$) can be expressed in terms of averages in v of f. More precisely, they are integrals with respect to an auxiliary parameter of such averages in v.

Therefore, the strategy of proof is now the following : in a first step, we show that averages in v of $Q^+(f,f)$ are somewhat smooth in x, and we keep track of the averaging function ζ in the estimate which expresses this smoothness. Then, in a second step, we approximate $Q^+(f,f)$ by $Q^+(f,f) *_v \zeta_\varepsilon$, where ζ_ε is a smoothing family of functions. The quantity $Q^+(f,f) *_v \zeta_\varepsilon$ is (somewhat) smooth in x according to the first step. It simply remains to use the properties of smoothness in v of $Q^+(f,f)$ (that is, the results of the previous section) to control the difference between $Q^+(f,f)$ and $Q^+(f,f) *_v \zeta_\varepsilon$, and to optimize the parameter ε.

We begin with the first part of this program.

4.3.1 Study of the averages (in velocity) of $Q^+(f,f)$

This part is devoted to the proof of the

Proposition 2 *Let $\zeta \in \mathcal{D}(\mathbb{R}^3_v)$, B satisfying assumption 1, and f_{in} such that (97) holds. Then we have, for any $T > 0$ and $h \in \mathbb{R}^3$,*

$$\int_{t,x} \left| \int_v [Q^+(f,f)(t,x+h,v) - Q^+(f,f)(t,x,v)]\zeta(v)dv \right|^2 dx dt$$
$$\leq K_T \, \|\zeta\|^2_{W^{1,\infty}(\mathbb{R}^3)} \, |h|^{2/5}, \tag{109}$$

where K_T is a constant that depends on T (more precisely on the constant C_T in (98) and on $\|B\|_{W^{1,\infty}(\mathbb{R}_+ \times [0,1])}$).

Proof: Let $\zeta \in \mathcal{D}(\mathbb{R}^3_v)$. We have

$$\int_{\mathbb{R}^3} Q^+(f,f)(v)\zeta(v) \, dv = \int_{v,v_*,\sigma} f(v')f(v'_*) \, B \, \zeta(v) d\sigma dv_* dv. \tag{110}$$

By changing pre/post collisional variables, eq. (110) becomes

$$\int_{\mathbb{R}^3} Q^+(f,f)(v) \, \zeta(v) \, dv = \int_{v,v_*} f(v)f(v_*) \int_\sigma B \, \zeta(v') \, d\sigma \, dv_* dv. \tag{111}$$

Let us set

$$Z(v,v_*) = \int_\sigma B \, \zeta(v') d\sigma, \tag{112}$$

which depends neither on t nor on x and belongs to $L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. As a matter of fact, we have

$$\|Z\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \leq 4\pi \, \|B\|_{L^\infty(\mathbb{R}^3 \times S^2)} \, \|\zeta\|_{L^\infty(\mathbb{R}^3)}.$$

Note that we still cannot directly express the quantity $\int_{\mathbb{R}^3} Q^+(f,f)(v)\zeta(v)dv$ in terms of averages in v of f, because Z is not a tensor product. As a consequence, we approximate Z by (integrals) of such tensor product.

This is done by taking a mollifying sequence $(\psi_\varepsilon)_{\varepsilon>0}$ of functions of v. Thanks to (111), we get

$$\int_{\mathbb{R}^3} Q^+(f,f)(v) \, \zeta(v) \, dv = \int_{v,v_*} f(v)f(v_*) \Big(\int_{w,w_*} Z(w,w_*)$$
$$\times \psi_\varepsilon(v-w)\psi_\varepsilon(v_*-w_*) dw_* dw \Big) \, dv_* dv$$
$$+ \int_{v,v_*} f(v)f(v_*) \left[\int_{w,w_*} (Z(v,v_*) - Z(w,w_*)) \right.$$

$$\times\, \psi_\varepsilon(v-w)\psi_\varepsilon(v_*-w_*)dw_*dw\Big]dv_*dv. \tag{113}$$

We name I_1 (respectively I_2) the first (respectively second) integral in (113). They are functions of $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^3$.

• *Estimate on* I_1.

The integral I_1 can be rewritten as

$$I_1 = \int_{w,w_*} Z(w,w_*)\, \rho_{\psi_\varepsilon(\cdot - w)}(f)(t,x)\, \rho_{\psi_\varepsilon(\cdot - w_*)}(f)(t,x)\, dw_*dw,$$

where $\rho_\psi(f)$ denotes the average quantity of f with respect to ψ.

Let us study the norm $\|\tau_h I_1 - I_1\|_{L^2([0,T]\times\mathbb{R}^3)}$, for $h \in \mathbb{R}^3$, with the notation $\tau_h g(x) = g(x+h)$.

The following equality holds :

$$\int_{t,x} |\tau_h I_1 - I_1|^2 dxdt$$

$$= \int_{t,x}\left|\int_{w,w_*} Z(w,w_*)[\rho_{\psi_\varepsilon(\cdot-w)}(f)(t,x+h)\rho_{\psi_\varepsilon(\cdot-w_*)}(f)(t,x+h)\right.$$
$$\left. -\rho_{\psi_\varepsilon(\cdot-w)}(f)(t,x)\rho_{\psi_\varepsilon(\cdot-w_*)}(f)(t,x)]dw_*dw\right|^2 dxdt.$$

We immediately get

$$\int_{t,x}|\tau_h I_1 - I_1|^2 dxdt \leq C\,\|Z\|^2_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \int_{t,x} dtdx \Big| \int_{w,w_*}$$
$$\Big|(\rho_{\psi_\varepsilon(\cdot-w)}(f)(t,x+h) - \rho_{\psi_\varepsilon(\cdot-w)}(f)(t,x))\rho_{\psi_\varepsilon(\cdot-w_*)}(f)(t,x+h)$$
$$+\,\rho_{\psi_\varepsilon(\cdot-w)}(f)(t,x)\,(\rho_{\psi_\varepsilon(\cdot-w_*)}(f)(t,x+h) - \rho_{\psi_\varepsilon(\cdot-w_*)}(f)(t,x))\Big|dw_*dw\Big|^2$$
$$\leq C\,\|Z\|^2_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}\left[\int_{t,x} dtdx\,\Big|\int_{w,w_*}\Big|((\tau_h - \mathrm{Id})\rho_{\psi_\varepsilon(\cdot-w)}(f))(t,x)\right.$$
$$\times\ \tau_h\rho_{\psi_\varepsilon(\cdot-w_*)}(f)(t,x)\Big|dw_*dw\Big|^2 dxdt$$
$$+\int_{t,x} dtdx\,\Big|\int_{w,w_*}\Big|((\tau_h - \mathrm{Id})\rho_{\psi_\varepsilon(\cdot-w_*)}(f))(t,x)$$

$$\rho_{\psi_\varepsilon(\cdot - w)}(f)(t,x)\Big| dw_* dw \Big|^2 dx dt\Big].$$

In the previous inequality, the two terms can be similarly treated. For example, let us study the second one, which we name J.

$$J = \int_{t,x} \left(\int_w \rho_{\psi_\varepsilon(\cdot - w)}(f)(t,x) dw \right)^2$$

$$\times \left(\int_{w_*} \Big| ((\tau_h - \mathrm{Id}) \rho_{\psi_\varepsilon(\cdot - w_*)}(f))(t,x) \Big| dw_* \right)^2 dx dt$$

$$\leq C_T \int_{t,x} \left(\int_{w_*} \Big| ((\tau_h - \mathrm{Id}) \rho_{\psi_\varepsilon(\cdot - w_*)}(f))(t,x) \Big| dw_* \right)^2 dx dt,$$

where C_T is the constant in (98). Let us choose $0 < \theta < \lambda < 1/2$. Using the notation T_λ as in subsection 4.2, we have

$$J \leq C_T \left(\int_{w_*} \mathrm{e}^{-\theta |w_*|^2} dw_* \right)$$

$$\times \left(\int_{t,x,w_*} ((\tau_h - \mathrm{Id}) \rho_{\psi_\varepsilon(\cdot - w_*)}(f))^2 (t,x) \mathrm{e}^{\theta |w_*|^2} dw_* dx dt \right)$$

$$\leq C_{T,\theta} |h| \int_{w_*} dw_* \ \mathrm{e}^{\theta |w_*|^2} \left\| \rho_{\psi_\varepsilon(\cdot - w_*) T_\lambda} \Big(\frac{f}{T_\lambda} \Big) \right\|^2_{L^2([0,T]; H^{1/2}(\mathbb{R}^3))}.$$

Then, thanks to the averaging lemma (lemma 2), we obtain

$$\begin{aligned} J \ \leq \ & C_{T,\theta,s} |h| \int_{w_*} dw_* \ \mathrm{e}^{\theta |w_*|^2} \\ & \times \Bigg[\int_{x,v_*} \frac{f_{\mathrm{in}}(x,v_*)^2}{T_\lambda(v_*)^2} \psi_\varepsilon(v_* - w_*)^2 T_\lambda(v_*)^2 (1 + |v_*|^2)^s dv_* dx \\ & + \int_{t,x,v_*} ((\partial_t + v_* \cdot \nabla_x) \frac{f}{T_\lambda})(t,x,v_*)^2 \\ & \times \psi_\varepsilon(v_* - w_*)^2 T_\lambda(v_*)^2 (1 + |v_*|^2)^s dv_* dx dt \Bigg]. \end{aligned}$$

Let us take care of the term with f_{in} (the other one is treated in the same way thanks to (107)). We notice that, for any $w_* \in \mathcal{B}(v_*, \varepsilon)$,

$$\mathrm{e}^{\theta |w_*|^2} \leq \mathrm{e}^{2\theta |v_*|^2} \mathrm{e}^{2\theta \varepsilon^2}.$$

We thus have

$$\int_{w_*} \mathrm{e}^{\theta |w_*|^2} \int_{x,v_*} \frac{f_{\mathrm{in}}(x,v_*)^2}{T_\lambda(v_*)^2} \psi_\varepsilon(v_* - w_*)^2 T_\lambda(v_*)^2 (1 + |v_*|^2)^s dv_* dx dw_*$$

$$\leq \int_{x,v_*} \frac{f_{\rm in}(x,v_*)^2}{T_\lambda(v_*)^2} T_\lambda(v_*)^2 (1+|v_*|^2)^s$$

$$\times \left(\int_{w_* \in \mathcal{B}(v_*,\varepsilon)} \mathrm{e}^{\theta |w_*|^2} \psi_\varepsilon(v_* - w_*)^2 dw_* \right) dv_* dx$$

$$\leq \int_{x,v_*} \frac{f_{\rm in}(x,v_*)^2}{T_\lambda(v_*)^2} T_{\lambda-\theta}(v_*)^2 (1+|v_*|^2)^s \mathrm{e}^{2\theta\varepsilon^2} \|\psi_\varepsilon\|_{L^2}^2 dv_* dx$$

$$\leq \frac{(\mathrm{e}^\theta M_{\lambda-\theta,s})^2}{\varepsilon^3} \left\| \frac{f_{\rm in}}{T_\lambda} \right\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2,$$

for $0 < \varepsilon < 1$.

Note that we have used that $\|\psi_\varepsilon\|_{L^2}^2 \leq \varepsilon^{-3}$ and $M_{\lambda-\theta,s}$ is defined by (102).

Hence we get, thanks to (103),

$$J \leq \frac{C_{\lambda,\theta,s}}{\varepsilon^3},$$

and finally

$$\|\tau_h I_1 - I_1\|_{L^2([0,T]\times\mathbb{R}^3)}^2 \leq C_{\lambda,\theta,s} \|Z\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}^2 \varepsilon^{-3} |h|. \tag{114}$$

• *Estimate on I_2.*

Let us now study the norm $\|\tau_h I_2 - I_2\|_{L^2([0,T]\times\mathbb{R}^3)}$, with the same notation τ_h as before. We successively have

$$\|\tau_h I_2 - I_2\|_{L^2([0,T]\times\mathbb{R}^3)}^2 = \int_{t,x} dt dx \Big| \int_{v,v_*} (f(t,x+h,v) f(t,x+h,v_*)$$

$$- f(t,x,v) f(t,x,v_*))$$

$$\times \Big(\int_{w,w_*} (Z(v,v_*) - Z(w,w_*)) \psi_\varepsilon(v-w) \psi_\varepsilon(v_*-w_*) dw_* dw \Big) dv_* dv \Big|^2$$

$$\leq C \|Z\|_{W^{1,\infty}(\mathbb{R}^3\times\mathbb{R}^3)}^2 \Big(\int_w |w| \psi_\varepsilon(w) dw \Big)^2$$

$$\times \int_{t,x} dt dx \Big(\int_{v,v_*} (\tau_h + \mathrm{Id}) (|f(t,x,v) f(t,x,v_*)|) dv_* dv \Big)^2. \tag{115}$$

Thanks to (98), the second integral term is bounded by a constant $K_T \geq 0$. Hence there exists a constant $C_T \geq 0$ such that

$$\|\tau_h I_2 - I_2\|^2_{L^2([0,T]\times I\!R^3)} \leq C_T \, \|Z\|^2_{W^{1,\infty}(I\!R^3\times I\!R^3)} \varepsilon^2. \tag{116}$$

• *Estimate on the average quantity.*

Under assumption 1, the following inequality clearly holds :

$$\|Z\|_{W^{1,\infty}(I\!R^3\times I\!R^3)} \leq C \, \|\zeta\|_{W^{1,\infty}(I\!R^3)}, \tag{117}$$

where C is a constant depending on T and $\|B\|_{W^{1,\infty}(I\!R_+\times[-1,1])}$. Consequently, using (113)–(117), we get, for $h \in I\!R^3$,

$$\int_{t,x} \Big| \int_v [Q^+(f,f)(t,x+h,v) - Q^+(f,f)(t,x,v)] \, \zeta(v) \, dv \Big|^2 dx\,dt$$
$$\leq K_T \, \|\zeta\|^2_{W^{1,\infty}(I\!R^3)} \, (\varepsilon^2 + \varepsilon^{-3}|h|),$$

that gives (109), if we choose $\varepsilon = |h|^{1/5}$.

Thus, we conclude the proof of proposition 2. □

4.3.2 Study of $Q^+(f,f)$

We turn back to the proof of our theorem.

Let us once again choose a mollifying sequence $(\psi_\delta)_{\delta>0}$ of functions of v. We obviously have, for all $\delta > 0$,

$$Q^+(f,f) = (Q^+(f,f) - \psi_\delta *_v Q^+(f,f)) + \psi_\delta *_v Q^+(f,f).$$

Note that, thanks to (109), for any $h \in I\!R^3$ and $\delta > 0$,

$$\int_{t,x} \Big| \int_w [Q^+(f,f)(t,x+h,w) - Q^+(f,f)(t,x,w)] \, \psi_\delta(v-w) dw \Big|^2 dx\,dt$$
$$\leq C \, \|\psi_\delta(v-\cdot)\|^2_{W^{1,\infty}(I\!R^3)} |h|^{2/5}$$
$$\leq C \, \delta^{-8} \, |h|^{2/5}. \tag{118}$$

On the other hand, we know that thanks to the regularizing properties of Q^+ (theorem 9), and thanks to the fact that $f \in L^\infty([0,T]\times \mathcal{B}_R; L^2_s(I\!R^3_v))$ (for all $s, R > 0$), $Q^+(f,f) \in L^\infty([0,T]\times \mathcal{B}_R; H^1(I\!R^3_v))$ and therefore

$$\|Q^+(f,f) - \psi_\delta *_v Q^+(f,f)\|_{L^2([0,T]\times\mathcal{B}_R\times\mathcal{B}_R)} \leq C\delta. \tag{119}$$

Using again the translations τ_h in the variable x ($h \in I\!R^3$), and assuming that $|h| \leq 1$, we successively have

$$\int_{(t,x,v)\in[0,T]\times\mathcal{B}_R\times\mathcal{B}_R} \Big|\tau_h Q^+(f,f) - Q^+(f,f)\Big|^2 dv dx dt$$
$$\leq C\Big[\int_{t,x,v}\Big|(Q^+(f,f) - \psi_\delta *_v Q^+(f,f))(t,x,v)\Big|^2 dv dx dt$$
$$+ \int_{t,x,v}\Big|(\tau_h(\psi_\delta *_v Q^+(f,f)) - \psi_\delta *_v Q^+(f,f))(t,x,v)\Big|^2 dv dx dt\Big]$$
$$\leq C_R\,(\delta^2 + |h|^{2/5}\delta^{-8}), \tag{120}$$

thanks to (118)–(119).

Then for a good choice of δ (that is, $\delta = |h|^{1/25}$) in (120), we find the following estimate :

$$\left(\int_0^T \int_{(\mathcal{B}_R)_x}\int_{(\mathcal{B}_R)_v} |\tau_h Q^+(f,f) - Q^+(f,f)|^2 dv dx dt\right)^{1/2} \leq C|h|^{1/25},$$

which ensures that $Q^+(f,f) \in L^2([0,T]\times(\mathcal{B}_R)_v; H^\alpha((\mathcal{B}_R)_x))$, for any $0 < \alpha < 1/25$.

Besides, we know that $Q^+(f,f) \in L^2([0,T]\times(\mathcal{B}_R)_x; H^1((\mathcal{B}_R)_v))$.

Then, by a standard interpolation result, we can state that for all $\alpha \in]0, 1/25[$,

$$Q^+(f,f) \in L^2([0,T]; H^\alpha_{\text{loc}}(I\!R^3 \times I\!R^3)). \tag{121}$$

4.4 Conclusion

Let us now conclude the proof of theorem 11. Note that if we use the notation $f^\#(t,x,v) = f(t,x+vt,v)$, formula (99) is (at least formally) easily rewritten as

$$\begin{aligned} f^\#(t,x,v) \quad = \quad & \exp\Big(-\int_0^t Lf^\#(\sigma,x,v)\,d\sigma\Big)\times \\ & \Big(f_{\text{ in}}(x,v) + \int_0^t \Big[Q^+(f,f)^\#(s,x,v) \\ & \times \exp\Big(\int_0^s Lf^\#(\sigma,x,v)\,d\sigma\Big)\Big]\,ds\Big). \end{aligned} \tag{122}$$

In (122), we name E_1 the first exponential term and E_2 the whole integral term with Q^+.

We first notice that since Lf has the same $H^{1/2}$ smoothness in both variables x and v, it is clear that $Lf^{\#} \in L^2([0,T]; H^{1/2}_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}^3))$. In the same way, $Q^+(f,f)^{\#}$ lies in $L^2([0,T]; H^{\alpha}_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}^3))$ for all $\alpha \in]0, 1/25[$.

Besides, we have, for any $h \in L^2([0,T]; H^{\alpha}(\mathcal{B}_R \times \mathcal{B}_R))$, $R > 0$, $\alpha \in]0, 1/25[$,

$$\int_0^T \Big\| \int_0^t h(\sigma) d\sigma \Big\|^2_{L^2([0,T]; H^{\alpha}(\mathcal{B}_R \times \mathcal{B}_R))} dt \leq T^2 \|h\|^2_{L^2([0,T]; H^{\alpha}(\mathcal{B}_R \times \mathcal{B}_R))}. \tag{123}$$

Using (123) with $h = Lf^{\#}$, we immediately obtain that for any $t \in [0,T]$,

$$\int_0^t Lf^{\#}(\sigma) d\sigma \in L^2([0,T]; H^{1/2}_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}^3)).$$

Its time derivative is exactly $Lf^{\#}$ which also lies in $L^2([0,T]; H^{1/2}_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}^3))$. Consequently, we have proven that

$$\int_0^t Lf^{\#}(\sigma)\, d\sigma \in H^1_{\text{loc}}(\mathbb{R}_+; H^{1/2}_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}^3)) \subset H^{1/2}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3).$$

Since $x \mapsto \mathrm{e}^x$ is a bounded C^{∞} function on $[-T \max Lf, T \max Lf]$, we can conclude that E_1 belongs to $H^{1/2}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$.

Then, we notice that E_2 is the integral of the product of two terms which are both in $A = L^2([0,T]; H^{\alpha}_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^{\infty}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ for all $\alpha \in]0, 1/25[$. The previous vector space A is in fact an algebra, so E_2 is the integral of a term that lies in A. Using once again (123), we find that E_2 belongs to $H^{\alpha}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ for all $\alpha \in]0, 1/25[$.

Since E_1 and E_2 are obviously in A, $\tilde{\Gamma}_1 = E_1$ and $\tilde{\Gamma}_2 = E_1 \times E_2$ lie in A too, so that both quantities belong to $H^{\alpha}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ for all $\alpha \in]0, 1/25[$.

Finally, from (122) back to the standard formulation, we obtain (99) with the required smoothness on both Γ_1 and Γ_2, because $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ have the same smoothness in the three variables t, x and v. $\square$

In this proof, we have only considered cross sections B lying in the space $W^{1,\infty}(\mathbb{R}_+ \times [-1,1])$, which covers the case of (cutoff) Maxwellian molecules and (cutoff) regularized soft potentials.

We briefly explain here how to transform the proof to get a result in the case of hard potentials (with angular cutoff) or hard spheres.

Note first that the solutions of [61], which have an exponential decay in both x and v, are replaced by solutions with an algebraic decay in at least one of the variables, like those of [11] or [65]. Then, throughout the proof, if the algebraic decay concerns the variable v, the function T_λ is replaced by $S_\lambda(v_*) = (1 + |v_*|^2)^{-\frac{\lambda}{2}}$. The estimate on $\frac{Q^+(f,f)}{S_\lambda}$ becomes then more intricate (but is still valid).

Then, one has to replace the estimates in $W^{1,\infty}$ by estimates in $C^{0,\beta}$ (except for hard spheres) because the cross sections of hard potentials are only Hölder continuous, not Lipschitz continuous.

Finally, the L^∞ estimates must be replaced by weighted L^∞ estimates because the cross sections of hard potentials (and hard spheres) tend to infinity when $|v - v_*|$ tends to infinity. At the end, the exponent in the Sobolev space is less than 1/25 (and may be very small for hard potentials close to Maxwellian molecules, because of the bad smoothness of the cross section for small relative velocities).

The situation for true soft potentials (that is, when one keeps the true singularity of the cross section for small relative velocities) is not so good, and one probably needs to find new estimates to prove a result of smoothness in such a case.

Finally, when one considers a cross section without cutoff, or the Landau kernel, a very different behaviour is expected, and will be described in the sequel.

5 The Fourier transform of the Boltzmann operator with Maxwellian molecules and applications

5.1 Introduction

Up to now, we have used the Fourier transform $\widehat{Q(f,f)}$ of Boltzmann's kernel $Q(f,f)$, but we have only written it in terms of f itself and not in terms of $\hat{f}$.

In this section, we shall use a formula, written down by Bobylev in [13], [16], which enables to express directly $\widehat{Q(f,f)}$ (or $\widehat{Q^+(f,f)}$) in terms of $\hat{f}$. This formula is computed in subsection 5.2.

However, this formula is easily tractable only for a special kind of cross sections, namely the Maxwellian molecules. We recall that in our terminology, it means that B depends only on the second variable. As a consequence, many results are valid only for that particular type of cross sections, and many others, whose validity is larger, are more easily proven in the case of Maxwellian molecules.

In subsection 5.2, we write down Bobylev's identity, which expresses $\widehat{Q(f,f)}$ in terms of $\hat{f}$. Then, in the remaining subsections, we treat only the case of Maxwellian molecules, and give at the same time results which are only valid for this cross section (study of explicit and eternal solutions, uniqueness in the non cutoff case) and results which have a larger validity, but which can be proven more easily when Maxwellian molecules are considered (a new proof of the regularization properties of $Q^+(f,f)$, and the study of the smoothness of the solutions of the non cutoff spatially homogeneous Boltzmann equation).

5.2 Bobylev's identity

We write down here the proof of an identity due to Bobylev, which enables to obtain a simple expression of the Fourier transform of Boltzmann collision operator (or even, separately, its positive and negative part) in terms of the Fourier transform of f. The proof is extracted from [15].

Theorem 12 *We consider Boltzmann's kernel Q in the case when B does not depend on $|v-v_*|$:*

$$Q(g,f)(v) = \int_{\mathbb{R}^N}\int_{S^{N-1}} \Big\{ g(v'_*)\, f(v') - g(v_*)\, f(v) \Big\}\, b\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\, d\sigma\, dv_*.$$

Then, the following formulas hold ($\hat{f}$ or $\mathcal{F}f$ both denote the Fourier transform of f in the variable v) :

$$\mathcal{F}\Big[Q^+(g,f)\Big](\xi) = \int_{S^{N-1}} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \hat{g}(\xi^-)\hat{f}(\xi^+)\, d\sigma\,, \qquad (124)$$

$$\mathcal{F}\Big[Q^-(g,f)\Big](\xi) = \int_{S^{N-1}} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \hat{g}(0)\hat{f}(\xi)\, d\sigma\,. \qquad (125)$$

In the previous formulas, we have used the shorthand notation

$$\xi^+ = \frac{\xi+|\xi|\sigma}{2}, \qquad \xi^- = \frac{\xi-|\xi|\sigma}{2}. \qquad (126)$$

Proof: We perform here the calculation of the Fourier transform of the gain term in a general Boltzmann collision operator :

$$Q^+(g,f)(v) = \int_{\mathbb{R}^N}\int_{S^{N-1}} B\Big(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\, g(v'_*)\, f(v')\, d\sigma\, dv_*.$$

First of all, for any test-function $\varphi(v)$, holds

$$\int_{\mathbb{R}^N} Q^+(g,f)(v)\,\varphi(v)\,dv = \int_{\mathbb{R}^{2N}\times S^{N-1}} B\Big(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)$$

$$\times\, g(v_*)\, f(v)\,\varphi(v')\,dv\,dv_*\,d\sigma\,.$$

Plugging $\varphi(v) = e^{-iv\cdot\xi}$ in this identity, we get

$$\mathcal{F}[Q^+(g,f)](\xi) = \int_{\mathbb{R}^{2N}\times S^{N-1}} g(v_*)\, f(v)$$

,

$$\times B\Big(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\, e^{-i\frac{v+v_*}{2}\cdot\xi} e^{-i\frac{|v-v_*|}{2}\sigma\cdot\xi}\,dv\,dv_*\,d\sigma\,.$$

A key remark by Bobylev is that

$$\int_{S^{N-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) e^{-i\frac{|v-v_*|}{2}\sigma\cdot\xi}\,d\sigma$$

$$= \int_{S^{N-1}} B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{-i\frac{|\xi|}{2}\sigma\cdot(v-v_*)}\,d\sigma.$$

This is a consequence of the general equality

$$\int_{S^{N-1}} F(k\cdot\sigma, \ell\cdot\sigma)\,d\sigma = \int_{S^{N-1}} F(\ell\cdot\sigma, k\cdot\sigma)\,d\sigma, \qquad |\ell| = |k| = 1$$

(due to the existence of an isometry on S^{N-1} exchanging ℓ and k).

Thus,

$$\mathcal{F}[Q^+(g,f)](\xi) = \int_{\mathbb{R}^{2N}\times S^{N-1}} g(v_*)\, f(v)\, B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right)$$

$$\times\, e^{-i\xi\cdot\frac{v+v_*}{2}} e^{-i|\xi|\sigma\cdot\frac{v-v_*}{2}}\,dv\,dv_*\,d\sigma$$

$$= \int_{\mathbb{R}^{2N}\times S^{N-1}} g(v_*)\, f(v)\, B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{-iv\cdot\xi^+} e^{-iv_*\cdot\xi^-}\,dv\,dv_*\,d\sigma\,,$$

where ξ^+ and ξ^- are defined by (126).

By the Fourier inversion formula, this is also

$$\frac{1}{(2\pi)^N}\int_{\mathbb{R}^{2N}\times S^{N-1}}\left\{\int_{\mathbb{R}^{2N}}\hat{g}(\eta_*)\hat{f}(\eta)B\left(|v-v_*|,\frac{\xi}{|\xi|}\cdot\sigma\right)\right.$$

$$\left. e^{iv_*\cdot\eta_*}e^{iv\cdot\eta}e^{-iv\cdot\xi^+}e^{-iv_*\cdot\xi^-}\,d\eta_*\,d\eta\right\}dv\,dv_*\,d\sigma$$

$$=\frac{1}{(2\pi)^N}\int_{\mathbb{R}^{2N}\times S^{N-1}}\hat{g}(\eta_*)\hat{f}(\eta)$$

$$\left[\int_{\mathbb{R}^{2N}}B\left(|v-v_*|,\frac{\xi}{|\xi|}\cdot\sigma\right)e^{iv_*\cdot(\eta_*-\xi^-)}e^{iv\cdot(\eta-\xi^+)}\,dv\,dv_*\right]d\sigma\,d\eta\,d\eta_*\,.$$

By the change of variables $q=v-v_*$,

$$\int_{\mathbb{R}^{2N}}B\left(|v-v_*|,\frac{\xi}{|\xi|}\cdot\sigma\right)e^{iv_*\cdot(\eta_*-\xi^-)}e^{iv\cdot(\eta-\xi^+)}\,dv\,dv_*$$

$$=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}B\left(|q|,\frac{\xi}{|\xi|}\cdot\sigma\right)e^{iv\cdot(\eta_*+\eta-\xi^--\xi^+)}e^{-iq\cdot(\eta_*-\xi^-)}\,dq\,dv$$

$$=(2\pi)^{N/2}\hat{B}\left(|\eta_*-\xi^-|,\frac{\xi}{|\xi|}\cdot\sigma\right)\delta[\eta=\xi-\eta_*],$$

where δ is the Dirac measure, and $\hat{B}(|\xi|,\cos\theta)=\int_{\mathbb{R}^N}B(|q|,\cos\theta)e^{-iq\cdot\xi}\,dq$ denotes the Fourier transform of B in the relative velocity variable.

Thus the Fourier transform of $Q^+(g,f)$ is given by

$$\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N\times S^{N-1}}\hat{g}(\eta_*)\hat{f}(\xi-\eta_*)\hat{B}\left(|\eta_*-\xi^-|,\frac{\xi}{|\xi|}\cdot\sigma\right)\,d\eta_*\,d\sigma\,.$$

Writing $\xi_*=\eta_*-\xi^-$, we find in the end

$$\mathcal{F}[Q^+(g,f)](\xi)=\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N\times S^{N-1}}\hat{g}(\xi^-+\xi_*)\hat{f}(\xi^+-\xi_*)$$

$$\times\hat{B}\left(|\xi_*|,\frac{\xi}{|\xi|}\cdot\sigma\right)d\xi_*\,d\sigma\,. \tag{127}$$

In the particular case considered here (that is, when $B(|z|,\cos\theta)=b(\cos\theta)$), we have

$$\hat{B}(|\xi_*|,\cos\theta)=(2\pi)^{N/2}\delta[\xi_*=0]b(\cos\theta),$$

and as a consequence

$$\mathcal{F}[Q^+(g,f)](\xi) = \int_{S^{N-1}} \hat{g}(\xi^-)\hat{f}(\xi^+) b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) d\sigma\,.$$

The formula for $\mathcal{F}[Q^-(g,f)](\xi)$ is then easily obtained by the same kind of computations (but much simpler). $\square$

We now write down a simpler form of the Fourier transform of Boltzmann's kernel (in the case of Maxwellian molecules) for functions which are radially symmetric (or, equivalently, for functions the Fourier transform of which is radially symmetric). We observe that

$$|\xi^+|^2 = |\xi|^2 \frac{1+\frac{\xi}{|\xi|}\cdot\sigma}{2}, \qquad |\xi^-|^2 = |\xi|^2 \frac{1-\frac{\xi}{|\xi|}\cdot\sigma}{2},$$

so that if we define θ by

$$\cos(2\theta) = \frac{\xi}{|\xi|}\cdot\sigma,$$

we obtain

$$|\xi^+|^2 = |\xi|^2\cos^2\theta, \qquad |\xi^-|^2 = |\xi|^2\sin^2\theta.$$

Then, the Fourier transform of Boltzmann's kernel (in the case of Maxwellian molecules) for functions which are radially symmetric writes (with $\xi \in I\!R$)

$$\mathcal{F}[Q^+(g,f)](\xi) = \int_{\theta=-\pi/2}^{\pi/2} \hat{g}(\xi\,\sin\theta)\,\hat{f}(\xi\,\cos\theta)\,\beta(|\theta|)\,d\theta, \tag{128}$$

$$\mathcal{F}[Q^-(g,f)](\xi) = \int_{\theta=-\pi/2}^{\pi/2} \hat{g}(0)\,\hat{f}(\xi)\,\beta(|\theta|)\,d\theta, \tag{129}$$

where

$$\beta(|\theta|) = \frac{1}{2}\sin(2|\theta|)\,b(\cos(2\theta))$$

(in dimension 3). Remember that $\hat{f}$ and $\hat{g}$ are even functions of ξ in the previous formulas.

Those formulas are sometimes called the Fourier transform of Kac's operator, since its corresponds to taking the Fourier transform in (31), that is, when $v \in I\!R$ and

$$Q(g,f)(v) = \int_{I\!R}\int_{-\pi}^{\pi} \Big\{ g(v\sin\theta + w\cos\theta)\,f(v\cos\theta - w\sin\theta) - g(w)\,f(v)\Big\}\beta(|\theta|)dwd\theta\,. \tag{130}$$

5.3 Explicit and eternal solutions of Boltzmann's equation with Maxwellian molecules

Using formulas (128) and (129) and making the change of variables

$$x = \frac{\xi^2}{2}, \qquad s = \cos^2\theta,$$

together with the change of function

$$\phi(t,x) = \hat{f}(t,\xi),$$

Boltzmann's equation for radially symmetric functions writes

$$\partial_t \phi(t,x) = \int_{s=0}^{1} \Big\{ \phi(t,sx)\,\phi(t,(1-s)x) - \phi(t,0)\,\phi(t,x) \Big\} G(s)\,ds, \qquad (131)$$

where G is related to b.

The systematic study of this equation was made by Bobylev and Cercignani. The results of this subsection are extracted from their articles [17] and [18].

First, we look for solutions to (131) of the form

$$\phi(t,x) = e^{-2\alpha\, x}\,\phi_0(x\, e^{-2\lambda\, t}),$$

for $\alpha, \lambda \in I\!R$.

The equation satisfied by ϕ_0 is

$$-2\lambda\, y\phi_0'(y) = \int_0^1 \Big\{ \phi_0(sy)\,\phi_0((1-s)y) - \phi_0(0)\,\phi_0(y) \Big\} G(s)\,ds. \qquad (132)$$

We see that $\phi_0(y) = (1+y)\,e^{-y}$ is a solution to eq. (132) as soon as

$$\lambda = \frac{1}{2} \int_0^1 s\,(1-s)\,G(s)\,ds.$$

As a consequence, we obtain solutions ϕ to eq. (131) of the form

$$\phi(t,x) = e^{-2\alpha\, x}\,(1 + x\, e^{-2\lambda\, t})\,\exp\Big(- x\, e^{-2\lambda\, t}\Big).$$

Those in turn lead to the following formula for the Fourier transform of the Boltzmann equation :

$$\hat f(t,\xi) = e^{-\alpha\,|\xi|^2}\,(1+\frac{1}{2}\,|\xi|^2\,e^{-2\lambda\,t})\,\exp\Big(-\frac{1}{2}\,|\xi|^2\,e^{-2\lambda\,t}\Big).$$

The well-known BKW mode (Cf. [13], [14] and [51]) is then recovered by taking the inverse Fourier transform of the previous formula (with $\alpha = \frac{1}{2}$, and in dimension 3):

$$f(t,v) = (2\pi\,(1-e^{-\lambda\,t}))^{-3/2}\Big(1+\frac{e^{-\lambda\,t}}{3\,(1-e^{-\lambda\,t})}\Big(\frac{|v|^2}{1-e^{-\lambda\,t}}-3\Big)\Big)$$
$$\times\exp\Big(-\frac{|v|^2}{2(1-e^{-\lambda\,t})}\Big).$$

This has long been the only (up to some transformations) known nonnegative (nontrivial) explicit solution to the (spatially homogeneous) Boltzmann equation.

However, Bobylev and Cercignani recently discovered (Cf. [17]) new nonnegative explicit solutions in the particular case when $G = 1$.

We only write here the simplest one. It is given by the formula

$$f(t,v) = 2^{-\frac{1}{2}}\,\pi^{-\frac{5}{2}}\,e^{-t}\int_0^\infty \frac{u\,e^{-u}}{\Big(u+\frac{|v|^2\,e^{-2t/3}}{2}\Big)^2}\,du\,. \tag{133}$$

This solution is said to be eternal. This means that it is defined and nonnegative for all times $t \in \mathbb{R}$.

This does not contradict the conjecture that all eternal (nonnegative) solutions with finite mass and energy of the (spatially homogeneous) Boltzmann equation are trivial (that is, Maxwellian). The reason for that is that the solution given by (133) has infinite energy.

In fact, Bobylev and Cercignani recently made a significant step towards this conjecture by proving the following result (Cf. [18]) :

Theorem 13 *Let f be a radially symmetric nonnegative eternal solution of the Boltzmann equation with Maxwellian molecules such that all its moments of even order*

$$m_n(t) = \int_{I\!R^N} f(t,v)\,|v|^{2n}\,dv$$

are finite for all $t \in I\!R$. Then, f is a (constant) Maxwellian.

Proof : We can suppose that $m_0 = 1$ and $m_1 = N$ without loss of generality (this is possible thanks to a multiplication and dilatation of f). Then, we want to prove that

$$f(t,v) = (2\pi)^{-N/2}\, e^{-\frac{|v|^2}{2}}.$$

We now use the Fourier transform of f and keep the notations (ϕ, s, G, etc.) of this subsection. For the sake of simplicity, we write down the proof only in the case when $G \equiv 1$.

The equation satisfied by ϕ is (131). The same equation is satisfied by ψ defined by

$$\psi(t,x) = e^x\,\phi(t,x),$$

that is

$$\partial_t\psi(t,x) = \int_{s=0}^{1} \Big\{\psi(t,sx)\,\psi(t,(1-s)x) - \psi(t,0)\,\psi(t,x)\Big\} G(s)\,ds. \qquad (134)$$

According to the definition of ψ, we simply want to prove that for all $t \in I\!R$, $x \in I\!R_+$, $\psi(t,x) = 1$.

Then, writing (with the convention that the derivatives concern the second variable)

$$\phi(t,x) = \sum_{n=0}^{+\infty} \frac{\phi^{(n)}(t,0)}{n!},$$

$$\psi(t,x) = \sum_{n=0}^{+\infty} \frac{\psi^{(n)}(t,0)}{n!},$$

we see that for all $n \geq 2$,

$$\partial_t\psi^{(n)}(t,0) - \lambda_n\,\psi^{(n)}(t,0) = \sum_{p+q=n,p,q\in[1,n-1]} \frac{n!}{p!\,q!}$$

$$\times\, \psi^{(p)}(t,0)\, \psi^{(q)}(t,0) \int_0^1 s^p\, (1-s)^q\, ds, \qquad (135)$$

with $\lambda_n = \frac{2}{n+1} - 1$.

We now suppose that we do not have $\psi(0,x) \equiv 1$ (that is, f is not a Maxwellian initially), so that there exists $p \in I\!N_*$ such that $\psi^{(i)}(0,0) = 0$ for $i = 1,..,p-1$, and $\psi^{(p)}(0,0) \neq 0$.

Then, thanks to (135), it is clear (by induction) that for all $t \in I\!R$, $\psi^{(i)}(t,0) = 0$ for $i = 1,..,p-1$. Again by induction, for all $t \in I\!R$, $\psi^{(i)}(t,0) = e^{\lambda_i\, t}\, \psi^{(i)}(0,0)$ for $i = p,..,2p-1$, and

$$\psi^{(2p)}(t,0) = \left[\psi^{(2p)}(0,0) - \frac{B_p\, \psi^{(p)}(0,0)^2}{2\lambda_p - \lambda_{2p}}\right] e^{\lambda_{2p}\, t} + \frac{B_p\, \psi^{(p)}(0,0)^2}{2\lambda_p - \lambda_{2p}}\, e^{2\lambda_p\, t},$$

with

$$B_p = \frac{(2p)!}{(p!)^2} \int_0^1 s^p\, (1-s)^p\, ds.$$

Then, we observe that $2\lambda_p < \lambda_{2p}$, so that

$$\frac{B_p\, \psi^{(p)}(0,0)^2}{2\lambda_p - \lambda_{2p}}\, e^{2\lambda_p\, t} < 0.$$

Because

$$\phi(x) = e^{-x}\, \psi(x),$$

one has for all $n \in I\!N$

$$\phi^{(n)}(t,0) = \sum_{a+b=n} \frac{n!}{a!\, b!}\, (-1)^a\, \psi^{(b)}(t,0),$$

so that

$$\phi^{(2p)}(t,0) = \sum_{b=p}^{2p} \frac{(2p)!}{(2p-b)!\, b!}\, (-1)^{2p-b}\, \psi^{(b)}(t,0)$$

$$= \sum_{b=p}^{2p-1} \frac{(2p)!}{(2p-b)!\, b!}\, (-1)^{2p-b}\, e^{2\lambda_b\, t}$$

$$+ \left[\psi^{(2p)}(0,0) - \frac{B_p\, \psi^{(p)}(0,0)^2}{2\lambda_p - \lambda_{2p}}\right] e^{\lambda_{2p}\, t} + \frac{B_p\, \psi^{(p)}(0,0)^2}{2\lambda_p - \lambda_{2p}}\, e^{2\lambda_p\, t}.$$

When $t \to -\infty$, the dominant term in the previous formula is the term in $e^{2\lambda_p\, t}$, and it is strictly negative.

This means that there exists a time T (negative and large enough in absolute value) such that $\phi^{(2p)}(T,0)$ is negative.

We now recall that expanding

$$\hat{f}(t,|k|) = \int e^{i|k|x_1} f(t,x)\,dx = \phi(t, \frac{|k|^2}{2})$$

in power series, we get for all $n \in \mathbb{N}$,

$$\frac{\phi^{(n)}(t,0)}{2^n\, n!} = \frac{(-1)^n}{2^n} \int x_1^{2n} f(t,x)\,dx,$$

so that the assumption that f be nonnegative entails the nonnegativity of $\phi^{(2p)}(t,0)$ for all $t \in \mathbb{R}$ and $p \in \mathbb{N}$, and we have a contradiction. Then, f is initially a Maxwellian and (thanks to a standard theorem of uniqueness), it will remain a Maxwellian for all times.

We also notice that in the computation above, there is no need that the power series (of ϕ or ψ) converge, nor is it compulsory for the equation on ϕ to be defined for all time : those are only used at the formal level to write equations on the moments of f, and could be removed from the proof. $\square$

Note that the only other known result concerning the eternal solutions of some spatially homogeneous kinetic equation is the result by Villani (Cf. [76]) for the Fokker-Planck-Landau equation.

5.4 Uniqueness for Boltzmann's equation with Maxwellian molecules without angular cutoff

We present here a result of stability of the (cutoff or non cutoff) spatially homogeneous Boltzmann equation with Maxwellian molecules in a weak norm due to Toscani and Villani (Cf. [69]).

In the non cutoff case, no other proof of uniqueness is known.

First, we define by

$$d_2(f,g) = \sup_{\xi \in \mathbb{R}^N} \frac{|\hat{f}(\xi) - \hat{g}(\xi)|}{|\xi|^2},$$

a distance between functions $f, g \in L^1_2(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} f(v) \begin{pmatrix} 1 \\ v \\ \frac{|v|^2}{2} \end{pmatrix} dv = \int_{\mathbb{R}^N} g(v) \begin{pmatrix} 1 \\ v \\ \frac{|v|^2}{2} \end{pmatrix} dv = \begin{pmatrix} 1 \\ 0 \\ N/2 \end{pmatrix}.$$

Note that for such functions f, g, the quantity $d_2(f, g)$ is indeed finite.

Then, the following property holds :

Theorem 14 *Let B be a cross section verifying $B(x, u) = b(u)$ (that is, of Maxwellian molecules type) with $|\sin\theta|\, b(\cos\theta) \leq K\,|\theta|^{-1-\gamma}$ and $K > 0$, $\gamma < 2$ (in other words, cutoff or non cutoff).*

Then, for all (nonnegative) energy-conserving solutions f, g of the spatially homogeneous Boltzmann equation (25) with respective initial data f_{in} and g_{in} satisfying

$$\int_{\mathbb{R}^N} f_{in}(v) \begin{pmatrix} 1 \\ v \\ \frac{|v|^2}{2} \end{pmatrix} dv = \int_{\mathbb{R}^N} g_{in}(v) \begin{pmatrix} 1 \\ v \\ \frac{|v|^2}{2} \end{pmatrix} dv,$$

(such solutions are known to exist thanks to theorem 1), one has the relation

$$\forall t \geq 0, \qquad d_2(f(t,\cdot), g(t,\cdot)) \leq d_2(f_{in}, g_{in}).$$

Proof : We can impose (up to a translation, a dilatation and a multiplication) that

$$\int_{\mathbb{R}^N} f_{in}(v) \begin{pmatrix} 1 \\ v \\ \frac{|v|^2}{2} \end{pmatrix} dv = \int_{\mathbb{R}^N} g_{in}(v) \begin{pmatrix} 1 \\ v \\ \frac{|v|^2}{2} \end{pmatrix} dv = \begin{pmatrix} 1 \\ 0 \\ N/2 \end{pmatrix}.$$

Then, thanks to the identites (124) and (125), we see that f and g satisfy

$$\forall t \geq 0, \qquad \hat{f}(t, 0) = \hat{g}(t, 0) = 1,$$

so that

$$\partial_t \hat{f}(\xi) = \int_{S^{N-1}} \Big(\hat{f}(\xi^-)\, \hat{f}(\xi^+) - \hat{f}(\xi)\Big)\, b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) d\sigma,$$

$$\partial_t \hat{g}(\xi) = \int_{S^{N-1}} \Big(\hat{g}(\xi^-)\, \hat{g}(\xi^+) - \hat{g}(\xi)\Big)\, b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) d\sigma,$$

and

$$\partial_t (\hat{f}(\xi) - \hat{g}(\xi)) = \int_{S^{N-1}} \Big[\Big(\hat{f}(\xi^-)\, \hat{f}(\xi^+) - \hat{g}(\xi^-)\, \hat{g}(\xi^+)\Big) - \Big(\hat{f}(\xi) - \hat{g}(\xi)\Big) \Big]\, b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) d\sigma.$$

But

$$\left|\frac{\hat f(\xi^-)\,\hat f(\xi^+) - \hat g(\xi^-)\,\hat g(\xi^+)}{|\xi|^2}\right| \le |\hat f(\xi^+)| \left|\frac{\hat f(\xi^-) - \hat g(\xi^-)}{|\xi^-|^2}\right| \frac{|\xi^-|^2}{|\xi|^2}$$

$$+ |\hat g(\xi^-)| \left|\frac{\hat f(\xi^+) - \hat g(\xi^+)}{|\xi^+|^2}\right| \frac{|\xi^+|^2}{|\xi|^2}$$

$$\le \sup_{\xi\in \mathbb{R}^N} \left|\frac{\hat f(\xi) - \hat g(\xi)}{|\xi|^2}\right| \left(\frac{|\xi^-|^2 + |\xi^+|^2}{|\xi|^2}\right)$$

$$\le \sup_{\xi\in \mathbb{R}^N} \left|\frac{\hat f(\xi) - \hat g(\xi)}{|\xi|^2}\right|.$$

Then, denoting $h(\xi) = \frac{\hat f(\xi) - \hat g(\xi)}{|\xi|^2}$, we obtain

$$\partial_t h(\xi) \le \int_{S^{N-1}} \Big[||h||_{L^\infty(\mathbb{R}^N)} - h(\xi) \Big]\, b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\, d\sigma.$$

Supposing momentarily that b is integrable (cutoff assumption), we immediately get that $d_2(f,g) = \sup_{\xi\in\mathbb{R}^N} |h(\xi)|$ decreases with t.

Since this estimate does not depend on b, it also holds in the non cutoff case (this is easily obtained by imposing a cutoff depending on a parameter such that, when this parameter goes to 0, the cutoff cross section converges to the non cutoff one). □

Note that the previous estimate immediately implies a property of uniqueness (as we already pointed out, such a property can easily be obtained without the Fourier transform in the cutoff case, but the proof above is the only one up to now in the non cutoff case).

5.5 Alternative proof for the properties of Q^+

We now propose a proof of the smoothing properties of Q^+ which uses Bobylev's identity and which is therefore particularly simple when Maxwellian molecules are considered. The assumption and the conclusion are close to that of theorems 8 and 9, but are not exactly the same.

We shall use the following formula to compute some integrals on the sphere S^{N-1} ($N \ge 2$). It deals with functions which only depend on one component: for any function β defined on $]-1,1[$,

$$\int_{S^{N-1}} \beta(\omega_N)\, d\omega = \frac{2\pi^{(N-1)/2}}{\Gamma(\frac{N-1}{2})} \int_{-1}^{1} \beta(u)(1-u^2)^{(N-3)/2}\, du. \tag{136}$$

We now state our result. The proof is close to the one used in [19] :

Theorem 15 *Assume that*

$$b \in L^2(]-1,1[,(1-u^2)^{(N-3)/2}du). \tag{137}$$

Then for any $f \in L^2_1(\mathbb{R}^N)$, $Q^+(f) \in \dot{H}^{(N-1)/2}(\mathbb{R}^N)$ *and*

$$\|Q^+(f)\|_{\dot{H}^{(N-1)/2}(\mathbb{R}^N)} \leq C_N \, \|b\|_{L^2(]-1,1[,(1-u^2)^{(N-3)/2}du)} \|f\|^2_{L^2_1(\mathbb{R}^N)}. \tag{138}$$

Proof : We know that

$$\widehat{Q^+(f,f)}(\xi) = \int\limits_{\sigma \in S^{N-1}} \widehat{f}\left(\frac{\xi - |\xi|\sigma}{2}\right) \widehat{f}\left(\frac{\xi + |\xi|\sigma}{2}\right) b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) d\sigma. \tag{139}$$

We have by Cauchy-Schwarz's inequality

$$\left|\widehat{Q^+(f,f)}(\xi)\right|^2 \leq \int\limits_{\sigma \in S^{N-1}} \left|\widehat{f}\left(\frac{\xi - |\xi|\sigma}{2}\right) \widehat{f}\left(\frac{\xi + |\xi|\sigma}{2}\right)\right|^2 d\sigma$$

$$\times \int\limits_{\sigma \in S^{N-1}} \left|b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\right|^2 d\sigma, \tag{140}$$

and the last integral can be computed by (136),

$$\int\limits_{\sigma \in S^{N-1}} \left|b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\right|^2 d\sigma = \frac{2\pi^{(N-1)/2}}{\Gamma(\frac{N-1}{2})} \int_{-1}^{1} |b(u)|^2 (1-u^2)^{(N-3)/2}\, du. \tag{141}$$

Then,

$$\int\limits_{\sigma \in S^{N-1}} \left|\widehat{f}\left(\frac{\xi - |\xi|\sigma}{2}\right) \widehat{f}\left(\frac{\xi + |\xi|\sigma}{2}\right)\right|^2 d\sigma$$

$$= \int\limits_{\sigma \in S^{N-1}} \int\limits_{r=|\xi|}^{\infty} -\frac{\partial}{\partial r}\left|\widehat{f}\left(\frac{\xi - r\sigma}{2}\right) \widehat{f}\left(\frac{\xi + r\sigma}{2}\right)\right|^2 dr d\sigma$$

$$\leq \int\limits_{\sigma \in S^{N-1}} \int\limits_{r=|\xi|}^{\infty} \left|\widehat{f}\left(\frac{\xi - r\sigma}{2}\right)\right|^2 \left|\widehat{f}\left(\frac{\xi + r\sigma}{2}\right)\right| \left|\nabla \widehat{f}\left(\frac{\xi + r\sigma}{2}\right)\right| dr d\sigma$$

$$= \int\limits_{|\eta|>|\xi|} \left|\hat{f}\left(\frac{\xi-\eta}{2}\right)\right| \left|\hat{f}\left(\frac{\xi+\eta}{2}\right)\right| \left|\nabla\hat{f}\left(\frac{\xi+\eta}{2}\right)\right| \frac{d\eta}{|\eta|^{N-1}}.$$

Therefore,

$$\int\limits_{\xi\in\mathbb{R}^N} d\xi\, |\xi|^{N-1} \left|\widehat{Q}^+(f,f)(\xi)\right|^2$$

$$\le C_N \iint\limits_{\xi,\eta\in\mathbb{R}^N} \left|\hat{f}\left(\frac{\xi-\eta}{2}\right)\right| \left|\hat{f}\left(\frac{\xi+\eta}{2}\right)\right| \left|\nabla\hat{f}\left(\frac{\xi+\eta}{2}\right)\right| d\xi d\eta$$

$$\le C_N \iint\limits_{\lambda,\mu\in\mathbb{R}^N} |\hat{f}(\lambda)|\, |\hat{f}(\mu)|\, |\widehat{v\,f}|(\mu)|\, d\mu d\lambda$$

$$\le C_N\, \|f\|_{L^2(\mathbb{R}^N)}\, \|v\, f\|_{L^2_1(\mathbb{R}^N)}.$$

□

As we shall see in the sequel, it is possible to extend this proof to non Maxwellian molecules cross sections.

5.6 Gain of smoothness for Kac equation without angular cutoff

In this subsection, we investigate the smoothness of the solutions of the spatially homogeneous Boltzmann equation when the cutoff assumption of Grad is not made. The result is quite different from that of the cutoff case, since we shall in fact prove that an immediate effect of smoothing occurs, as in the heat equation.

In order to put into evidence this effect, we investigate here the simplest nontrivial model, that is Kac's equation (defined by (130)) or, equivalenty, Boltzmann's equation with Maxwellian molecules in a radially symmetric context.

We shall even restrict our attention to a typical non cutoff cross section, that is

$$\beta(|\theta|) = |\sin\theta|^{-2}\cos\theta \mathbf{1}_{|\theta|\le\pi/4}\,, \tag{142}$$

rather than try to give general conditions.

We state a theorem which was first proven in [26]. The proof given here is however extracted from [29].

Theorem 16 *We consider Kac's operator Q defined by (130), together with the cross section (142).*

Then, for all measurable even initial datum $f^{in} \geq 0$ a.e. on $\mathbb{R}$ satisfying

$$E(f_{in}) := \int_{\mathbb{R}} (1 + v^2 + |\log f^{in}|) f^{in} dv < +\infty, \tag{143}$$

the Cauchy problem

$$\partial_t f(t, v) = Q(f, f)(t, v), \qquad f(0, \cdot) = f_{in} \tag{144}$$

has an a.e. even nonnegative solution f such that

$$\sup_{t>0} \int_{\mathbb{R}} (1 + v^2 + |\log f(t, v)|)\, f(t, v)\, dv < +\infty\,. \tag{145}$$

In addition, for all $\tau > 0$,

$$f \in L^\infty_{loc}([\tau, +\infty[, H^\infty(\mathbb{R}_v))\,. \tag{146}$$

Proof: We admit the existence of an even a.e. nonnegative solution to eq. (144) such that the conservation of mass and energy holds, and such that the entropy decreases. Moreover, we shall write down the estimates on f as if it were smooth. In order to justify all our computations, we should in fact write them on the solution of an approximated problem. We shall not do that here for the sake of simplicity.

According to formulas (128) and (129), we see that

$$\mathcal{F}[Q(g, f)](\xi) = \int_{\theta=-\pi/2}^{\pi/2} \left[\hat{g}(\xi\, \sin\theta)\, \hat{f}(\xi\, \cos\theta) - \hat{g}(0)\, \hat{f}(\xi)\right] \beta(|\theta|)\, d\theta.$$

Then, for all $\alpha \geq 0$,

$$\int_{\mathbb{R}} \mathcal{F}[Q(g, f)](\xi) \hat{f}(\xi) |\xi|^{2\alpha} d\xi = A + B, \tag{147}$$

with

$$A = \int_{\mathbb{R}} \int_{-\pi/4}^{\pi/4} \Big[\hat{f}(\xi \cos\theta)\, \hat{g}(\xi \sin\theta)\, \hat{f}(\xi)$$

$$- \tfrac{1}{2} \hat{g}(0)\, \left(|\hat{f}(\xi)|^2 + |\hat{f}(\xi\cos\theta)|^2\right) \Big]\, |\xi|^{2\alpha}\, |\sin\theta|^{-2} \cos\theta\, d\theta d\xi \tag{148}$$

and

$$B = \frac{1}{2} \int_{\mathbb{R}} \int_{-\pi/4}^{\pi/4} \hat{g}(0) \left(|\hat{f}(\xi \cos\theta)|^2 - |\hat{f}(\xi)|^2 \right) |\xi|^{2\alpha} |\sin\theta|^{-2} \cos\theta \, d\theta d\xi \, . \tag{149}$$

Changing variables by $\xi \cos\theta \mapsto \zeta$ shows that

$$\begin{aligned} |B| &= \frac{1}{2} \left| \int_{\mathbb{R}} \int_{-\pi/4}^{\pi/4} \hat{g}(0) |\hat{f}(\zeta)|^2 |\zeta|^{2\alpha} [(\cos\theta)^{-2\alpha} - 1] |\sin\theta|^{-2} \cos\theta d\theta \right| \\ &\leq E_\alpha \int_{\mathbb{R}} g(v) \, dv \, \|f\|^2_{H^{2\alpha}}, \end{aligned} \tag{150}$$

with

$$E_\alpha = \tfrac{1}{2} \int_{-\pi/4}^{\pi/4} [(\cos\theta)^{-2\alpha} - 1] \, |\sin\theta|^{-2} \cos\theta \, d\theta < +\infty \, . \tag{151}$$

The most important estimate is the one concerning A:

$$\begin{aligned} A &\leq -\tfrac{1}{2} \int_{\mathbb{R}} \int_{-\pi/4}^{\pi/4} \left(|\hat{f}(\xi)|^2 + |\hat{f}(\xi \cos\theta)|^2 \right) \\ &\quad \times \left(\hat{g}(0) - |\hat{g}(\xi \sin\theta)| \right) |\xi|^{2\alpha} \frac{\cos\theta}{|\sin\theta|^2} \, d\theta d\xi \\ &\leq -\tfrac{1}{2} \int_{\mathbb{R}} \int_{-\pi/4}^{\pi/4} |\hat{f}(\xi)|^2 \left(\hat{g}(0) - |\hat{g}(\xi \sin\theta)| \right) |\xi|^{2\alpha} |\sin\theta|^{-2} \cos\theta \, d\theta d\xi \end{aligned} \tag{152}$$

(since $g \geq 0$ a.e., $\hat{g}(0) = \|g\|_{L^1} \geq |\hat{g}(\xi)|$ for all $\xi \in \mathbb{R}$).

We now use the change of variables $(\xi, \theta) \mapsto (\xi, \xi \sin\theta)$ (this is where the special form (142) of the cross section helps) and get

$$\begin{aligned} A &\leq -\tfrac{1}{2} \int_{\mathbb{R}} \int_{-|\xi|/\sqrt{2}}^{|\xi|/\sqrt{2}} \left(\hat{g}(0) - |\hat{g}(u)| \right) |\hat{f}(\xi)|^2 |\xi|^{2\alpha+1} \frac{du}{|u|} d\xi \\ &\leq -\tfrac{1}{2} \int_{-1}^{1} \int_{\mathbb{R}} g(x) \left(1 \pm \cos(ux) \right) dx du \left(\|f\|^2_{H^{\alpha+1/2}} - \sqrt{2} \|f\|^2_{H^\alpha} \right) . \end{aligned} \tag{153}$$

As a consequences of estimates (150) and (153), we see that

$$\int_{\mathbb{R}} \mathcal{F}[Q(g,f)](\xi) \, \hat{f}(\xi) \, |\xi|^{2\alpha} \, d\xi \leq -C_{g,\alpha} \|f\|^2_{H^{\alpha+1/2}} + D_{g,\alpha} \|f\|^2_{H^\alpha}, \tag{154}$$

where $C_{g,\alpha}$ and $D_{g,\alpha}$ are nonnegative constants depending only on $\alpha > 0$ and $E(g)$ (defined in (143)).

We now take the Fourier transform (in v) to both sides of (144) and multiply the resulting equation by $\hat{f}(t,\xi)|\xi|^{2\alpha}$ (remember that $\hat{f}$ is real because f is even).

We know that thanks to estimate (154),

$$\frac{d}{dt}||f(t)||^2_{H^\alpha} \leq -C_{f,\alpha}||f(t)||^2_{H^{\alpha+1/2}} + D_{f,\alpha}||f(t)||^2_{H^\alpha}. \qquad (155)$$

Here, $C_{f,\alpha}$ and $D_{f,\alpha}$ only depend on α because the evolution semigroup of (144) conserves the mass and energy of f and decreases the H function.

Using an interpolation of H^α between $H^{\alpha+1/2}$ and H^{-d} for d large enough (typically $d > 1/2$ so that $L^1_2 \subset H^{-d}$, estimate (155) becomes (for some $s_\alpha, K_\alpha, L_\alpha > 0$),

$$\frac{d}{dt}||f(t)||^2_{H^\alpha} \leq -K_\alpha\, ||f(t)||^{2+s_\alpha}_{H^\alpha} + L_\alpha. \qquad (156)$$

Then, using a Gronwall type inequality, we see that for all $\alpha, t_0, T > 0$,

$$\sup_{t_0 \leq t \leq T} ||f(t)||_{H^\alpha} < +\infty.$$

Note that the method used here is very close to that of Nash for the parabolic equations.

The proof described in this subsetcion applies to the 3D homogeneous (non radially symmetric) Boltzmann equation for Maxwell molecules without angular cutoff: for all a.e. nonnegative measurable initial data with finite mass, energy and entropy, the number density f satisfies $f(t,\cdot) \in C^\infty(\mathbb{R}^3)$ for all $t > 0$. This is partly proven (in 2D) in [28].

6 Extensions in the case of other cross sections

6.1 Introduction

One could think that though somehow complicated, the formula giving the Fourier transform of $Q(f,f)$ in terms of the Fourier transform of f when the cross section is not that of Maxwellian molecules will enable to extend the results of the previous section.

However, it turns out that this idea is hard to put in application. Among the rare works using this formula, one can quote [27] and [66].

In fact, in order to extend the theorems of the previous section, it seems a better idea to find estimates in the standard space in which appears the cross section of Maxwellian molecules, and only then, to take the Fourier transform.

In this section, we present two applications of this vague idea. The first one enables to extend the proof of the regularity properties of Q^+ obtained in the previous section. The second one deals with the non cutoff spatially homogeneous Boltzmann equation.

Finally, we conclude this introduction by pointing out the analogy between the role of the Maxwellian molecules (with respect to other cross sections) and the role of the linear PDEs with constant coefficients (with respect to the linear PDEs with variable coefficients). The ideas developed in this section have their origin in this analogy.

6.2 Properties of Q^+

We now propose an extension of the result of subsection 5.5 in the case of hard potentials. We obtain a result which is close to that of theorem 9, but still with an assumption and a conclusion slightly different. The theorem and its proof are extracted from [19].

We shall make on the cross section the following assumption :

Assumption 2 : We suppose that B takes the form

$$B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) = b_1\Big(|v-v_*|\Big)\, b_2\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right), \qquad (157)$$

where b_1 and b_2 are functions defined on $]0,\infty[$ and $]-1,1[$ respectively, and satisfy for some $K_b \geq 0$, $\alpha_b \geq 0$,

$$\forall x > 0, \quad |b_1(x)| \leq K_b\,(1+x)^{\alpha_b}, \qquad (158)$$

and

$$b_2 \in L^2(]-1,1[, (1-u^2)^{(N-3)/2}du). \qquad (159)$$

Then, the following result holds :

Theorem 17 *Under assumption 2, for any* $f \in L^2_{1+\alpha_b}(I\!\!R^N)$, $Q^+(f,f) \in \dot{H}^{(N-1)/2}(I\!\!R^N)$, *and there exists a constant* $C_N > 0$ *such that*

$$\|Q^+(f,f)\|_{\dot{H}^{(N-1)/2}(I\!\!R^N)} \leq C_N\, K_b\, \|b_2\|_{L^2(]-1,1[,(1-u^2)^{(N-3)/2}du)} \|f\|^2_{L^2_{1+\alpha_b}(I\!\!R^N)}. \tag{160}$$

Proof : We first define the operator Q^+ for functions of two variables $F(v_1,v_2)$, $v_1, v_2 \in I\!\!R^N$ by

$$Q^+(F)(v) = \iint\limits_{\substack{v_* \in I\!\!R^N \\ \sigma \in S^{N-1}}} F\left(\frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma\right) \tag{161}$$

$$\times\, b_2\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) d\sigma dv_*.$$

Then, theorem 17 is the direct consequence of the following proposition :

Proposition 3 *For the linear operator (161), we have*
(i) If $b_2 \in L^1(]-1,1[,(1-u^2)^{(N-3)/2}du)$, *then for any* $F \in L^1(I\!\!R^N \times I\!\!R^N)$, $Q^+(F) \in L^1(I\!\!R^N)$ *and*

$$\|Q^+(F)\|_{L^1(I\!\!R^N)} \leq \frac{2\pi^{(N-1)/2}}{\Gamma(\frac{N-1}{2})} \|b_2\|_{L^1(]-1,1[,(1-u^2)^{(N-3)/2}du)} \|F\|_{L^1(I\!\!R^N \times I\!\!R^N)}. \tag{162}$$

(ii) If $b_2 \in L^2(]-1,1[,(1-u^2)^{(N-3)/2}du)$, *then for any* $F \in L^2(I\!\!R^N \times I\!\!R^N)$ *such that* $(v_2 - v_1)F \in L^2(I\!\!R^N \times I\!\!R^N)$, *the integral (161) is absolutely convergent for a.e.* v, $Q^+(F) \in \dot{H}^{(N-1)/2}(I\!\!R^N)$ *and*

$$\|Q^+(F)\|_{\dot{H}^{(N-1)/2}(I\!\!R^N)} \leq C_N\, \|b_2\|_{L^2(]-1,1[,(1-u^2)^{(N-3)/2}du)}$$

$$\times\, \|F\|^{1/2}_{L^2} \|(v_2 - v_1)F\|^{1/2}_{L^2}. \tag{163}$$

Let us postpone the proof of Proposition 3 and deduce Theorem 17.

Proof of Theorem 17. Let us define

$$F(v_1, v_2) = f(v_1)\, f(v_2)\, b_1\left(|v_2 - v_1|\right). \tag{164}$$

Then, it is clear that $Q^+(f,f) = Q^+(F)$. Now, by (158) we have

$$\begin{aligned} |F(v_1,v_2)| &\leq |f(v_1)||f(v_2)|K_b(1+|v_2-v_1|)^{\alpha_b} \\ &\leq K_b|f(v_1)||f(v_2)|(1+|v_1|+|v_2|)^{\alpha_b} \\ &\leq K_b\,|(1+|v_1|)^{\alpha_b} f(v_1)|\,|(1+|v_2|)^{\alpha_b} f(v_2)|\,. \end{aligned} \tag{165}$$

Therefore,

$$\|F\|_{L^1} \leq K_b\,\|f\|^2_{L^1_{\alpha_b}}, \qquad \|F\|_{L^2} \leq K_b\,\|f\|^2_{L^2_{\alpha_b}}, \tag{166}$$

and since

$$\begin{aligned} |(v_2-v_1)F(v_1,v_2)| &\leq |v_1||F(v_1,v_2)| + |v_2||F(v_1,v_2)| \\ &\leq K_b\Big|(1+|v_1|)^{1+\alpha_b} f(v_1)\Big|\Big|(1+|v_2|)^{\alpha_b} f(v_2)\Big| \\ &+K_b\Big|(1+|v_1|)^{\alpha_b} f(v_1)\Big|\Big|(1+|v_2|)^{1+\alpha_b} f(v_2)\Big|, \end{aligned}$$

we also have

$$\|(v_2-v_1)F\|_{L^2} \leq 2K_b\|f\|_{L^2_{\alpha_b}}\|f\|_{L^2_{1+\alpha_b}}. \tag{167}$$

Now since $b_2 \in L^2$ by (159), we can apply Proposition 3 (ii), and we obtain that $Q^+(f,f) = Q^+(F) \in \dot{H}^{(N-1)/2}$, and

$$\|Q^+(f,f)\|_{\dot{H}^{(N-1)/2}} \leq C_N\,\|b_2\|_{L^2}\,K_b\,\|f\|^{3/2}_{L^2_{\alpha_b}}\,\|f\|^{1/2}_{L^2_{1+\alpha_b}}, \tag{168}$$

and (160) follows since $\|f\|_{L^2_{\alpha_b}} \leq \|f\|_{L^2_{1+\alpha_b}}$. $\square$

Proof of Proposition 3. Estimate (i) is easy, and we only prove (ii).

By a computation similar to that of subsection 5.2, we get

$$\widehat{Q^+(F)}(\xi) = \int_{\sigma\in S^{N-1}} \widehat{F}\left(\frac{\xi-|\xi|\sigma}{2}, \frac{\xi+|\xi|\sigma}{2}\right)\, b_2\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\,d\sigma. \tag{169}$$

Then, the computation closely follows that of theorem 15.

We have by Cauchy-Schwarz's inequality

$$\left|\widehat{Q^+(F)}(\xi)\right|^2 \leq \int_{\sigma\in S^{N-1}} \left|\widehat{F}\left(\frac{\xi-|\xi|\sigma}{2}, \frac{\xi+|\xi|\sigma}{2}\right)\right|^2 d\sigma \int_{\sigma\in S^{N-1}} \left|\beta\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\right|^2 d\sigma, \tag{170}$$

and the last integral can be computed by (136).

Then,

$$\int\limits_{\sigma\in S^{N-1}} \left|\widehat{F}\left(\frac{\xi-|\xi|\sigma}{2},\frac{\xi+|\xi|\sigma}{2}\right)\right|^2 d\sigma$$

$$\leq \int\limits_{|\eta|>|\xi|} \left|\widehat{F}\left(\frac{\xi-\eta}{2},\frac{\xi+\eta}{2}\right)\right| \left|\left(\nabla_2\widehat{F}-\nabla_1\widehat{F}\right)\left(\frac{\xi-\eta}{2},\frac{\xi+\eta}{2}\right)\right| \frac{d\eta}{|\eta|^{N-1}}$$

where $\nabla_1\widehat{F}$ and $\nabla_2\widehat{F}$ are the gradients of $\widehat{F}$ with respect to the first and second variables. Therefore,

$$\int\limits_{\xi\in\mathbb{R}^N} d\xi\,|\xi|^{N-1} \int\limits_{\sigma\in S^{N-1}} \left|\widehat{F}\left(\frac{\xi-|\xi|\sigma}{2},\frac{\xi+|\xi|\sigma}{2}\right)\right|^2 d\sigma$$

$$\leq 2^N(2\pi)^{2N}\|F\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}\|(v_2-v_1)F\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)},$$

and together with (170), we obtain (163). □

6.3 Gain of smoothness in the non cutoff case

6.3.1 Introduction and presentation of the estimate

As specified in the general introduction of this section, we shall not try here to use the formula which gives the Fourier transform of $Q(f,f)$ in terms of the Fourier transform of f for non Maxwellian molecules. Instead, we shall choose a quantity (the entropy dissipation) which is monotonous with respect to the cross section, so that it is possible to estimate it in terms of the same quantity for Maxwellian molecules. Then, a computation close to that of subsection 5.6 yields an estimate of regularity (typically, some Sobolev norm of $\sqrt{f}$ can be estimated by the entropy dissipation).

In this subsection, we consider only the dimension three, and we take a cross section B which satisfy the two following assumption (for all $x\geq 0$, $\theta\in[0,\pi]$) :

$$K_0\,|\theta|^{-1-\nu} \leq \sin\theta\, B(x,cos\theta) \leq K_1\,(1+|x|)\,|\theta|^{-1-\nu}, \tag{171}$$

for some $K_0, K_1>0$ and $\nu\in]0,2[$.

This is a typical assumption of non cutoff hard potentials (including Maxwellian molecules), except that usually for hard potentials, the cross section takes the value 0 for $x=0$. This last difficulty leads to tremendous technicalities but can be overcome. We shall not present those difficulties here. This subsection presents works which are included in [4]. In this reference can be found a much more complete overview of the problems tackled here.

We shall prove here the following estimate :

Theorem 18 *Under assumption (171) on the cross section, one has*

$$D(f) \geq c_1 \|\sqrt{f}\|^2_{H^{\nu/2}} - c_2 \|f\|^2_{L^1_2}. \tag{172}$$

for some constants c_1 and c_2 which may depend on K_0, ν and (only) on the mass, entropy and energy of f.

6.3.2 Proof of the estimate

First we use the monotonicity of D with respect to the cross section B in order to replace B by $b \equiv b(\frac{v-v_*}{|v-v_*|} \cdot \sigma)$ defined by

$$\sin\theta \, b(\cos\theta) = K_0 \, |\theta|^{-1-\nu}. \tag{173}$$

We get

$$D(f) = -\int_{\mathbb{R}^{2N} \times S^{N-1}} \Big(f(v'_*) \, f(v') - f(v_*) \, f(v) \Big) \log f(v) \, B \, dv \, dv_* \, d\sigma \tag{174}$$

$$\geq -\int_{\mathbb{R}^{2N} \times S^{N-1}} \Big(f(v'_*) \, f(v') - f(v_*) \, f(v) \Big) \log f(v) \, b \, dv \, dv_* \, d\sigma. \tag{175}$$

Then, we rewrite $D(f)$ using the standard pre/post collisional change of variables :

$$D(f) \geq -\int_{\mathbb{R}^{2N} \times S^{N-1}} \Big(f(v'_*) \, f(v') - f(v_*) \, f(v) \Big) \log f(v) \, b \, dv \, dv_* \, d\sigma \tag{176}$$

$$\geq \int_{\mathbb{R}^{2N} \times S^{N-1}} f(v_*) \, f(v) \, \log \frac{f(v)}{f(v')} \, b \, dv \, dv_* \, d\sigma$$

$$= \int_{\mathbb{R}^{2N} \times S^{N-1}} f(v_*) \, \left(f(v) \, \log \frac{f(v)}{f(v')} - f(v) + f(v') \right) b \, dv \, dv_* \, d\sigma$$

$$+ \int_{\mathbb{R}^{2N} \times S^{N-1}} f(v_*) \, (f(v) - f(v')) \, b \, dv \, dv_* \, d\sigma \, .$$

This decomposition splits $D(f)$ into two parts, the first of which is signed and retains all the smoothness control. As for the second, it involves strong cancellations due to the presence of the term $f(v) - f(v')$.

Under our assumptions on the cross-section, a general lemma (called cancellation lemma) of [4] gives a bound for the second term on the right,

$$\int f(v_*)\,(f(v) - f(v'))\, b\, dv\, dv_*\, d\sigma \le c_2\, \|f\|^2_{L^1_2}.$$

For the first term, we use the inequality

$$x \log \frac{x}{y} - x + y \ge (\sqrt{x} - \sqrt{y})^2\,,$$

which can be proven easily using the fact that it is homogeneous of degree one.

Hence

$$D(f) + c_2\, \|f\|^2_{L^1_2} \ge \int f(v_*)\,(\sqrt{f(v')} - \sqrt{f(v)})^2\, b\, dv\, dv_*\, d\sigma. \tag{177}$$

From now on, we let

$$F(v) \quad = \quad \sqrt{f(v)}$$

and we use the notation F' for $F(v')$.

Then we use the following result (written in an arbitrary dimension N) :

Lemma 3 *The following Plancherel-type identity holds for arbitrary functions $g \in L^1(\mathbb{R}^N)$, $F \in L^2(\mathbb{R}^N)$:*

$$\int_{\mathbb{R}^{2N}} \int_{S^{N-1}} g(v_*)\,(F' - F)^2\, b\left(\frac{v - v_*}{|v - v_*|}\cdot\sigma\right)\, dv\, dv_*\, d\sigma \tag{178}$$

$$= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} \Big[\hat{g}(0)|\hat{F}(\xi)|^2 + \hat{g}(0)|\hat{F}(\xi^+)|^2$$

$$- \hat{g}(\xi^-)\hat{F}(\xi^+)\overline{\hat{F}}(\xi) - \overline{\hat{g}}(\xi^-)\overline{\hat{F}}(\xi^+)\hat{F}(\xi)\Big]\, b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\, d\xi\, d\sigma\,,$$

with the notations of (126).

Proof of Lemma 3 :
Expanding the quadratic term in (178) gives three terms,

$$F'^2 - 2FF' + F^2 \,. \tag{179}$$

From now on, we denote by Q_b (and Q_b^+) Boltzmann's operator (and its positive part) with the cross section b (that of Maxwellian molecules).

We begin with the middle term. By the pre/post collisional change of variables and Parseval's identity,

$$\int b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*)\, F'F\, dv\, dv_*\, d\sigma \;=\; \int Q_b^+(g,F)F\, dv$$

$$= \frac{1}{(2\pi)^N}\int \mathcal{F}\Big[Q_b^+(g,F)\Big]\overline{\hat{F}}d\xi \,.$$

Then, we invoke Bobylev's identity (124) and deduce that

$$\int b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*)\, F'F\, dv\, dv_*\, d\sigma$$

$$= \frac{1}{(2\pi)^N}\int\, b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\hat{g}(\xi^-)\hat{F}(\xi^+)\overline{\hat{F}}(\xi) d\xi\, d\sigma \,.$$

Of course, this expression is also equal to its own complex conjugate. This shows how to compute the cross-products in (178).

Next, we note that, since $\int_{S^{N-1}} b(k\cdot\sigma)\, d\sigma$ does not depend on the unit vector k,

$$\int b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*)\, F^2\, dv\, dv_*\, d\sigma \;=\; \int\, d\sigma\, \int g(v_*)\, dv_* \,\int F^2\, dv \tag{180}$$

$$= \frac{1}{(2\pi)^N}\int b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\hat{g}(0)\,|\hat{F}|^2(\xi)\, d\sigma d\xi,$$

where we have applied the usual Plancherel identity.

For the term involving F'^2, we first make the change of variables $(v,v_*)\to(v-v_*,v_*)$, and then $v\to v'$ to obtain

$$\int\int g(v_*)\, b(\frac{v}{|v|}\cdot\sigma)\left|\tau_{-v_*}F(\frac{v+|v|\sigma}{2})\right|^2 dv\, d\sigma\, dv_* \tag{181}$$

$$= \int \int g(v_*)\, b(\psi(v',\sigma))\, \frac{2^{N-1}}{(\frac{v'}{|v'|}\cdot\sigma)^2}\, |\tau_{-v*}F(v')|^2\, dv'\, d\sigma\, dv_* \,,$$

where

$$\psi(v',\sigma) \quad = \quad 2\left(\frac{v'}{|v'|}\cdot\sigma\right)^2 - 1 \,,$$

and $\tau_{-v_*}F = F(v_* + \cdot)$.

Because $|\mathcal{F}(\tau_h F)| = |\mathcal{F}(F)|$, and using the fact that $\int_{S^{N-1}} b(k\cdot\sigma)\, d\sigma$ does not depend on k, we obtain

$$\frac{1}{(2\pi)^N}\int g(v_*)\Big(\int b(\psi(\xi,\sigma))\frac{2^{N-1}}{\left(\frac{\xi}{|\xi|}\cdot\sigma\right)^2}|\hat{F}(\xi)|^2 d\xi\, d\sigma\Big)dv_* \,.$$

Finally we note that the inner integral does not depend on v_*, so that, reversing the change of variables, we can rewrite the last expression as

$$\frac{1}{(2\pi)^N}\hat{g}(0)\int b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left|\hat{F}\left(\frac{\xi+|\xi|\sigma}{2}\right)\right|^2 d\xi\, d\sigma.$$

Putting all the pieces together, we conclude the proof of the identity. $\square$

As a consequence, we see that

$$\int_{\mathbb{R}^{2N}}\int_{S^{N-1}} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\, g(v_*)\,(F'-F)^2\, dv\, dv_*\, d\sigma$$

$$\geq \frac{1}{2(2\pi)^N}\int_{\mathbb{R}^N}|\hat{F}(\xi)|^2\left\{\int_{S^{N-1}} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)(\hat{g}(0)-|\hat{g}(\xi^-)|)\, d\sigma\right\} d\xi \,.$$

Then, we use the following result :

Lemma 4 *Suppose that b satisfies assumption (173). Then, there exists a positive constant C_g depending only on the mass, energy and entropy of g and b such that for $|\xi|\geq 1$,*

$$\int_{S^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)(\hat{g}(0)-|\hat{g}(\xi^-)|)\, d\sigma \geq C_g\,|\xi|^\nu. \tag{182}$$

This lemma is itself a consequence of the two lemmas below.

Lemma 5 *There exists a positive constant C'_g, depending only on the mass, energy and entropy of g such that for all $\xi \in \mathbb{R}^3$,*

$$\hat{g}(0) - |\hat{g}(\xi)| \geq C'_g\,(|\xi|^2 \wedge 1).$$

Proof of lemma 5 : Note first that for some $\theta \in \mathbb{R}$,

$$\hat{g}(0) - |\hat{g}(\xi)| = \int_{\mathbb{R}^3} g(v)\,(1 - \cos(v\cdot\xi + \theta))\,dv$$

$$= 2\int_{\mathbb{R}^3} g(v)\,\sin^2\left(\frac{v\cdot\xi+\theta}{2}\right)\,dv$$

$$\geq 2\,\sin^2\varepsilon \int_{\{|v|\leq r, \forall p\in\mathbb{Z}, |v\cdot\xi+\theta-2\,p\,\pi|\geq 2\,\varepsilon\}} g(v)\,dv$$

$$\geq 2\,\sin^2\varepsilon\left\{||g||_{L^1(\mathbb{R}^3)} - \frac{||g||_{L^1_1(\mathbb{R}^3)}}{r} - \int_{|v|\leq r, \exists p\in\mathbb{Z}, |v\cdot\frac{\xi}{|\xi|}+\frac{\theta}{|\xi|}-p\,\frac{2\,\pi}{|\xi|}|\leq 2\,\frac{\varepsilon}{|\xi|}} g(v)\,dv\right\}$$

$$\geq 2\,\sin^2\varepsilon\left\{||g||_{L^1(\mathbb{R}^3)} - \frac{||g||_{L^1_1(\mathbb{R}^3)}}{r} - \sup_{|A|\leq\frac{4\,\varepsilon}{|\xi|}\,(2\,r)^2\,(1+\frac{r\,|\xi|}{\pi})} \int_A g(v)\,dv\right\}. \quad (183)$$

When $|\xi| \geq 1$, we obtain our lemma with

$$C'_g = 2\,\sin^2\varepsilon\left\{||g||_{L^1(\mathbb{R}^3)} - \frac{||g||_{L^1_1(\mathbb{R}^3)}}{r} - \sup_{|A|\leq 4\,\varepsilon\,(2\,r)^2+\frac{2\varepsilon}{\pi}(2\,r)^3} \int_A g(v)\,dv\right\},$$

$\varepsilon > 0$ and $r > 0$ being chosen in such a way that this quantity is positive.

When $|\xi| \leq 1$, we put $\delta = \frac{\varepsilon}{|\xi|}$ in (183), and set

$$C'_g = 2\,\delta^2 \inf_{|\xi|\leq 1}\left|\frac{\sin^2(\delta\,|\xi|)}{\delta^2\,|\xi|^2}\right|$$

$$\times\left\{||g||_{L^1(\mathbb{R}^3)} - \frac{||g||_{L^1_1(\mathbb{R}^3)}}{r} - \sup_{|A|\leq 4\,\delta\,(2\,r)^2\,(1+\frac{r}{\pi})} \int_A g(v)\,dv\right\},$$

$\delta > 0$ and $r > 0$ being chosen in such a way that this quantity is positive. $\square$

Lemma 6 *There exists a constant $K(\nu)$, such that if*

$$\sin\theta\, b(\cos\theta) \quad \sim \quad \frac{K}{\theta^{1+\nu}} \qquad as \quad \theta \to 0, \qquad\qquad \nu > 0$$

then for all $\xi \in \mathbb{R}^3$, $|\xi| \geq 1$,

$$\int_{S^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)(|\xi^-|^2 \wedge 1)\,d\sigma \geq K(\nu)|\xi|^\nu.$$

Proof of lemma 6 :
We first note that

$$|\xi^-|^2 = \frac{|\xi|^2}{2}\left(1 - \frac{\xi}{|\xi|}\cdot\sigma\right).$$

Passing to spherical coordinates, we find for some $\theta_0 > 0$,

$$\begin{aligned}\int_{S^2} b(\frac{\xi}{|\xi|}\cdot\sigma)(|\xi^-|^2 \wedge 1)\,d\sigma &= 2\pi \int_0^{\frac{\pi}{2}} \sin\theta\, b(\cos\theta) \\ &\times\left[\frac{|\xi|^2}{2}(1-\cos\theta)\wedge 1\right]d\theta \\ &\geq \pi\, K \int_0^{\theta_0}\left(\frac{|\xi|^2\theta^2}{2}\wedge 1\right)\frac{d\theta}{\theta^{1+\nu}}.\end{aligned}$$

By the change of variables $\theta \to |\xi|\theta$, this integral is also

$$|\xi|^\nu \int_0^{\theta_0}\left(\frac{\theta^2}{2}\wedge 1\right)\frac{d\theta}{\theta^{1+\nu}},$$

so that when $|\xi| \geq 1$, lemma 6 holds with

$$K(\nu) = K\,\pi\int_0^{\theta_0}(\frac{\theta^2}{2} - 1)\,\frac{d\theta}{\theta^{1+\nu}}.$$

□

6.3.3 Regularity for the spatially homogeneous Boltzmann equation without cutoff

Let B be a cross section satisfying assumption (171), and f a solution of (25), (26) given by theorem 1.

A straightforward application of Theorem 18 shows that such a solution satisfies the smoothness estimate

$$\sqrt{f} \in L^2([0,T]; H^{\nu/2}_{\text{loc}}(\mathbb{R}^N_v)). \tag{184}$$

If we suppose moreover that B is smooth (and corresponds to hard potential) with respect to the first variable, then it is possible (at least in dimension two) to prove that f lies in Schwartz's space $\mathcal{S}$.

7 Inhomogeneous Dissipative equations

7.1 Introduction

We now wish to investigate the interaction of the free transport operator and of the non cutoff Boltzmann operator. Unfortunately, there is at the present time no good setting to study the smoothness of the solution of this equation (the renormalized solutions with a defect measure of Alexandre and Villani (Cf. [5]) do not seem to be regular enough). As a consequence, we turn to simplified models keeping the same features.

We begin with the classical linear model of Vlasov-Fokker-Planck with a confining potential, which models particles interacting with a thermal bath. This is a linear second order PDE, for which it is possible to use the theory of Hörmander of hypoellipticity (Cf. [44], [49], [50], [24]). We propose here a direct computation by Fourier transform when the potential is quadratic (this enables to find a classical explicit solution in this case), or close to quadratic (then, this computation enables to directly find the smoothness in all variables even when the time tends to infinity).

Then, we introduce a model which is quadratic, but close to linear (in the sense that the collision operator is a product of a function depending only on t and x by a linear operator). We prove that some smoothness in all variables occurs as soon as $t > 0$.

7.2 Vlasov-Fokker-Planck equation with quadratic potential

We consider in this subsection the Vlasov-Fokker-Planck equation with a quadratic confining potential, that is, equation

$$\partial_t f + v \cdot \nabla_x f - x \cdot \nabla_v f - \nabla_v \cdot (\nabla_v f + v\, f) = 0. \tag{185}$$

We perform here a classical computation which enables to obtain the explicit (Fourier transform of the) solution to this equation, once an initial datum is given.

We first write down the Fourier transform in x and v of eq. (185). We denote by ξ and η the corresponding Fourier variables, and by $\hat{f}$ the Fourier transform of f. p This equation writes

$$\partial_t \hat{f} + \eta \cdot \nabla_\xi \hat{f} + (\eta - \xi) \cdot \nabla_\eta \hat{f} + |\eta|^2\, \hat{f} = 0. \tag{186}$$

We introduce the characteristic differential system associated to eq. (186) :

$$\dot{\xi} = \eta, \qquad \xi(0) = \xi_0, \tag{187}$$

$$\dot{\eta} = \eta - \xi, \qquad \eta(0) = \eta_0, \tag{188}$$

the solution of which is given by

$$(\xi(t), \eta(t)) = \frac{2}{\sqrt{3}} e^{\frac{t}{2}} \left[\left(\frac{\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2} t\right) - \frac{1}{2} \sin\left(\frac{\sqrt{3}}{2} t\right) \right) \xi_0 + \sin\left(\frac{\sqrt{3}}{2} t\right) \eta_0, \right.$$

$$\left. - \sin\left(\frac{\sqrt{3}}{2} t\right) \xi_0 + \left(\frac{\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2} t\right) + \frac{1}{2} \sin\left(\frac{\sqrt{3}}{2} t\right) \right) \eta_0 \right]. \tag{189}$$

Then, the solution of equation (186) satisfies

$$\frac{d}{dt} \hat{f}(t, \xi(t), \eta(t)) = -|\eta(t)|^2 \hat{f}(t, \xi(t), \eta(t)), \tag{190}$$

so that

$$\hat{f}(t, \xi(t), \eta(t)) = \hat{f}(0, \xi_0, \eta_0) \exp\left(\frac{1}{6} (|\eta_0|^2 + |\xi_0|^2 - 4\, \xi_0 \cdot \eta_0)\, e^t \cos(\sqrt{3}\, t) \right.$$

$$+ \frac{\sqrt{3}}{6} (|\xi_0|^2 - |\eta_0|^2)\, e^t \sin(\sqrt{3}\, t)$$

$$\left. + \frac{2}{3} (-|\xi_0|^2 - |\eta_0|^2 + \xi_0 \cdot \eta_0)\, e^t + \frac{1}{2} (|\xi_0|^2 + |\eta_0|^2) \right). \tag{191}$$

Noticing now that equations (189) can be solved in the form

$$\xi_0 = e^{-\frac{t}{2}} \left(\left[\cos\left(\frac{\sqrt{3}}{2} t\right) + \frac{\sqrt{3}}{3} \sin\left(\frac{\sqrt{3}}{2} t\right) \right] \xi - 2 \frac{\sqrt{3}}{3} \sin\left(\frac{\sqrt{3}}{2} t\right) \eta \right),$$

$$\eta_0 = e^{-\frac{t}{2}} \left(2 \frac{\sqrt{3}}{3} \sin\left(\frac{\sqrt{3}}{2} t\right) \xi + \left[\cos\left(\frac{\sqrt{3}}{2} t\right) - \frac{\sqrt{3}}{3} \sin\left(\frac{\sqrt{3}}{2} t\right) \right] \eta \right).$$

We obtain in this way the final explicit form of the Fourier transform of eq. (185) :

$$\hat{f}(t, \xi, \eta) = \hat{f}\left[0, e^{-\frac{t}{2}} \left((\cos\left(\frac{\sqrt{3}}{2} t\right) + \frac{\sqrt{3}}{3} \sin\left(\frac{\sqrt{3}}{2} t\right)) \xi - 2 \frac{\sqrt{3}}{3} \sin\left(\frac{\sqrt{3}}{2} t\right) \eta \right), \right.$$

$$e^{-\frac{t}{2}} \left(2\,\frac{\sqrt{3}}{3}\,\sin\left(\frac{\sqrt{3}}{2}t\right)\,\xi + (\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{3}\,\sin\left(\frac{\sqrt{3}}{2}t\right))\,\eta\right)\Big]\,e^{A(t,\xi,\eta)},$$

where

$$A(t,\xi,\eta) = \Big(-\frac{1}{2} + \frac{2}{3}\,e^{-t} - \frac{1}{6}\,e^{-t}\,\cos\left(\sqrt{3}t\right) + \frac{\sqrt{3}}{6}\,e^{-t}\,\sin\left(\sqrt{3}t\right)\Big)\,|\xi|^2$$

$$+\Big(-\frac{1}{2} + \frac{2}{3}\,e^{-t} - \frac{1}{6}\,e^{-t}\,\cos\left(\sqrt{3}t\right) - \frac{\sqrt{3}}{6}\,e^{-t}\,\sin\left(\sqrt{3}t\right)\Big)\,|\eta|^2$$

$$-\frac{4}{3}\,e^{-t}\,\sin\left(\frac{\sqrt{3}}{2}t\right)^2\,\xi\cdot\eta.$$

Then, it is possible (by studying the quadratic form appearing in the previous formula : this is done in lemma 7 below) to prove that f is smooth as soon as $t > 0$.

The idea of the previous computation can be summarized in the following remark : the Fourier transform changes a linear partial differential equation with constant coefficients into an ordinary differential equation (the Fourier transform is not taken here with respect to the time variable). It also changes a linear partial differential equation with affine coefficients into a first order partial differential equation. Such an equation can then be solved with the methods of characteristics.

7.3 Vlasov-Fokker-Planck equation with a potential close to quadratic

We now introduce a confining potential

$$V(x) = \frac{|x|^2}{2} + \Phi(x), \tag{192}$$

where $\Phi \in H^\infty(\mathbb{R}^N)$.

It is not possible to find an explicit solution to the corresponding Vlasov-Fokker-Planck equation

$$\partial_t f + v\cdot\nabla_x f - \nabla_x V(x)\cdot\nabla_v f - \nabla_v\cdot(\nabla_v f + v\,f) = 0, \tag{193}$$

as in the previous subsection, but we still can obtain an hypoellipticity property which is uniform when $t\to\infty$, using a computation close to what we did in the previous subsection.

More precisely, we prove the following proposition :

Proposition 4 *Let $f \in C(\mathbb{R}_t^+, L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N))$ be a solution of eq. (193), with $V(x)$ given by (192). Then, for any $t_0 > 0$, the function f lies in the space $L^\infty([t_0, +\infty); C_b^\infty(\mathbb{R}^N \times \mathbb{R}^N))$, i.e. has all its derivatives in x and v bounded, uniformly for $t \geq t_0 > 0$.*

Proof : We first establish a convenient representation formula. We rewrite equation (193) as

$$\partial_t f + v \cdot \nabla_x f - x \cdot \nabla_v f - \nabla_v \cdot (\nabla_v f + v\, f) = \nabla\Phi(x) \cdot \nabla_v f, \tag{194}$$

and denote by

$$\hat{f}(t, \xi, \eta) = \int_{\mathbb{R}^N \times \mathbb{R}^N} e^{-i\,(x\cdot\xi + v\cdot\eta)}\, f(t, x, v)\, dv\, dx \tag{195}$$

the Fourier transform of f.

Eq. (194) becomes

$$\partial_t \hat{f} + \eta \cdot \nabla_\xi \hat{f} + (\eta - \xi) \cdot \nabla_\eta \hat{f} + |\eta|^2\, \hat{f} = i\, \eta \cdot \widehat{\nabla\Phi\, f}. \tag{196}$$

We introduce (as in the previous section) the characteristic differential system associated to the first-order differential part of the left-hand side of (196) :

$$\dot{\xi} = \eta, \tag{197}$$

$$\dot{\eta} = \eta - \xi, \tag{198}$$

the solution of which is given by the flow

$$\begin{aligned} T_t(\xi, \eta) = \frac{2}{\sqrt{3}} e^{\frac{t}{2}} \Bigg[&\left(\frac{\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2} t\right) - \frac{1}{2} \sin\left(\frac{\sqrt{3}}{2} t\right) \right) \xi + \sin\left(\frac{\sqrt{3}}{2} t\right) \eta, \\ &- \sin\left(\frac{\sqrt{3}}{2} t\right) \xi + \left(\frac{\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2} t\right) + \frac{1}{2} \sin\left(\frac{\sqrt{3}}{2} t\right) \right) \eta \Bigg] \\ &\equiv [T_t^1(\xi, \eta), T_t^2(\xi, \eta)]. \end{aligned}$$

The solution of equation (196) can be written under the (semi–explicit) Duhamel form

$$\hat{f}(t, \xi, \eta) = \hat{f}_0(T_{-t}(\xi, \eta))\, e^{-\int_0^t |T^2_{\sigma - t}(\xi, \eta)|^2\, d\sigma} \tag{199}$$

$$+\, i \int_0^t T^2_{s-t}(\xi,\eta)\, \widehat{\nabla\Phi\, f}(s, T_{s-t}(\xi,\eta))\, e^{-\int_s^t |T^2_{\sigma-t}(\xi,\eta)|^2\, d\sigma}\, ds.$$

After the change of variables $\sigma \to t-\sigma$, $s \to t-s$, we end up with the so-called Duhamel representation of $\hat f$:

$$\hat f(t,\xi,\eta) = \hat f_0(T_{-t}(\xi,\eta))\, e^{-\int_0^t |T^2_{-\sigma}(\xi,\eta)|^2\, d\sigma}$$

$$+\, i \int_0^t T^2_{-s}(\xi,\eta)\, \widehat{\nabla\Phi\, f}(t-s, T_{-s}(\xi,\eta))\, e^{-\int_0^s |T^2_{-\sigma}(\xi,\eta)|^2\, d\sigma}\, ds. \tag{200}$$

We now give two lemmas.

Lemma 7 *There exists $K>0$, such that for any $s \geq 0$, $\xi, \eta \in I\!R^N$, one has*

$$\int_0^s |T^2_{-\sigma}(\xi,\eta)|^2\, d\sigma \geq K\left(\inf(s,1)^3\, |\xi|^2 + \inf(s,1)\, |\eta|^2\right). \tag{201}$$

Proof of Lemma 7 : It is obviously enough to prove the lemma for $s \in [0, s_0]$ for some $s_0 < 1$.

But for $s \in [0, s_0]$, we have

$$\int_0^s |T^2_{-\sigma}(\xi,\eta)|^2\, d\sigma \geq \frac{4}{3}\, e^{-1} \int_0^s \Bigg| \sin(\frac{\sqrt3}{2}\sigma)\, \xi$$

$$+ \left(\frac{\sqrt3}{2}\cos(\frac{\sqrt3}{2}\sigma) - \frac12 \sin(\frac{\sqrt3}{2}\sigma)\right)\, \eta\Bigg|^2\, d\sigma$$

$$\geq \frac23\, e^{-1}\Bigg((s - \frac{\sin(\sqrt3\, s)}{\sqrt3})\, |\xi|^2 + (1 - \cos(\sqrt3\, s) + \frac{\sin(\sqrt3\, s)}{\sqrt3} - s)\, \xi\cdot\eta$$

$$+(\frac12 \frac{\sin(\sqrt3\, s)}{\sqrt3} + s + \frac12 \cos(\sqrt3\, s) - \frac12)\, |\eta|^2\Bigg)$$

$$= \frac23\, e^{-1}\left(\alpha_1(s)\, (s^3\, |\xi|^2) + 2\,\alpha_2(s)\, (s^2\, \xi\cdot\eta) + \alpha_3(s)\, (s\, |\eta|^2)\right), \tag{202}$$

where

$$\alpha_1(s) = \frac{s - \frac{\sin(\sqrt3\, s)}{\sqrt3}}{s^3}, \qquad \alpha_2(s) = \frac{1 - \cos(\sqrt3\, s) + \frac{\sin(\sqrt3\, s)}{\sqrt3} - s}{2\, s^2}, \tag{203}$$

$$\alpha_3(s) = \frac{\frac12 \frac{\sin(\sqrt3\, s)}{\sqrt3} + s + \frac12 \cos(\sqrt3\, s) - \frac12}{s}. \tag{204}$$

Then, $\alpha_1(0) = 1$, $\alpha_2(0) = 3/4$, $\alpha_3(0) = 3/2$.

The eigenvalues of the matrix

$$\mathcal{M}(s) = \begin{pmatrix} \alpha_1(s) & \alpha_2(s) \\ \alpha_2(s) & \alpha_3(s) \end{pmatrix}$$

are strictly positive for $s = 0$, and by continuity, are bounded below by $K > 0$ for $s \in [0, s_0]$ if s_0 is small enough.

For such parameters s, we get

$$\int_0^s |T^2_{-\sigma}(\xi,\eta)|^2 \, d\sigma \geq \frac{2}{3} \, e^{-1} \, K \, (s^3 \, |\xi|^2 + s \, |\eta|^2), \tag{205}$$

and the lemma is proven. □

Lemma 8 *Let $s_0 \in [0,1]$ and*

$$L_{s_0}(\xi,\eta) = \int_0^{s_0} (s \, |\xi| + |\eta|) \, e^{-K \, (s^3 \, |\xi|^2 + s \, |\eta|^2)} ds. \tag{206}$$

Then there exists $C > 0$ (depending only on K) such that

$$|L_{s_0}(\xi,\eta)| \leq \frac{C}{1 + |\xi|^{1/3} + |\eta|}. \tag{207}$$

Proof : Thanks to the change of variables $u = s \, |\xi|^{2/3}$ and $v = s \, |\eta|^2$, we get

$$\begin{aligned} \int_0^{+\infty} s \, |\xi| \, e^{-K \, (s^3 \, |\xi|^2 + s \, |\eta|^2)} ds &\leq \int_0^{+\infty} s \, |\xi| \, e^{-K \, s^3 \, |\xi|^2} \, ds \\ &\leq |\xi|^{-1/3} \int_0^{+\infty} u \, e^{-K \, u^3} \, du, \end{aligned} \tag{208}$$

and

$$\begin{aligned} \int_0^{+\infty} |\eta| \, e^{-K \, (s^3 \, |\xi|^2 + s \, |\eta|^2)} ds &\leq \int_0^{+\infty} |\eta| \, e^{-K \, s \, |\eta|^2} ds \\ &\leq |\eta|^{-1} \int_0^{+\infty} e^{-K \, v} \, dv. \end{aligned} \tag{209}$$

On the other hand, if we denote

$$C_1 = \sup_{u \in [0,+\infty)} u^{3/2} \, e^{-K \, u^3}, \qquad C_2 = \sup_{v \in [0,+\infty)} v^{1/2} \, e^{-K \, v}, \tag{210}$$

we find

$$\int_0^{+\infty} s\,|\xi|\,e^{-K\,(s^3\,|\xi|^2+s\,|\eta|^2)}ds \leq C_1 \int_0^{+\infty} s^{-1/2}\,e^{-K\,s\,|\eta|^2}\,ds$$

$$\leq C_1\,|\eta|^{-1} \int_0^{+\infty} v^{-1/2}\,e^{-K\,v}\,dv, \tag{211}$$

and

$$\int_0^{+\infty} |\eta|\,e^{-K\,(s^3\,|\xi|^2+s\,|\eta|^2)}ds \leq C_2 \int_0^{+\infty} s^{-1/2}\,e^{-K\,s^3\,|\xi|^2}\,ds$$

$$\leq C_2\,|\xi|^{-1/3} \int_0^{+\infty} u^{-1/2}\,e^{-K\,u^3}\,du. \tag{212}$$

Grouping estimates (208), (209), (211) and (212), we conclude the proof of lemma 8. □

End of the proof of Proposition 4 : By mass conservation,

$$\sup_{t\geq 0}\sup_{\xi,\eta\in\mathbb{R}^N} |\hat{f}(t,\xi,\eta)| \leq \|f_0\|_{L^1(\mathbb{R}^N\times\mathbb{R}^N)}. \tag{213}$$

We shall show that if

$$\sup_{t\geq 0} |\hat{f}(t,\xi,\eta)| \leq \frac{C_k}{(1+|\xi|^2+|\eta|^2)^k}$$

($k\in\mathbb{R}^+$), then for any $t_0>0$,

$$\sup_{t\geq t_0} |\hat{f}(t,\xi,\eta)| \leq \frac{C'_k}{(1+|\xi|^2+|\eta|^2)^{k+\frac{1}{6}}}. \tag{214}$$

The conclusion will follow by induction.

We first note that in view of (213) and lemma 7, estimate (214) holds with $\hat{f}$ replaced by

$$A(t,\xi,\eta) = \hat{f}_0(T_{-t}(\xi,\eta))\,e^{-\int_0^t |T^2_{-\sigma}(\xi,\eta)|^2\,d\sigma}.$$

Thus, according to the Duhamel representation, we only need to estimate

$$B(t,\xi,\eta) = \int_0^t T^2_{-s}(\xi,\eta)\,\widehat{\nabla\Phi f}(t-s,T_{-s}(\xi,\eta))\,e^{-\int_0^s |T^2_{-\sigma}(\xi,\eta)|^2\,d\sigma}\,ds. \tag{215}$$

With C_k denoting various constants depending on one another, we have

$$|\widehat{\nabla\Phi f}(t,\xi,\eta)| = \left|\int \widehat{\nabla\Phi}(\xi_*)\,\hat{f}(t,\xi-\xi_*,\eta)\,d\xi_*\right|$$

$$\le \int_{|\xi_*|\le\frac12|\xi|} d\xi_*\,|\widehat{\nabla\Phi}(\xi_*)|\frac{C_k}{(1+|\xi|^2+|\eta|^2)^k}$$

$$+\int_{|\xi_*|\ge\frac12|\xi|} d\xi_*\,|\widehat{\nabla\Phi}(\xi_*)|\frac{C_k}{(1+|\eta|^2)^k}$$

$$\le \frac{C_k}{(1+|\xi|^2+|\eta|^2)^k}\|\widehat{\nabla\Phi}\|_{L^1}$$

$$+\frac{C_k}{(1+|\xi|^2)^k(1+|\eta|^2)^k}\int_{I\!R^N}(1+|\xi_*|^2)^k|\widehat{\nabla\Phi}(\xi_*)|\,d\xi_*.$$

Since

$$\int_{I\!R^N}|\widehat{\nabla\Phi}(\xi_*)|(1+|\xi_*|^2)^k\,d\xi_*$$

$$\le \left[\int_{I\!R^N}|\widehat{\nabla\Phi}(\xi_*)|^2(1+|\xi_*|^2)^{2k+N+1}\,d\xi_*\right]^{1/2}\left[\int_{I\!R^N}\frac{d\xi_*}{(1+|\xi_*|^2)^{N+1}}\right]^{1/2}$$

$$\le C_k\|\Phi\|_{H^{2k+N+2}},$$

we find

$$\sup_{t\ge0}|\widehat{\nabla\Phi}f(t,\xi,\eta)|\le\frac{C_k}{(1+|\xi|^2+|\eta|^2)^k}.$$

Let $s_0\le\inf(1,t_0)$ be an intermediate time that will be chosen later on.

We write for $t\ge t_0$,

$$|B(t,\xi,\eta)|\le\int_0^t|T^2_{-s}(\xi,\eta)|\,|\widehat{\nabla\Phi}\,f(t-s,T_{-s}(\xi,\eta))|\,e^{-\int_0^s|T^2_{-\sigma}(\xi,\eta)|^2\,d\sigma}\,ds$$

$$\le\int_{s_0}^t 2\,e^{-s/2}(s|\xi|+|\eta|)\,ds\,C_k\,e^{-K(s_0^3|\xi|^3+s_0|\eta|^2)}$$

$$+\int_0^{s_0}(s|\xi|+|\eta|)\frac{C_k}{(1+|T_{-s}(\xi,\eta)|^2)^k}e^{-K(s^3|\xi|^2+s|\eta|^2)}\,ds.$$

By continuity of the flow $t\mapsto T_t(\xi,\eta)$, and its linearity with respect to ξ,η, we can choose $s_0\in(0,\inf(t_0,1))$ in such a way that for all $s\in[0,s_0]$,

$$|T_{-s}(\xi,\eta)|^2\ge\frac12(|\xi|^2+|\eta|^2).$$

Then, for $t\ge t_0$,

$$|B(t,\xi,\eta)|\le C_k\,(|\xi|+|\eta|)\,e^{-K(s_0^3|\xi|^3+s_0|\eta|^2)}$$

$$+\frac{C_k}{(1+|\eta|^2+|\xi|^2)^k}\int_0^{s_0}(s|\xi|+|\eta|)e^{-K(s^3|\xi|^3+s|\eta|^2)}\,ds.$$

The last integral is bounded by

$$\int_0^{+\infty}(s|\xi|+|\eta|)e^{-K(s^3|\xi|^3+s|\eta|^2)},$$

and we conclude by Lemma 8. $\square$

We recall that the hypoellipticity of linear operators of the form $\partial_t + v\cdot\nabla_x - \Delta_v$ is a standard topic [77], which has been systematically studied by Hörmander [44] for instance. In particular, his celebrated theorem of hypoellipticity applies here to show that solutions become immediately C^∞ (and would apply also for much more general linear operators). But we are aware of no study of the uniformity in time of these bounds, whereas the previous computation easily yields this uniformity.

7.4 A Space Inhomogeneous Model without Cutoff Assumption

We now consider a space inhomogeneous Boltzmann equation of the form (16). We suppose that the collision operator is singular.

We suggest the following strategy to obtain a priori regularity estimates on f (steps 2 and 3 below consisting of regularity lemmas analogous to the compactness results in [5], [54], [55]) :

1] use the entropy production (estimated by the H theorem) to control fractional derivatives of the number density in the velocity variable;

2] apply the Velocity Averaging method (see [38], [34]) to obtain smoothness in (t,x,v) on quantities of the form

$$\int f(t,x,w)\chi(v,w)dw \tag{216}$$

for any smooth test function χ; moreover, estimate the norm (in some Sobolev or Besov space) of such velocity average in terms of χ;

3] replace χ by a suitable approximation of the Dirac mass at $v=w$ and use the results of steps 1 and 2 above to finally obtain some regularity on f itself in the variables (t,x,v).

Step 1 above is the result of the study of the previous section. At the present stage, it is however very unclear how to apply steps 2 and 3 of the strategy above to the Boltzmann equation itself. This requires more ideas and probably tremendous technicalities.

However, the method above successfully applies to the caricature of the Boltzmann equation described by equations (32) and (34). which we supplement with the initial data

$$f(0,x,v) = f_0(x,v)\,, \qquad (x,v) \in \mathbb{T}^1 \times \mathbb{T}^1\,. \tag{217}$$

We introduce the assumption on the cross section β that for some $\beta_1, \beta_2 > 0$, $\gamma \in]1,3[$,

$$\beta_1 |\theta|^{-\gamma} \leq \beta(|\theta|) \leq \beta_2 |\theta|^{-\gamma}\,, \qquad \theta \in]-\pi,\pi[\,. \tag{218}$$

Definition. Let β satisfy (218) and $f_0 \geq 0 \in L^\infty(\mathbb{T}^1 \times \mathbb{T}^1)$. An entropic solution of (34), (217) is a function $f \geq 0 \in L^\infty(\mathbb{R}_+^* \times \mathbb{T}^1 \times \mathbb{T}^1) \cap C(\mathbb{R}^+; \mathcal{D}'(\mathbb{T}^1 \times \mathbb{T}^1))$ satisfying (34), (217) in the sense of distributions as well as the following entropy relation : for all $T > 0$,

$$\frac{1}{2} \int_0^T \int_{\mathbb{T}^1} \rho_f(t,x) \left(\iint_{\mathbb{T}^1 \times \mathbb{T}^1} |f(t,x,v+\theta) - f(t,x,v)|^2 \beta(\theta)\, d\theta dv \right) dx dt$$
$$\leq \frac{1}{2} \iint_{\mathbb{T}^1 \times \mathbb{T}^1} |f_0(x,v)|^2\, dx dv - \frac{1}{2} \iint_{\mathbb{T}^1 \times \mathbb{T}^1} |f(T,x,v)|^2\, dx dv\,. \tag{219}$$

Our main result is the

Theorem 19 *Let β satisfy (218) and $f_0 \geq 0 \in L^\infty(\mathbb{T}^1 \times \mathbb{T}^1)$. The Cauchy problem (34), (217) admits an entropic solution $f \in H_{loc}^{s(\gamma)-\epsilon}(\mathbb{R}_+^* \times \mathbb{T}^1 \times \mathbb{T}^1)$ for all $\epsilon > 0$ with*

$$s(\gamma) = \frac{\gamma - 1}{2\,(\gamma+1)\,(\gamma+3)}\,. \tag{220}$$

If $f_0 \geq R_0$ a.e. for some $R_0 > 0$, the value in the right hand side of (220) can be replaced by the better regularity index

$$s(\gamma) = \frac{\gamma - 1}{2\,(\gamma+1)^2}\,. \tag{221}$$

The proof of Theorem 19 proceeds through steps 1-3 above.

We finally say a few words about the most interesting model, namely the true inhomogeneous Boltzmann equation without cutoff. Then, the only existing setting is that of renormalized solutions with a defect measure.

As explained in Lions [57], a smoothness estimate in the v variable like the one in Theorem 18, combined with a so-called renormalized formulation of the spatially inhomogeneous equation (16), is enough to prove that solutions (or approximate solutions) (f_n) of (16) enjoy a property of immediate strong compactification, in the following sense. If the sequence of initial data $(f_0^n)_{n\in\mathbb{N}}$ satisfies only the physically natural bounds

$$\sup_{n\in\mathbb{N}} \int f_0^n(x,v)(1+|x|^2+|v|^2+\log f_0^n(x,v))\,dx\,dv < +\infty,$$

(and is therefore weakly compact in $L^1(\mathbb{R}^N\times\mathbb{R}^N)$), then for all time $t>0$ the sequence $(f^n(t,\cdot,\cdot))$ is *strongly* compact in $L^1(\mathbb{R}^N\times\mathbb{R}^N)$ (i.e., converges a.e., up to extraction).

This property is what remains of the gain of smoothness in all variables when renormalized solutions are concerned.

The strategy runs as follows : first, by the use of a renormalized formulation [5] and [33], and velocity-averaging lemmas [38] and [34], one proves that suitable quantities of the form $\beta(f^n)*_v\phi_\delta$, where ϕ_δ $(\delta>0)$ is a mollifier in the velocity space only, are strongly compact. Then, by truncation arguments, the smoothness estimate in v applies out of a set of small measure in (t,x), (where $\|f^n(t,x,\cdot)\|_{L^1_2}$ may be infinite, etc.). Out of these particular sets, the velocity smoothness entails that $\beta(f^n)*_v\phi_\delta$ is very close to $\beta(f^n)$, uniformly in n, as δ goes to 0, and this is enough to prove strong compactness of $\beta(f^n)$, which in turn implies pointwise convergence of f^n if β is chosen to be one–to–one.

References

[1] Agoshkov, Spaces of functions with differential-difference characteristics and the smoothness of the solution of the transport equation. *Dokl. Acad. Nauk. SSSR, 276*, 6 (1984), 1289–1293.

[2] Alexandre, R. Around 3D Boltzmann nonlinear operator without angular cutoff, a new formulation. *Math. Methods Numer. Anal., 34*, 3 (2000), 575–590.

[3] Alexandre, R. Some solutions of the Boltzmann equation without angular cutoff. *J. Statist. Phys., 104*, 1-2 (2001), 327–358.

[4] Alexandre, R., Desvillettes, L., Villani, C., Wennberg, B. Entropy dissipation and long-range interactions. *Arch. Rat. Mech. Anal., 52*, 4, (2000), 327–355.

[5] Alexandre, R., and Villani, C. On the Boltzmann equation for long-range interactions. *Comm. Pure Appl. Math., 55*, 1, (2002), 30–70.

[6] Andreasson, H. Regularity of the gain term and strong L^1 convergence to equilibrium for the relativistic Boltzmann equation. *SIAM J. Math. Anal., 27*, 5, (1996), 1586–1605.

[7] Aoki, K., Bardos, C., Dogbe, C., Golse, F. A note on the propagation of boundary induced discontinuities in kinetic theory. *Math. Models Methods Appl. Sci.*, **11**, n. 9, (2001), 1581–1595.

[8] Arkeryd, L. On the Boltzmann equation, I and II, *Arch. Rat. Mech. Anal., 45*, (1972), 1–34.

[9] Arkeryd, L. Intermolecular forces of infinite range and the Boltzmann equation. *Arch. Rat. Mech. Anal. 77* (1981), 11–21.

[10] N. Bellomo, A. Palczewski, G. Toscani. *Mathematical topics in nonlinear kinetic theory.* World Scientific, Singapore.

[11] N. Bellomo, G. Toscani. On the Cauchy problem for the nonlinear Boltzmann equation, global existence, uniqueness and asymptotic stability. *J. Math. Phys.*, **26**: 334–338, 1985.

[12] M. Bézard. *Régularité L^p précisée des moyennes dans les équations de transport*, Bull. Soc. Math. France, **122**, (1994), 29–76.

[13] A.V. Bobylev. *The Fourier transform method in the theory of the Boltzmann equation for Maxwell molecules*, Sov. Phys. Dokl., **20**, (1976), 820–822.

[14] A.V. Bobylev. *Exact solutions of the Boltzmann equation*, Sov. Phys. Dokl., **20**, (1976), 822–824.

[15] Bobylev, A. Exact solutions of the nonlinear Boltzmann equation and the theory of relaxation of a Maxwellian gas, Teor. Math. Phys., **60**, (1984), 280–310.

[16] Bobylev, A. The theory of the nonlinear, spatially uniform Boltzmann equation for Maxwellian molecules. *Sov. Sci. Rev. C. Math. Phys. 7* (1988), 111–233.

[17] Bobylev, A., Cercignani, C. Exact eternal solutions of the Boltzmann equation. *J. Statist. Phys. 106* 5/6 (2002), 1019–1038.

[18] Bobylev, A., Cercignani, C. Self-similar solutions of the Boltzmann equation and their applications. *J. Statist. Phys. 106* 5/6 (2002), 1039–1071.

[19] Bouchut, F., and Desvillettes, L. A proof of the smoothing properties of the positive part of Boltzmann's kernel. *Rev. Mat. Iberoamericana 14*, 1 (1998), 47–61.

[20] F. Bouchut, L. Desvillettes. Averaging lemmas without time Fourier transform and application to discretized kinetic equations. *Proc. Roy. Soc. Ed.*, **129A** : 19–36, 1999.

[21] Boudin, L., and Desvillettes, L. On the Singularities of the Global Small Solutions of the full Boltzmann Equation, Monatschefte für Mathematik, **131**, No.2, (2000), 91–108.

[22] Cercignani, C., Illner, R., and Pulvirenti, M. *The Mathematical Theory of Dilute Gases.* Springer, 1994.

[23] C. Cercignani. *The Boltzmann equation and its applications.* Springer, New York, 1988.

[24] Chaleyat-Maurel, M. La condition d'hypoellipticité d'Hörmander. In *Astérisque*, no. 84–85. Soc. Math. Fr., 1981, pp. 189–202.

[25] S. Chapman and T.G. Cowling. *The mathematical theory of non–uniform gases.* Cambridge Univ. Press., London, 1952.

[26] Desvillettes, L. About the regularizing properties of the non-cut-off Kac equation. *Comm. Math. Phys. 168*, 2 (1995), 417–440.

[27] Desvillettes, L. Regularization for the non-cutoff 2D radially symmetric Boltzmann equation with a velocity dependent cross section. *Transp. Theory Stat. Phys. 25*, 3-5 (1996), 383–394.

[28] DESVILLETTES, L. Regularization properties of the 2-dimensional non radially symmetric non cutoff spatially homogeneous Boltzmann equation for Maxwellian molecules. *Transp. Theory Stat. Phys. 26*, 3 (1997), 341–357.

[29] DESVILLETTES, L., GOLSE, F. On the Smoothing Properties of a Model Boltzmann Equation without Grad's Cutoff Assumption, To appear in the Proceedings of the 21^{st} Rarefied Gas Dynamics Conference, 1998, Marseille.

[30] L. DESVILLETTES, F. GOLSE. On a model Boltzmann equation without angular cutoff. *Diff. Int. Eq.*, **13**, n.4-6, (2000), 567–594.

[31] L. DESVILLETTES, S. MISCHLER. About the Splitting Algorithm for Boltzmann and B.G.K. Equations. *Math. Models Methods in Appl. Sciences*, **6**, n.8, (1996), 1079–1101.

[32] DESVILLETTES, L., AND WENNBERG, B. Work in preparation.

[33] DIPERNA, R., AND LIONS, P.L. On the Cauchy problem for the Boltzmann equation : Global existence and weak stability. *Ann. Math. 130* (1989), 312–366.

[34] R.J. DIPERNA, P.-L. LIONS. *Global weak solutions of Vlasov–Maxwell systems*, Comm. Pure Appl. Math., **42**, (1989), 729–757.

[35] DIPERNA, R., LIONS, P.L., AND MEYER, Y. L^p regularity of velocity averages. *Ann. IHP 114* (1991), 271–287.

[36] P. GÉRARD. *Moyennes de solutions d'équations aux dérivées partielles*, Séminaire de l'Ecole Polytechnique, exposé n.11, (1986–1987).

[37] F. GOLSE. *Quelques résultats de moyennisation pour les équations aux dérivées partielles*, Rend. Sem. Mat. Univ. Pol. Torino, Fascicolo Speciale 1988 "Hyperbolic equations" (1987), 101-123.

[38] F. GOLSE, P.-L. LIONS, B. PERTHAME, R. SENTIS *Regularity of the moments of the solution of a transport equation*, J. Funct. Anal., **76**, (1988), 110–125.

[39] F. GOLSE, B. PERTHAME, R. SENTIS. *Un résultat de compacité pour les équations de transport et application au calcul de la limite de la valeur propre principale d'un opérateur de transport*, C. R. Acad. Sc., Série I, **301**, (1985), 341–344.

[40] Goudon, T. On Boltzmann equations and Fokker-Planck asymptotics: influence of grazing collisions. *J. Stat. Phys. 89*, 3-4 (1997), 751–776.

[41] T. Goudon. Generalized invariant sets for the Boltzmann equation. *Math. Models Methods Appl. Sci.*, **7**: 457–476, 1997.

[42] H. Grad. Principles of the kinetic theory of gases. *Flügge's Handbuch der Physik*, **12**: 205–294, 1958.

[43] K. Hamdache. Existence in the large and asymptotic behaviour for the Boltzmann equation. *Japan J. Appl. Math.*, **2**: 1–15, 1985.

[44] Hörmander, L. Hypoelliptic second order differential equations. *Acta Math. 119* (1967), 147–171.

[45] R. Illner, M. Shinbrot. Global existence for a rare gas in an infinite vacuum. *Comm. Math. Phys.*, **95**: 117–126, 1984.

[46] K. Imai, T. Nishida. Global solutions to the initial value problem for the nonlinear Boltzmann equation. *Publ. RIMS Kyoto Univ.*, **12**: 229–239, 1976.

[47] M. Kac. Probability and related topics in the Physical Sciences, New–York, (1959).

[48] S. Kaniel, M. Shinbrot. The Boltzmann equation I: uniqueness and global existence. *Comm. Math. Phys.*, **58**: 65–84, 1978.

[49] Kohn, J. Pseudo-differential operators and hypoellipticity. In *Proc. Symp. Pure Math.* (1969), vol. 23, AMS Providence, RI, pp. 61–69.

[50] Kohn, J. J. Pseudo-differential operators and hypoellipticity. In *Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971)* (Providence, R.I., 1973), Amer. Math. Soc., pp. 61–69.

[51] M. Krook, T.T. Wu, *Formation of Maxwellian tails*, Phys. Rev. Letters, **36**, (1976), 1107.

[52] P.-E. Jabin, B. Perthame. Regularity in kinetic formulations via averaging lemmas. Preprint.

[53] O. Lanford III. *Time evolution of large classical systems*, Lecture Notes in Physics, Springer Verlag, **38**, 1–111, (1975).

[54] LIONS, P.L. Compactness in Boltzmann's equation via Fourier integral operators and applications, I. *J. Math. Kyoto Univ. 34*, **2** (1994), 391–427.

[55] LIONS, P.L. Compactness in Boltzmann's equation via Fourier integral operators and applications, II. *J. Math. Kyoto Univ.* **34** , 2 (1994), 429–461.

[56] LIONS, P.L. On Boltzmann and Landau equations. *Phil. Trans. R. Soc. Lond. A*, **346** (1994), 191–204.

[57] LIONS, P.L. Regularity and compactness for Boltzmann collision operators without angular cut-off. *C.R. Acad. Sci. Paris* **326**, *Série I*, 1 (1998), 37–41.

[58] P.-L. LIONS *Régularité optimale des moyennes en vitesses*, C. R. Acad. Sc., Série I, **320**, (1995), 911–915.

[59] P.-L. LIONS, B. PERTHAME, *Lemmes de moments, de moyenne et de dispersion*, C. R. Acad. Sc., Série I, **314**, (1992), 801–806.

[60] LU, X. A direct method for the regularity of the gain term in the Boltzmann equation. *J. Math. Anal. Appl. 228*, 2 (1998), 409–435.

[61] MISCHLER, S., AND PERTHAME, B. Boltzmann equation with infinite energy: renormalized solutions and distributional solutions for small initial data and initial data close to a Maxwellian. *SIAM J. Math. Anal. 28*, 5 (1997), 1015–1027.

[62] MORGENSTERN, D. General existence and uniqueness proof for spatially homogeneous solutions of the Maxwell-Boltzmann equation in the case of Maxwellian molecules. *Proc. Nat. Acad. Sci. U.S.A. 40* (1954), 719–721.

[63] B. PERTHAME, *Higher moments for kinetic equations: the Vlasov-Poisson and Fokker-Planck cases*, Math. Methods in the Applied Sc., **13**, (1990), 441–452.

[64] B. PERTHAME, P.E. SOUGANIDIS. A limiting case for velocity averaging. *Ann. Sci. Ecole Normale Sup.*, 31 (4) (1998), 591–598.

[65] J. POLEWCZAK. Classical solutions of the Boltzmann equation in all $\mathbb{R}^3$: asymptotic behavior of solutions. *J. Stat. Phys.*, **50**: 611–632, 1988.

[66] Proutière, A. New results of regularization for weak solutions of Boltzmann equation. Preprint, 1996.

[67] Stein, E.M. *Harmonic analysis, Real-variable methods, orthogonality, and oscillatory integrals* Princeton University Press, 1993.

[68] G. Toscani. On the nonlinear Boltzmann equation in unbounded domains. *Arch. Rational Mech. Anal.*, **95** : 37–49, 1986.

[69] G. Toscani, C. Villani. Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas. *J. Stat. Phys.*, **94** : 619–637, 1999.

[70] Truesdell, C., and Muncaster, R. *Fundamentals of Maxwell's kinetic theory of a simple monoatomic gas.* Academic Press, New York, 1980.

[71] S. Ukai, K. Asano. On the Cauchy problem of the Boltzmann equation with a soft potential. *Publ. RIMS Kyoto Univ.*, **18** : 57–99, 1982.

[72] A. Vasseur *Démonstration de la convergence de la méthode du splitting dans le cas à une branche sans utiliser la contraction BV*, preprint Univ. Paris 6.

[73] Villani, C. Regularity estimates via the entropy dissipation for the spatially homogeneous Boltzmann equation without cut-off. *Rev. Mat. Iberoam. 15*, 2 (1999), 335–352.

[74] Villani, C. *Contribution à l'étude mathématique des équations de Boltzmann et de Landau en théorie cinétique des gaz et des plasmas.* PhD thesis, Univ. Paris-Dauphine, 1998.

[75] Villani, C. On a new class of weak solutions for the spatially homogeneous Boltzmann and Landau equations. *Arch. Rat. Mech. Anal. 143*, 3 (1998), 273–307.

[76] Villani, C. On the spatially homogeneous Landau equation for Maxwellian molecules. *Math. Meth. Mod. Appl. Sci.*, 8 (1998), 957–983.

[77] Weber, M. The fundamental solution of a degenerate partial differential equation of parabolic type. *Trans. Amer. Math. Soc. 71* (1951), 24–37.

[78] Wennberg, B. Regularity in the Boltzmann equation and the Radon transform. *Comm. P.D.E.* **19**, 11–12 (1994), 2057–2074.

[79] Wennberg, B. The geometry of binary collisions and generalized Radon transforms. *Arch. Rational Mech. Anal. 139*, 3 (1997), 291–302.