CONVERGENCE TOWARDS
THE THERMODYNAMICAL EQUILIBRIUM

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1 Collision kernels and entropy production

Collision kernels are standard objects of rational mechanics.

One of the most important is Boltzmann's kernel of rarefied gases (Cf. [Ce], [Ch, Co], [Tr, Mu]), defined by

$$Q_1(f)(v) = \int_{v_* \in \mathbb{R}^3} \int_{\sigma \in S^2} \Big\{ f(v')f(v'_*) - f(v)f(v_*) \Big\}$$

$$\times B_1\left(|v - v_*|, \sigma \cdot \frac{v - v_*}{|v - v_*|} \right) d\sigma dv_*, \tag{1.1}$$

where

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \tag{1.2}$$

$$v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \tag{1.3}$$

B_1 is a nonnegative cross section and $f \equiv f(v) \geq 0$ is the density of particles of velocity $v \in \mathbb{R}^3$.

A classical simplified kernel is the so–called Kac's kernel (Cf. [K], [MK]), defined by

$$Q_2(f)(v) = \int_{v_* \in \mathbb{R}} \int_{\theta = -\pi}^{\pi} \Big\{ f(v \cos\theta - v_* \sin\theta)\, f(v \sin\theta + v_* \cos\theta)$$

$$- f(v)\, f(v_*) \Big\}\, B_2(|\theta|)\, \frac{d\theta}{2\pi} dv_*, \tag{1.4}$$

and here $v \in I\!R$, B_2 is a nonnegative cross section and $f \equiv f(v) \geq 0$ is the density of particles of a one–dimensional gas where the mass and energy are conserved but not the momentum.

In the context of semiconductors (Cf. [BA, Deg, Ge]), Boltzmann's kernel is replaced (as far as electrons–electrons collisions are concerned) by

$$Q_3(f)(v) = \int_{v_* \in B} \int_{v' \in B} \int_{v'_* \in B} \Big\{ f(v')\, f(v'_*)\, (1 - f(v))(1 - f(v_*))$$

$$- f(v)\, f(v_*)\, (1 - f(v'))\, (1 - f(v'_*)) \Big\}\, \delta(\varepsilon(v) + \varepsilon(v_*) = \varepsilon(v') + \varepsilon(v'_*))$$

$$\times\, \delta(v + v_* - v' - v'_* \in L^*)\, B_3(v, v_*, v', v'_*)\, dv'_* dv' dv_*, \tag{1.5}$$

where L is the lattice of the semiconductor, $B = I\!R^3/L^*$ is the Brillouin zone, $\varepsilon : B \to I\!R^+$ is the energy band, $f \equiv f(v) \in [0,1]$ is the density of electrons with wave number v submitted to the Pauli principle, and B_3 is a nonnegative cross section satisfying the microreversibility assumption

$$\forall v, v_*, v', v'_* \in B, \qquad B_3(v, v_*, v', v'_*) = B_3(v_*, v, v'_*, v') = B_3(v', v'_*, v, v_*). \tag{1.6}$$

Finally, the Fokker–Planck–Landau kernel of plasma physics is a limit of the kernel Q_1 when the collisions become grazing (Cf. [Ars, Bu], [Des 2], [Des, Vi 1]).

It reads

$$Q_4(f)(v) = div_v \int_{v_* \in I\!R^3} \Big\{ |v - v_*|^2\, Id - (v - v_*) \otimes (v - v_*) \Big\}$$

$$\Big\{ f(v_*)\, \nabla f(v) - f(v) \nabla f(v_*) \Big\}\, B_4(|v - v_*|)\, dv_*, \tag{1.7}$$

where $v \in I\!R^3$, $f \equiv f(v) \geq 0$ and B_4 is a nonnegative cross section.

The classical H–theorem of Boltzmann states that for all $f \geq 0$ such that the integrals make sense,

$$- \int_v Q_i(f)(v)\, H_i(f(v))\, dv \geq 0, \tag{1.8}$$

where $H_i(x) = \log x$ for $i = 1, 2, 4$ and $H_3(x) = \log(\frac{x}{1-x})$.

In other words, the entropy production is nonnegative. Moreover (under the additional assumption $B_i > 0$ a.e.), there is equality in inequality (1.8) if and only if $f \in \mathcal{M}_i$, where $\mathcal{M}_i$ is the set of Maxwellian (that is, Gaussian) functions of v when $i = 1, 4$, $\mathcal{M}_2$ is the set of centered (that is, of mean 0) Maxwellian functions of v, and $\mathcal{M}_3$ is the set of Fermi–Dirac functions of v.

The entropy production estimates are quantitative versions of the H–theorem.

A first kind of such estimates is of the form

$$- \int_v Q_i(f)(v)\, H_i(f(v))\, dv \geq \phi(d(f, \mathcal{M}_i)), \tag{1.9}$$

where ϕ is a continuous function such that $\phi(0) = 0$, and $d(\cdot, \mathcal{M}_i)$ is some distance to the set $\mathcal{M}_i$. Roughly speaking, such a formula shows how the entropy production can be seen as a distance to the thermodynamical equilibrium.

Another kind of entropy production estimate is of the form

$$- \int_v Q_i(f)(v)\, H_i(f(v))\, dv \geq \phi\Big(\int_v f(v)\, H_i(f(v))\, dv - \int_v f(v)\, H_i(M_f(v)) \Big), \tag{1.10}$$

where M_f is the function belonging to $\mathcal{M}_i$ which has the same mass $\int_v f(v)\, dv$, impulsion (except for $i = 2$) $\int_v f(v)\, v\, dv$ and kinetic energy $\int_v f(v)\, \frac{|v|^2}{2}\, dv$ as f.

Remembering that the entropy $\int_v f(v)\, H_i(f(v))\, dv$ is always larger than $\int_v f(v)\, H_i(M_f(v))\, dv$, and that the equality occurs if and only if $f = M_f$, we see that this is once again a way of controling the distance to the thermodynamical equilibrium by the entropy production.

In section 2, we recall some of the existing entropy production estimates for the kernels Q_i, $i = 1, .., 4$, and give a simplified proof of one of them in the case $i = 2$.

Then, we observe that there are two classical situations in which there is convergence towards the equilibrium in kinetic theory, and the entropy production estimates can help to control quantitatively the speed of this convergence.

In section 3, we briefly describe the first situation, namely the study of the limit $t \to \infty$ in Boltzmann–type equations (and especially in the spatially homogeneous case) and show how to apply at this level the estimates of section 2.

Finally, in section 4, we investigate another limit in which the thermodynamical equilibrium is reached, namely the Chapman–Enskog asymptotics of the Boltzmann equation. Once again, the estimates of section 2 are used.

2 New proofs of a class of entropy production estimates

We give here an entropy production estimate for each kernel $Q_i, i = 1, .., 4$. Many variants of these estimates can be found, and also many quite different estimates (Cf. [Carl, Carv] for example).

Theorem 1: *Let $i = 1, 2$. We suppose that there exists constants $b_i > 0$ such that $B_i \geq b_i$ a.e.*

Then, for all $\delta, D \geq 0$, one can find $K_D, K'_D > 0$ such that for all $f \equiv f(v)$ satisfying

$$f(v) \geq \delta\, e^{-D\,|v|^2}, \tag{2.1}$$

the following estimate holds:

$$- \int_v Q_i(f)(v)\, H_i(f(v))\, dv \geq b_i\, \delta^2\, K_D$$

$$\times S\Big(K'_D \inf_{\mu \in \log \mathcal{M}_i} \int_v |H_i(f(v)) - \mu(v)|\, e^{-2\,D\,|v|^2}\, dv \Big), \tag{2.2}$$

with $S(x) = \frac{x^2}{1+|x|}$.

Theorem 2: *We suppose that there exists a constant $b_3 > 0$ such that $B_3 \geq b_3$ a.e.*

Then, for all $\beta > 0$, one can find $K_\beta > 0$ such that for all $f \equiv f(v)$ satisfying

$$\beta \leq f(v) \leq 1 - \beta, \tag{2.3}$$

the following estimate holds:

$$- \int_v Q_3(f)(v)\, H_3(f(v))\, dv \geq K_\beta \inf_{\mu \in \frac{\log \mathcal{M}_3}{1+\log \mathcal{M}_3}} \int_v |H_3(f(v)) - \mu(v)|\, dv. \tag{2.4}$$

Theorem 3: *We suppose that there exists a constant $b_4 > 0$ such that $B_4 > b_4$ a.e.*

Then for all $f \equiv f(v) \geq 0$, there exists a constant $K_f > 0$ depending only on the mass $\int_v f(v)\, dv$, energy $\int_v f(v) \frac{|v|^2}{2}\, dv$ and entropy $\int_v f(v)\, H_4(f(v))\, dv$

$$- \int_v Q_4(f)(v)\, H_4(f(v))\, dv \geq K_f \int_{v \in \mathbb{R}^3} \Big\{ f(v)\, H_4(f(v))$$
$$- M_f(v)\, H_4(M_f(v)) \Big\}\, dv. \tag{2.5}$$

The constant K_f can be computed explicitly.

The proof of theorem 2 can be found in [BA, Des, Ge] and that of theorem 3 in [De, Vi 2]. A variant of theorem 1 was proven in [Des 1] (Cf. also [Wen]).

Note that the estimates of theorem 1 and 2 belong to the first class described in section 1, whereas the estimate of theorem 3 belongs to the second class.

We give here a new proof of theorem 1 in the case when $i = 2$. It is far simpler than that of [Des 1] for at least two reasons: first, only one derivation is performed (instead of 3 in [Des 1]), secondly, the open mapping theorem is not used any more. As a consequence, the constants K_D, K'_D becomes explicit.

Proof of theorem 1 (case $i = 2$): We first observe that

$$- \int_v Q_2(f)(v)\, H_2(f(v))\, dv = \frac{1}{4} \int_{v \in \mathbb{R}} \int_{v_* \in \mathbb{R}} \int_{\theta=-\pi}^{\pi} \Big\{ f(v\, \cos\theta - v_*\, \sin\theta)$$
$$\times f(v\, \sin\theta + v_*\, \cos\theta) - f(v)\, f(v_*) \Big\} \Big\{ \log(f(v\, \cos\theta - v_*\, \sin\theta)$$
$$\times f(v\, \sin\theta + v_*\, \cos\theta)) - \log(f(v)\, f(v_*)) \Big\}\, B_2(|\theta|)\, \frac{d\theta}{2\pi} dv_* dv. \tag{2.6}$$

This is the standard form of the entropy production for Kac's model.

Thanks to the assumptions of theorem 1, we immediately get

$$- \int_v Q_2(f)(v)\, H_2(f(v))\, dv \geq b_2\, \frac{\delta^2}{4} \int_{v \in \mathbb{R}} \int_{v_* \in \mathbb{R}} e^{-D\, (v^2 + v_*^2)}$$
$$\int_{\theta=-\pi}^{\pi} \lambda \Big(\log f(v\, \cos\theta - v_*\, \sin\theta) + \log f(v\, \sin\theta + v_*\, \cos\theta)$$
$$- \log f(v) - \log f(v_*) \Big)\, \frac{d\theta}{2\pi} dv_* dv, \tag{2.7}$$

where

$$\lambda(x) = x\,(e^x - 1). \tag{2.8}$$

But

$$\lambda(x) \geq e^{-1}\,S(|x|), \tag{2.9}$$

so that thanks to the convexity of S, Jensen's inequality yields

$$- \int_v Q_2(f)(v)\,H_2(f(v))\,dv \geq b_2\,\delta^2\,\frac{e^{-1}}{4}\,\frac{D}{\pi}\,S\bigg(\frac{\pi}{D}\int_{v\in I\!R}\int_{v_*\in I\!R} e^{-D\,(v^2+v_*^2)}$$

$$\bigg|\int_{\theta=-\pi}^{\pi}[\log f(v\,\cos\theta - v_*\,\sin\theta) + \log f(v\,\sin\theta + v_*\,\cos\theta)]\frac{d\theta}{2\pi}$$

$$- \log f(v) - \log f(v_*)\bigg|\,dvdv_*\bigg). \tag{2.10}$$

But the function

$$r(v,v_*) = \int_{\theta=-\pi}^{\pi}[\log f(v\,\cos\theta - v_*\,\sin\theta) + \log f(v\,\sin\theta + v_*\,\cos\theta)]\,\frac{d\theta}{2\pi} \tag{2.11}$$

depends only on $v^2 + v_*^2$, so that

$$\left(v_*\,\frac{\partial}{\partial v} - v\,\frac{\partial}{\partial v_*}\right) r = 0. \tag{2.12}$$

Then, if we denote

$$k(v,v_*) = \log f(v) + \log f(v_*) - r(v,v_*), \tag{2.13}$$

one immediately gets

$$v_*\,\frac{f'(v)}{f(v)} - v\,\frac{f'(v_*)}{f(v_*)} = v_*\,\frac{\partial k}{\partial 1}(v,v_*) - v\,\frac{\partial k}{\partial 2}(v,v_*). \tag{2.14}$$

Integrating (2.14) with respect to v_* against the function $v_* \mapsto v_*\,e^{-2\,D\,v_*^2}$, one gets

$$\frac{f'(v)}{f(v)}\int_{v_*\in I\!R} v_*^2\,e^{-2\,D\,v_*^2}\,dv_* - v\int_{v_*\in I\!R} v_*\,e^{-2\,D\,v_*^2}\,\frac{f'(v_*)}{f(v_*)}\,dv_*$$

$$= \int_{v_*\in I\!R} v_*^2\,\frac{\partial k}{\partial 1}(v,v_*)\,e^{-2\,D\,v_*^2}\,dv_* + \int_{v_*\in I\!R}(1 - 4\,D\,v_*^2)\,e^{-2\,D\,v_*^2}\,k(v,v_*)\,dv_*. \tag{2.15}$$

Integrating then (2.15) with respect to v, one can find $\mu \in \log \mathcal{M}_2$ such that

$$H_2(f(v)) - \mu(v) = \Big(\int_{v_* \in I\!R} v_*^2 \, e^{-2\,D\,v_*^2} \, dv_* \Big)^{-1} [\int_{v_* \in I\!R} v_*^2 \, k(v, v_*) \, e^{-2\,D\,v_*^2} \, dv_*$$

$$+ \int_0^v \int_{v_* \in I\!R} (1 - 4\,D\,v_*^2) \, e^{-2\,D\,v_*^2} \, k(u, v_*) \, dv_* du], \tag{2.16}$$

so that (thanks to the monotonicity of $x \mapsto (1+2\,D\,x^2)\, e^{-2\,D\,x^2}$ when $x \geq 0$),

$$\int_{v \in I\!R} |H_2(f(v)) - \mu(v)| \, e^{-2\,D\,v^2} \, dv \leq \frac{\sqrt{\pi}}{2} \, (2\,D)^{-\frac{3}{2}} \Big\{ \int_{v \in I\!R} \int_{v_* \in I\!R} |k(v, v_*)|$$

$$\times v_*^2 \, e^{-2\,D\,(v^2 + v_*^2)} \, dv_* dv + \int_{u \in I\!R} \int_{|v| \geq |u|} \frac{dv}{1 + 2\,D\,v^2} \int_{v_* \in I\!R} (1+2\,D\,u^2)\,(1+4\,D\,v_*^2)$$

$$\times e^{-2\,D\,(u^2+v_*^2)} |k(u, v_*)| \, dv_* du \Big\}$$

$$\leq 4\,\sqrt{\pi}\,(2\,D)^{-\frac{3}{2}} \, \sup(1, D)^2 \, (1 + \frac{\pi}{\sqrt{2\,D}}) \int_{v \in I\!R} \int_{v_* \in I\!R} (1 + v^2 + v_*^2)^2$$

$$\times e^{-2\,D\,(v^2+v_*^2)} \, |k(v, v_*)| \, dv dv_*$$

$$\leq 2^{\frac{5}{2}} \, e^{-2} \, \sqrt{\pi} \, D^{-\frac{7}{2}} \, \sup(1, D)^2 \, (1 + \frac{\pi}{\sqrt{2\,D}}) \, e^D$$

$$\times \int_{v \in I\!R} \int_{v_* \in I\!R} e^{-\,D\,(v^2+v_*^2)} \, |k(v, v_*)| \, dv dv_*, \tag{2.17}$$

and (2.2) holds (for $i = 2$) with the explicit constant

$$K_D = (e\,\pi)^{-1} \, D \qquad K'_D = 2^{\frac{3}{2}} \, e^{-1} \, \frac{D^{-3/2}}{\sqrt{\pi}} \, \sup(1, D)^2 \, (1 + \frac{\pi}{\sqrt{2\,D}}) \, e^D. \tag{2.18}$$

3 Application to the study of the long–time behaviour in kinetic equations

We now look to the applications of the estimates of section 2 when one deals with the long time behaviour of (spatially homogeneous) kinetic equations.

The most precise estimate is obtained in the case of the Fokker–Planck–Landau equation and is a consequence of theorem 3. One proves in [Des, Vi

2] (thm. 7) the following estimate for the speed of convergence towards the equilibrium:

Theorem 4: *Let $f_{in} \in L^1(\mathbb{R}^3)$ be a nonnegative initial datum such that the total mass $\int f_{in}(v)\,dv$ is 1 and the total kinetic energy $\int f_{in}(v)\,\frac{|v|^2}{2}\,dv$ is $\frac{3}{2}$. Then there exists an explicitely computable constant C depending only on the initial entropy $\int f_{in}(v)\,H_3(f_{in}(v))\,dv$ such that if $f(t,\cdot)$ is the (unique) solution of the spatially homogeneous Fokker–Planck–Landau equation*

$$\partial_t f(t,v) = Q_3(f)(t,v), \tag{3.1}$$

$$f(0,v) = f_{in}(v), \tag{3.2}$$

with a cross section B_3 such that

$$1 \leq B_3(|v-v_*|) \leq K\,(1+|v-v_*|) \tag{3.3}$$

for some constant $K>0$, then

$$||f(t,\cdot) - M_{f_{in}}||_{L^1(\mathbb{R}^3)} \leq C\,e^{-\frac{t}{3}}. \tag{3.4}$$

Many variants of this theorem can be found in [Des, Vi 2]. It can be considered as "almost" optimal in the sense that the coefficient $\frac{1}{3}$ in the exponential in eq. (3.4) is of the same order of magnitude as the spectral gap in the corresponding linearized equation.

The idea of the proof consists in using theorem 3 together with the entropy estimate

$$\partial_t \int_{v\in\mathbb{R}^3} f(t,v)\,H_3(f(t,v))\,dv = \int_{v\in\mathbb{R}^3} Q_3(f)(t,v)\,H_3(f(t,v))\,dv. \tag{3.5}$$

In the case when $i=1,2$, the entropy production estimates given in section 2 do not take into account the entropy itself (i.-e. they are of the first kind). As a consequence, the corresponding estimates of convergence towards the equilibrium are far worse. One can typically prove that for a well chosen $D>0$,

$$||f(t,\cdot) - M_{f_{in}}||_{L^1(\mathbb{R}^3,e^{-D\,|v|^2}\,dv)} = O\Big(\frac{1}{\sqrt{t}}\Big), \tag{3.6}$$

where $f(t, \cdot)$ is the unique solution to the homogeneous equation

$$\partial_t f(t,v) = Q_i(f)(t,v), \tag{3.7}$$

$$f(0,v) = f_{in}(v), \tag{3.8}$$

with a cross section B_i such that

$$K_1 \le B_i(|v-v_*|) \le K_2\,(1+|v-v_*|) \tag{3.9}$$

for some constants $K_1, K_2 > 0$.

Note that a Maxwellian lower bound like (2.1) (for a certain $\delta, D > 0$ depending on the initial datum) is proven (under reasonable assumptions on the cross section B_1) in [Plv, Wenn].

Estimate (3.6) is very far from optimal and is to be compared to the estimates of [Ark], [Carl, Carv] and [Tosc, Vil], which are based on quite different approaches and involve different norms.

4 Application to the study of the Chapman–Enskog asymptotics

The Chapman–Enskog asymptotics consists in finding solutions of the approximated equation (sometimes called the Hilbert expansion at order 2)

$$\frac{\partial f_\varepsilon}{\partial t} + v\cdot\nabla_x f_\varepsilon = \frac{1}{\varepsilon}Q_1(f_\varepsilon) + O(\varepsilon^2), \tag{4.1}$$

under the form of an asymptotic expansion when $\varepsilon \to 0$. Because of the $\frac{1}{\varepsilon}$ in front of the collision kernel, we are once again in a situation in which, when $\varepsilon \to 0$, there is convergence towards the thermodynamical equilibrium.

One can show (at the formal level, like in [Ch, Co], [Ba], or for solutions defined for finite times (Cf. [Ka, Ma, Ni])) that there exists

$$s_\varepsilon = O(1) \tag{4.2}$$

such that

$$f_\varepsilon(t,x,v) = M_\varepsilon(t,x,v)(1+\varepsilon q_\varepsilon(t,x,v) + \varepsilon^2 s_\varepsilon(t,x,v)) \tag{4.3}$$

is a solution of (4.1).

In formula (4.3), M_ε denotes a Maxwellian function of v,

$$M_\varepsilon(t,x,v) = \frac{\rho_\varepsilon(t,x)}{(2\pi T_\varepsilon(t,x))^{\frac{3}{2}}} e^{-\frac{|v-u_\varepsilon(t,x)|^2}{2T_\varepsilon(t,x)}}, \tag{4.4}$$

and the macroscopic quantities $\rho_\varepsilon, u_\varepsilon, T_\varepsilon$ satisfy the Navier-Stokes equations of compressible perfect monoatomic gases, with a viscosity depending only on T_ε and of order ε. The dependance of the viscosity with respect to the temperature is related to the cross section B_1 appearing in Q_1. Precise formulas can be found in [Ba].

In formula (4.3), q_ε is a function of t, x and $\frac{v-u_\varepsilon}{\sqrt{T_\varepsilon}}$ whose dependance with respect to the third variable is fixed (and depends in fact on the cross section B_1). Once again, precise formulas can be found in [Ba].

On the other hand, if a solution of (4.1) can be written under the form

$$f_\varepsilon(t,x,v) = M_\varepsilon(t,x,v)(1+\varepsilon f_{1\varepsilon}(t,x,v)), \tag{4.5}$$

where

$$\int_{v\in \mathbb{R}^3} f_{1\varepsilon}(t,x,v)\, M_\varepsilon(t,x,v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} dv = O(\varepsilon), \tag{4.6}$$

$$M_\varepsilon = O(1), \qquad g_\varepsilon = O(1), \tag{4.7}$$

and where M_ε is a Maxwellian function of v, then it seems classical (at the formal level, Cf. [Des 3] for example) that f_ε can be written under the form (4.3), (4.4), (with the macroscopic quantities $\rho_\varepsilon, u_\varepsilon, T_\varepsilon$ satisfying the Navier-Stokes equations as above) so that the Chapman–Enskog asymptotics holds.

We show here (at the formal level) thanks to the entropy production estimates that any solution of (4.1) such that the initial datum $f_\varepsilon(0,x,v)$ does not depend on ε (or more generally is in $O(1)$)can also be written under the form (4.3), (4.4). In other words, the whole Chapman–Enskog asymptotics can be recovered from the Hilbert expansion at order 2 under no extra assumptions.

According to the remark above, it is enough to prove that (4.1) implies (4.5) – (4.7).

We first note that thanks to the properties of conservation of the mass, impulsion and energy, we get

$$\int_{x\in \mathbb{R}^3}\int_{v\in \mathbb{R}^3} f_\varepsilon(t,x,v)\begin{pmatrix}1\\ v_1\\ v_2\\ v_3\\ |v|^2\end{pmatrix} dvdx = \int_{x\in \mathbb{R}^3}\int_{v\in \mathbb{R}^3} f_\varepsilon(0,x,v)\begin{pmatrix}1\\ v_1\\ v_2\\ v_3\\ |v|^2\end{pmatrix} dvdx. \tag{4.8}$$

Therefore (since $f_\varepsilon \geq 0$),

$$f_\varepsilon = O(1) \tag{4.9}$$

for the $L^\infty([0,+\infty[_t; L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v))$ norm.

Integrating eq. (4.1) against $\log f_\varepsilon$, we also get

$$\int_{x\in \mathbb{R}^3}\int_{v\in \mathbb{R}^3} f_\varepsilon\, H_1(f_\varepsilon)(t,x,v)\, dvdx - \int_{x\in \mathbb{R}^3}\int_{v\in \mathbb{R}^3} f_\varepsilon\, H_1(f_\varepsilon)(0,x,v)\, dvdx$$

$$= \frac{1}{\varepsilon}\int_{s=0}^{t}\int_{x\in \mathbb{R}^3}\int_{v\in \mathbb{R}^3} Q_1(f_\varepsilon)\, H_1(f_\varepsilon)(s,x,v)\, dvdxds. \tag{4.9}$$

Thanks to the H-theorem, the right–hand side of (4.9) is nonpositive, so that the entropy decreases:

$$\int_{x\in \mathbb{R}^3}\int_{v\in \mathbb{R}^3} f_\varepsilon\, H_1(f_\varepsilon)(t,x,v)\, dvdx \leq \int_{x\in \mathbb{R}^3}\int_{v\in \mathbb{R}^3} f_\varepsilon\, H_1(f_\varepsilon)(0,x,v)\, dvdx. \tag{4.10}$$

But it also implies that

$$\int_{s=0}^{t}\int_{x\in \mathbb{R}^3}\int_{v\in \mathbb{R}^3} -Q_1(f_\varepsilon)\, H_1(f_\varepsilon)(s,x,v)\, dvdxds = O(\varepsilon). \tag{4.10}$$

Then, we use theorem 1 to get the estimate

$$\log f_\varepsilon = m_\varepsilon + O(\sqrt{\varepsilon}), \tag{4.11}$$

where m_ε is the logarithm of a Maxwellian and the $O(\sqrt{\varepsilon})$ is in the topology of $L^1_{\mathrm{loc}}([0,+\infty[_t\times \mathbb{R}^3_x \times \mathbb{R}^3_v)$.

Note that (4.11) rigorously holds only under the hypothesis that f_ε is bounded from below by a given Maxwellian (and that the cross section B_1 is also bounded from below, but this last assumption can be relaxed, Cf. [Wenn]). This seems extremely difficult to prove, except maybe in the context of solutions defined on a small interval of time. Therefore, from now on, we proceed in the computation only at the formal level.

Thanks to (4.11), we get

$$f_\varepsilon = M_{1\varepsilon}\,(1+\sqrt{\varepsilon}\,p_\varepsilon), \tag{4.12}$$

where $M_{1\varepsilon}$ is a Maxwellian function of v,

$$M_{1\varepsilon} = O(1), \qquad p_\varepsilon = O(1), \tag{4.13}$$

and the $O(1)$ holds in the topology of $L^1_{\text{loc}}([0,+\infty[_t \times I\!R^3_x \times I\!R^3_v)$. Introducing the ansatz (4.12), (4.13) in (4.1), we get

$$\begin{aligned}&(\frac{\partial}{\partial t} + v\cdot\nabla_x)M_{1\varepsilon} + \sqrt{\varepsilon}\,(\frac{\partial}{\partial t} + v\cdot\nabla_x)(M_{1\varepsilon}p_\varepsilon)\\ &= \frac{1}{\varepsilon}Q_1(M_{1\varepsilon}) + \frac{2}{\sqrt{\varepsilon}}Q_1(M_{1\varepsilon}, M_{1\varepsilon}p_\varepsilon) + Q_1(M_{1\varepsilon}p_\varepsilon).\end{aligned} \tag{4.14}$$

In eq. (4.14), $Q(a,b)$ denotes the bilinear symmetric operator associated to the quadratic operator Q_1.

But Q_1 vanishes on the set of Maxwellians, so that

$$Q_1(M_{1\varepsilon}, M_{1\varepsilon}p_\varepsilon) = O(\sqrt{\varepsilon}). \tag{4.15}$$

Note however that the $O(\sqrt{\varepsilon})$ in formula (4.15) is only to be taken in the sense of distributions (in t, x).

We know (Cf. [Ce]) that the spectrum of the associated self adjoint operator

$$L^\varepsilon = -M_{1\varepsilon}^{-1}\,Q(M_{1\varepsilon}, M_{1\varepsilon}\,\cdot\,) \tag{4.16}$$

is included in the interval $[x_0, +\infty[$ (with $x_0 > 0$ under assumption (3.9) on B_1), except for the eigenvalue 0 which is of order 5 and whose associated eigenspace is

$$\text{Ker }(L^\varepsilon) = \text{Vect}(1, v_1, v_2, v_3, |v|^2). \tag{4.17}$$

Therefore

$$p_\varepsilon = p_{1\varepsilon} + \sqrt{\varepsilon}\,t_\varepsilon, \tag{4.18}$$

where

$$p_{1\varepsilon} \in \text{Vect}(1, v_1, v_2, v_3, |v|^2), \tag{4.19}$$

and

$$p_{1\varepsilon} = O(1), \qquad t_\varepsilon = O(1). \tag{4.20}$$

Finally,

$$f_\varepsilon = M_{1\varepsilon}(1 + \sqrt{\varepsilon}\, p_{1\varepsilon} + \varepsilon\, t_\varepsilon). \tag{4.21}$$

But $M_{1\varepsilon}$ is a Maxwellian function of v, so that it can be written under the form

$$M_{1\varepsilon}(t,x,v) = \frac{\rho_{1\varepsilon}(t,x)}{(2\pi\, T_{1\varepsilon}(t,x))^{\frac{3}{2}}} e^{-\frac{|v-u_{1\varepsilon}(t,x)|^2}{2\, T_{1\varepsilon}(t,x)}}. \tag{4.22}$$

We compute then, when

$$\delta\rho_\varepsilon = O(1), \qquad \delta u_\varepsilon = O(1), \qquad \delta T_\varepsilon = O(1), \tag{4.23}$$

the quantity

$$M_{2\varepsilon}(v) = \frac{\rho_{1\varepsilon} + \sqrt{\varepsilon}\,\delta\rho_\varepsilon}{(2\pi\, T_{1\varepsilon} + 2\pi\,\sqrt{\varepsilon}\,\delta T_\varepsilon)^{\frac{3}{2}}} e^{-\frac{|v-u_{1\varepsilon}-\sqrt{\varepsilon}\,\delta u_\varepsilon|^2}{2T_{1\varepsilon}+2\sqrt{\varepsilon}\,\delta T_\varepsilon}}$$

$$= M_{1\varepsilon}(v)\Big\{1 + \sqrt{\varepsilon}\Big((\frac{\delta\rho_\varepsilon}{\rho_{1\varepsilon}} - \frac{3}{2}\frac{\delta T_\varepsilon}{T_{1\varepsilon}}) + \frac{v-u_{1\varepsilon}}{T_{1\varepsilon}}\,\delta u_\varepsilon + \frac{|v-u_{1\varepsilon}|^2}{2T_{1\varepsilon}^2}\,\delta T_\varepsilon\Big)\Big\} + O(\varepsilon). \tag{4.24}$$

Choosing $\delta\rho_\varepsilon, \delta u_\varepsilon, \delta T_\varepsilon$ in such a way that

$$p_{1\varepsilon} = (\frac{\delta\rho_\varepsilon}{\rho_{1\varepsilon}} - \frac{3}{2}\frac{\delta T_\varepsilon}{T_{1\varepsilon}}) + \frac{v-u_{1\varepsilon}}{T_{1\varepsilon}}\,\delta u_\varepsilon + \frac{|v-u_{1\varepsilon}|^2}{2T_{1\varepsilon}^2}\,\delta T_\varepsilon, \tag{4.25}$$

and this is possible thanks to (4.19), (4.20), we see that we can get

$$f_\varepsilon = M_{2\varepsilon}(1 + \varepsilon g_\varepsilon), \tag{4.26}$$

where

$$M_{2\varepsilon} = O(1), \qquad g_\varepsilon = O(1), \tag{4.27}$$

and $M_{2\varepsilon}$ is a Maxwellian function of v.

Therefore, we obtain (4.5) and (4.7), but not necessarily (4.6). In order to get this last estimate, we perturb the parameters of the Maxwellian $M_{2\varepsilon}$ by functions of order of magnitude $O(\varepsilon)$, and we proceed as in (4.24) – (4.26).

Finally, we see that the Chapman Enskog expansion (4.3), (4.4) is a consequence of the Hilbert expansion at order 2 (4.1) (when the initial datum is independant of ε).

A different asymptotics, namely the one leading from the Boltzmann equation of semiconductors (Cf. [BA, Des, Ge]) to an energy transport model, makes use of related arguments (but for the kernel Q_3).

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