# HYPOCOERCIVITY : THE EXAMPLE OF LINEAR TRANSPORT

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ABSTRACT. The concept of hypocoercivity is useful when one deals with the large time behavior of PDEs in which dissipative effects can be easily obtained only with respect to part of the variables. In order to illustrate it, we show how it works on a very simple linear transport equation in a periodic box, in a situation where it is also possible to resort to spectral theory. Then, we recall the estimates obtained thanks to the concept of hypocoercivity by various authors for kinetic equations.

## 1. HYPOCOERCIVITY

### 1.1. Entropy and entropy dissipation.

We recall here how to obtain explicit estimates of convergence (in large time) toward equilibrium for a PDE (or an ODE, or an integral equation) in which dissipative effects are predominant : the equation is then sometimes called coercive.

We suppose that $f$ is solution of the equation

$$\partial_t f = A\, f, \tag{1}$$

where $A$ can be either linear or nonlinear, and involve either derivatives or integrals.

We suppose that there exists a (bounded below) Lyapounov functional $H \equiv H(f)$ (usually called entropy (or opposite of the entropy)) and a functional $D \equiv D(f)$ (usually called entropy dissipation or entropy production) such that

$$\partial_t H(f) = -D(f) \le 0, \tag{2}$$

$$D(f) = 0 \quad \Longleftrightarrow \quad A\, f = 0 \quad \Longleftrightarrow \quad f = f_{eq}, \tag{3}$$

where $f_{eq}$ is a given function.

Note that very often, $f_{eq}$ is uniquely defined only once a finite number of conserved quantities (along the flow of eq. (1)) is fixed.

We suppose moreover that

$$D(f) \geq \Phi(H(f) - H(f_{eq})), \tag{4}$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\Phi(x) = 0 \iff x = 0$. One looks in general for a function $\Phi$ which increases as much as possible near 0.

Note that since estimate (4) will be used only when $f$ is a solution at time $t$ of eq. (1), it has to be proven only for such $f$. In particular, in the (very common) case when there are conserved quantities in the evolution of eq. (1), it is enough to prove estimate (4) when the corresponding quantities are fixed.

Assuming that estimate (4) holds, we get thanks to formula (2) the differential inequality

$$\partial_t(H(f) - H(f_{eq})) \leq -\Phi(H(f) - H(f_{eq})). \tag{5}$$

Then, thanks to Gronwall's lemma,

$$H(f(t)) - H(f_{eq}) \leq R(t), \tag{6}$$

where $R$ is the reciprocal of a primitive of $-1/\Phi$.

Finally, when $H$ is well behaved, we obtain

$$||f - f_{eq}|| \leq S(t), \tag{7}$$

where $S$ is related to $R$, and $||\ ||$ is some norm which depends on the problem.

When one can take $\Phi(x) = Cst\, x$, one gets $R(t) \leq e^{-Cst\, t}$, so that exponential decay toward the equilibrium can be proven. Sometimes, it is however only possible to take $\Phi(x) = Cst_\varepsilon\, x^{1+\varepsilon}$ for some (or all) $\varepsilon > 0$, and consequently $R(t) = Cst_\varepsilon\, t^{-1/\varepsilon}$. When this holds for all $\varepsilon > 0$, we say that the decay is almost exponential.

Finally, since $H(f)$ is minimal when $f = f_{eq}$, we see that generically one can take $R(t) = Cst\, S(t)^2$. Then, an exponential (resp. almost exponential) decay of the entropy towards its minimum entails an exponential (resp. almost exponential) decay of $f$ itself towards the equilibrium.

1.2. **Examples of coercivity.** When $A$ is a linear integral operator, the formalism of the previous chapter can sometimes be used with a functional $H$ which is quadratic. For example, if $f \equiv f(v) \geq 0$, with $v \in [-1/2, 1/2]$ and

$$Af(v) = \int_{-1/2}^{1/2} f(w)\, dw - f(v), \tag{8}$$

one has $f_{eq}(v) = \int_{-1/2}^{1/2} f(0,v)\, dv$. In the sequel we shall assume that $\int_{-1/2}^{1/2} f(0,v)\, dv = 1$ so that $f_{eq}(v) = 1$.

Then, we introduce $H(f) = \int_{-1/2}^{1/2} f^2\, dv$, so that $H(f_{eq}) = 1$. We see that eq. (2) holds with $D(f) = 2\, \int_{-1/2}^{1/2} |f-1|^2\, dv$. We also immediately obtain estimate (4) with $\Phi(x) = 2\, x$. Finally, we get the exponential convergence, which is in agreement with the (very simple) explicit solution of eq. (1) in this case.

In many cases, it leads to interesting developments to take a functional $H$ which uses logarithms, such as the "physical" entropy $\int f \log f$ or variants of this quantity.

We begin with the case of the Fokker-Planck equation. We consider $f \equiv f(v) \geq 0$, with $v \in \mathbb{R}^N$, and

$$Af(v) = \nabla \cdot (\nabla f + v\, f). \tag{9}$$

We consider once again only the case when $\int_{v \in \mathbb{R}^N} f(0,v)\, dv = 1$, and define the (relative) entropy (or free energy) by :

$$H(f) = \int_{v \in \mathbb{R}^N} f(v) \log \frac{f(v)}{M(v)}\, dv,$$

with

$$M(v) = \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{N/2}}. \tag{10}$$

Differentiating $H(f)$ along the flow (1) with $A$ given by (9), we see that (2) holds, with

$$D(f) = \int_{\mathbb{R}^N} f \left| \nabla \log \frac{f}{M} \right|^2 dv. \tag{11}$$

That is, the dissipation of the (relative) entropy $H(f)$ is exactly the so-called relative Fisher information $D(f)$.

Then, (3) holds with $f_{eq} = M$ (and $H(f_{eq}) = 0$).

In this situation (and still assuming that $\int_{v\in\mathbb{R}^N} f(0,v)\,dv = 1$), estimate (4) can be obtained with $\Phi(x) = 2\,x$ thanks to the logarithmic Sobolev inequality of Gross [14], that is :

$$D(f) \geq 2\,H(f). \tag{12}$$

As a consequence, we can take $R(t) = e^{-2\,t}$ in eq. (6), so that the exponential decay is still obtained.

Using now the Csiszár-Kullback-Pinsker inequality (Cf. [6] and [18]) :

$$H(f) \geq \frac{1}{2}\,||f - f_{eq}||^2_{L^1}, \tag{13}$$

we see that we can take $S(t) = 2\,e^{-t}$ in (7). This result is sharp since $-1$ is the first nonpositive eigenvalue of $A$. Note finally that as in the previous example, the exponential decay can be directly seen on the explicit solution of the Fokker-Planck equation. Note that this result was first described in the vocabulary of kinetic theory in [23].

We now consider the (spatially homogeneous) Boltzmann equation of rarefied gases. One still takes $f \equiv f(t,v) \geq 0$, with $v \in \mathbb{R}^N$ ($f(t,v)$ is the number density of molecules of a gas which at time $t$ have velocity $v$), but now the operator $A$ is a quadratic and integral operator modeling the binary collisions between the molecules. It is defined by :

$$Af(v) = Q(f)(v) = \int\int_{v_*\in\mathbb{R}^N,\sigma\in S^{N-1}} \Big\{ f(v')\,f(v'_*) - f(v)\,f(v_*) \Big\} \tag{14}$$

$$\times\, B\bigg( |v - v_*|, \frac{v - v_*}{|v - v_*|} \cdot \sigma \bigg)\, d\sigma dv_*,$$

with

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\,\sigma,$$

$$v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\,\sigma,$$

and $B > 0$ (a.e.) is a cross section depending on the interaction between the molecules. For more details about this kernel, we refer to [5].

The conserved quantities for this model are the mass $\int f\,dv$, the momentum $\int f\,v\,dv$ and the energy $\int f\,\frac{|v|^2}{2}\,dv$. One defines the (relative) entropy by

$$H(f) = \int_{v\in\mathbb{R}^N} f(v)\,\log\bigg(\frac{f(v)}{M_f(v)}\bigg)\,dv, \tag{15}$$

where

$$M_f(v) = \rho_f \, \frac{e^{-\frac{|v-u_f|^2}{2\,T_f}}}{(2\pi\,T_f)^{N/2}} \tag{16}$$

is the Maxwellian function with same mass, momentum and energy as $f$ :

$$\begin{pmatrix} \rho_f \\ \rho_f\,u_f \\ \frac{1}{2}\rho_f\,|u_f|^2 + \frac{N}{2}\,\rho_f\,T_f \end{pmatrix} = \int_{v\in\mathbb{R}^N} f(t,v) \begin{pmatrix} 1 \\ v_i \\ \frac{|v|^2}{2} \end{pmatrix} dv.$$

Differentiating (1) along the flow (with $A$ given by (14)), we get (2) with

$$D(f) = \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{S^{N-1}} \Big\{ f(v')\,f(v'_*) - f(v)\,f(v_*) \Big\}$$

$$\times \log\left(\frac{f(v')\,f(v'_*)}{f(v)\,f(v_*)}\right) B\bigg(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\bigg)\, d\sigma dv_* dv. \tag{17}$$

Then (still assuming that $B > 0$ a.e.), (3) holds with $f_{eq} = M_{f(0,\cdot)}$ (and $H(f_{eq}) = 0$).

Another possible presentation of this result is that

$$\forall v \in \mathbb{R}^N, \qquad Af(v) = 0 \qquad \Longleftrightarrow \qquad D(f) = 0$$

$$\Longleftrightarrow \qquad f(v) = \exp\left(a + b\cdot v - c\,|v|^2\right)$$

for some $a \geq 0, c > 0$, and $b \in \mathbb{R}^N$. This is exactly Boltzmann's H theorem.

In this situation, estimate (4) can be obtained (for all physically relevant cross sections $B$) with $\Phi(x) = Cst_\varepsilon\,x^{1+\varepsilon}$ for all $\varepsilon > 0$, provided that $f$ belongs to a set $Z$ of functions which satisfy smoothness conditions detailed in [29] together with the precise assumptions on $B$ (for a quadratic cross section, one can even take $\varepsilon = 0$). Since for many initial data and cross sections $B$ (including the physically relevant cases), it is possible to prove that the solution $t \mapsto f(t)$ of eq. (1), (14) remains in $Z$, one can take $R(t) = Cst_\varepsilon\,t^{-1/\varepsilon}$ and, still using Csiszár-Kullback-Pinsker's inequality (13), $S(t) = Cst_\varepsilon\,t^{-1/(2\varepsilon)}$.

We refer to [4], [8], [24], [25] and [29] for a complete picture of the state of the art concerning the use of the entropy/entropy dissipation method for the spatially homogeneous Boltzmann equation. Those results should be compared with what is obtained thanks to spectral theory and linearization (Cf. in particular [1]). The latest development on this subject consists in mixing the entropy method and the spectral theory, in order to get at the same time the exponential decay and explicitly computable constants (Cf. [19]).

Note finally that the entropy/entropy dissipation method can be used in other fields than kinetic theory. We refer for example to [11] for an application in the context of nonlinear elliptic PDEs.

1.3. **Description of the method for hypocoercivity.** We now investigate a situation in which the dissipative effects hold only with respect to part of the variables. We assume nevertheless that thanks to the coupling between the variables, a unique equilibrium is expected (up to the conserved quantities).

More precisely, we still consider the abstract equation (1), and we assume that a couple $H, D$ of entropy and entropy dissipation satisfies (2).

However, we do not suppose anymore that eq. (3) holds, but assume instead that

$$D(f) = 0 \iff f \in \mathcal{M}, \tag{18}$$

where $\mathcal{M}$ is an infinite-dimensional set of functions, and that

$$f \in \mathcal{M} \quad \text{and} \quad Af = 0 \qquad \Rightarrow \qquad f = f_{eq}, \tag{19}$$

where $f_{eq}$ is a given function (once again, a finite number $n$ of quantities can be conserved along (1), so that $f_{eq}$ may in fact depend upon $n$ parameters). This situation corresponds to what we call hypocoercivity (by analogy with the hypoellipticity).

Assumption (18) basically means that the dissipation phenomena in eq. (1) concern only some part of the variables of $f$, and drives $f$ to what is called in kinetic theory the *local equilibria*, that is, the elements of $\mathcal{M}$. Then, assumption (19) means that the *local equilibria* are unstable by the flow along eq. (1), except when they are *global* (that is, when $f = f_{eq}$).

In order to obtain explicit rates of convergence to equilibrium in the hypocoercive case, one cannot hope to use an estimate like (5), since it is not compatible with assumption (18). One has somehow to make quantitative the assumptions (18) and (19).

We suggest to look for an intermediate functional $K(f) \geq 0$ for which

$$D(f) \geq \Phi(K(f)) \tag{20}$$

with

$$K(f) = 0 \qquad \iff \qquad f \in \mathcal{M}, \tag{21}$$

and which moreover, when differentiated twice along the flow of eq. (1), yields a differential inequality of the form

$$\frac{d^2}{dt^2}K(f) \geq \Psi_1(H(f) - H(f_{eq})) - \Psi_2(K(f)), \tag{22}$$

where $\Phi, \Psi_1, \Psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are functions such that the property $\Phi(x) = 0 \iff x = 0$ and $\Psi_i(x) = 0 \iff x = 0$ holds.

Usually, (22) can be proven only under certain assumptions (typically, of smoothness) on $f$. One then has to prove that those assumptions are compatible with the properties of the solutions of eq. (1).

Before explaining how to use formula (22), let us say that looking for an estimate on the second derivative in time of $K(f)$ is reasonable since if at some time $t_0 \geq 0$, one has $f(t_0) \in \mathcal{M}$, then $K(f)(t_0) = 0$ because of (21), and $\frac{d}{dt}K(f)(t_0) = 0$ because $K$ is nonnegative.

Suppose now that estimates (20) and (22) hold. Using eq. (2), we can rewrite them as a system of two differential inequalities :

$$-\frac{d}{dt}(H(f) - H(f_{eq})) \geq \Phi(K(f)), \tag{23}$$

$$\frac{d^2}{dt^2}K(f) \geq \Psi_1(H(f) - H(f_{eq})) - \Psi_2(K(f)). \tag{24}$$

When $\Phi$, $\Psi_1$ and $\Psi_2$ behave nicely (that is, when they grow fast enough near 0), this system will imply (thanks to a lemma which replaces Gronwall's lemma in this context) that

$$H(f(t)) - H(f_{eq}) \leq R(t),$$

for some function $R$ related to $\Phi$, $\Psi_1$ and $\Psi_2$, and (if $H$ has good properties of coercivity)

$$||f(t) - f_{eq}|| \leq S(t)$$

for some norm $||\ ||$.

The heuristic reason why such a lemma may hold is the following : as long as $f$ is not close to $\mathcal{M}$, $\Phi(K(f))$ will be large, so that thanks to (23), $H(f)$ decreases in a controlled way to $H(f_{eq})$. Now if at some point $t_0$, $f(t_0)$ is close to $\mathcal{M}$, then $\Psi_1(H(f) - H(f_{eq})) \geq c$ for some $c > 0$, and we use (24) under the form

$$\frac{d^2}{dt^2}K(f) + \Psi_2(K(f)) \geq c.$$

Then, if $\Psi_2$ behaves nicely, it is possible to show that $K(f)$ cannot remain close to 0 (except for a small interval of time), so that $f(t)$ will not stay close to $\mathcal{M}$ for long, and we can again use (23).

A complete proof in the case when $\Phi(x) = Cst\, x$, $\Psi_1(x) = Cst\, x$, $\Psi_2(x) = Cst_\varepsilon\, x^{1-\varepsilon}$ can be found in [9]. In this particular case, one can take $R(t) = Cst_\varepsilon\, t^{1/\varepsilon - 1}$.

In order to clarify the above strategy, we shall first show how it works on a very simple transport equation, for which the computations are (relatively) simple. Note that for this equation, a direct study of the spectrum yields the large time behavior, so that the concept of hypocoercivity is not really unavoidable.

## 2. The example of a linear transport equation

In this section, we consider the simplest possible situation of hypocoercivity. We introduce therefore a one-dimensional transport model in a periodic box, and we present the computations in details.

We take $t \in \mathbb{R}_+$, $x \in \mathbb{T}$ the one-dimensional torus, and $v \in [-1/2, 1/2]$. For a function $g \equiv g(t,x,v)$, we introduce the average in velocity

$$\bar{g}(t,x) = \int_{-1/2}^{1/2} g(t,x,v)\, dv.$$

In the sequel, we shall systematically use the identity $\bar{\bar{g}} = \bar{g}$ (remember that $g \mapsto \bar{g}$ is an average) and the Cauchy-Schwarz inequality under the form

$$\int_{\mathbb{T}} |\bar{g}(t,x)|^2\, dx \le \int_{\mathbb{T}} \int_{-1/2}^{1/2} |g(t,x,v)|^2\, dv dx$$

or sometimes under the equivalent form

$$\int_{\mathbb{T}} \int_{-1/2}^{1/2} |\bar{g}(t,x)|^2\, dv dx \le \int_{\mathbb{T}} \int_{-1/2}^{1/2} |g(t,x,v)|^2\, dv dx.$$

Then, we introduce the transport operator

$$Af(x,v) = -v\,\partial_x f(x,v) + \bar{f}(x) - f(x,v).$$

Eq. (1) becomes

$$\partial_t f(t,x,v) + v\,\partial_x f(t,x,v) = \bar{f}(t,x) - f(t,x,v). \tag{25}$$

This equation models a set of particles on a line which are scattered isotropically.

We choose an initial datum $f(0,x,v) \ge 0$ which is smooth (we shall see in the sequel which smoothness is needed in order to get the estimates of convergence toward the equilibrium). Then, we can find a unique solution to eq. (25) for $t \in \mathbb{R}_+$, which is such that $f(t,x,v) \ge 0$ and $f(t,\cdot,\cdot)$ is smooth (the smoothness is propagated along the flow of eq. (25), as we shall see later).

Note that the unique conserved quantity associated to eq. (25) is the total mass $\int_{\mathbb{T}} \int_{-1/2}^{1/2} f(t,x,v)\, dvdx$. In the sequel, we restrict ourselves to the case when $\int_{\mathbb{T}} \int_{-1/2}^{1/2} f(0,x,v)\, dvdx = 1$.

As in subsection 1.2, we introduce a quadratic entropy (but this time, we also integrate with respect to the variable $x$) :

$$H(f)(t) = \frac{1}{2} \int_{\mathbb{T}} \int_{-1/2}^{1/2} |f(t,x,v)|^2\, dvdx.$$

Then, along the flow of eq. (25), we get (thanks to the identity $\int_{\mathbb{T}} \int_{-1/2}^{1/2} \bar{f}\,(\bar{f} - f)\, dvdx = 0$),

$$\frac{d}{dt} H(f)(t) = \int_{\mathbb{T}} \int_{-1/2}^{1/2} f(t,x,v)\, \partial_t f(t,x,v)\, dvdx$$

$$= \int_{\mathbb{T}} \int_{-1/2}^{1/2} f(t,x,v)\,(-v\,\partial_x f(t,x,v) + \bar{f}(t,x) - f(t,x,v))\, dvdx$$

$$= \int_{\mathbb{T}} \int_{-1/2}^{1/2} f(t,x,v)\,(\bar{f}(t,x) - f(t,x,v))\, dvdx$$

$$= -\int_{\mathbb{T}} \int_{-1/2}^{1/2} |\bar{f}(t,x) - f(t,x,v)|^2\, dvdx.$$

As a consequence, eq. (2) holds with

$$D(f)(t) = \int_{\mathbb{T}} \int_{-1/2}^{1/2} |\bar{f}(t,x) - f(t,x,v)|^2\, dvdx.$$

Then,

$$D(g) = 0 \qquad \Longleftrightarrow \qquad g \in \mathcal{M},$$

where $\mathcal{M}$ is the set of local equilibria, consisting here of the functions which do not depend upon the variable $v$ (that is, they are equal to their average in $v$).

It is then obvious that if $g \equiv g(x,v)$ lies in $\mathcal{M}$ and $Ag = 0$, then $-v\,\partial_x g(x) = 0$, so that $g$ is constant, and after taking into account the conservation of mass, $g = f_{eq} = 1$. We see therefore that (19) holds, and we are in the typical situation of hypocoercivity.

Our method is particularly simple for this transport equation because we can take the intermediate functional $K(f) := D(f)$. Therefore, estimate (20) immediately holds. This is at variance with a large part of the other results in this direction, even in a linear context.

In order to obtain an estimate of the type (22), we need to compute the second derivative of $K(f)$ with respect to time along the flow of eq. (25).

The experience shows that in order to get (22), the first computation to perform consists in looking for the second derivative of $K(f)$ *at a time* $t_0$ *when* $f(t_0)=\bar f(t_0)$, *i.-e. at local equilibrium*. For such a time,

$$\begin{aligned}
\frac{d^2}{dt^2}K(f) &= \frac{d}{dt}\bigg( \frac{d}{dt}\int_{\mathbb{T}}\int_{-1/2}^{1/2} |f-\bar f|^2\, dv dx \bigg) \\
&= \frac{d}{dt}\bigg( \int_{\mathbb{T}}\int_{-1/2}^{1/2} 2\,(f-\bar f)\,\frac{d}{dt}(f-\bar f)\, dv dx \bigg) \\
&= 2\int_{\mathbb{T}}\int_{-1/2}^{1/2} |\frac{d}{dt}(f-\bar f)|^2\, dv dx \\
&= 2\int_{\mathbb{T}}\int_{-1/2}^{1/2} |-v\,\partial_x f + \bar f - f + \partial_x(\overline{v\,f})|^2\, dv dx \\
&= 2\int_{\mathbb{T}}\int_{-1/2}^{1/2} |-v\,\partial_x \bar f + \partial_x(\overline{v\,\bar f})|^2\, dv dx \\
&= 2\int_{-1/2}^{1/2} |v|^2\, dv \int_{\mathbb{T}} |\partial_x \bar f|^2\, dx \\
&= \frac{1}{6}\int_{\mathbb{T}} |\partial_x \bar f|^2\, dx \\
&\ge \frac{1}{6}\int_{\mathbb{T}} |\bar f - \int_{\mathbb{T}} \bar f|^2\, dx \\
&\ge \frac{1}{3}\,(H(f) - H(f_{eq})). \qquad (26)
\end{aligned}$$

Then, we can see that (22) (with $\Psi_1 = Cst$) is a perturbation of estimate (26) when we do not suppose anymore that $f(t_0)=\bar f(t_0)$.

We begin the computation of $\frac{d^2}{dt^2}K(f)$ without assuming anymore that $f(t_0)=\bar f(t_0)$.

We shall use the following formulas (obtained from equation (25) by taking averages in $v$) :

$$\partial_t \bar f + \partial_x(\overline{v\,f}) = 0,$$

$$\partial_t(\overline{v\,f}) + \partial_x(\overline{v^2\,f}) = -\overline{v\,f},$$

$$\partial_{tt} f = v^2\,\partial_{xx} f - v\,\partial_x \bar f + 2\,v\,\partial_x f - \partial_x(\overline{v\,f}) - \bar f + f,$$

$$\partial_{tt}\bar f = \partial_{xx}(\overline{v^2\,f}) + \partial_x(\overline{v\,f}).$$

Then,

$$\frac{d^2}{dt^2}K(f) = \frac{d}{dt}\bigg( \frac{d}{dt}\int_{\mathbb{T}}\int_{-1/2}^{1/2} |f-\bar f|^2\, dv dx \bigg)$$

$$= 2 \int_{\mathbb{T}}\int_{-1/2}^{1/2} |\frac{d}{dt}(f-\bar f)|^2\, dv dx + 2 \int_{\mathbb{T}}\int_{-1/2}^{1/2} (f-\bar f)\, \frac{d^2}{dt^2}(f-\bar f)\, dv dx$$

$$= 2 \int_{\mathbb{T}}\int_{-1/2}^{1/2} |-v\,\partial_x f + \bar f - f + \partial_x(\overline{v\, f})|^2\, dv dx$$

$$+ 2 \int_{\mathbb{T}}\int_{-1/2}^{1/2} (f-\bar f)\, \Big( v^2\,\partial_{xx} f - v\,\partial_x \bar f + 2\,v\,\partial_x f - 2\,\partial_x(\overline{v\,f}) - \bar f + f - \partial_{xx}(\overline{v^2\,f}) \Big)\, dv dx.$$

Using the elementary inequality

$$2\,(a+b)^2 \geq a^2 - 2\,b^2, \tag{27}$$

we obtain

$$\frac{d^2}{dt^2}K(f) \geq \int_{\mathbb{T}}\int_{-1/2}^{1/2} |v\,\partial_x f|^2\, dv dx$$

$$-4 \int_{\mathbb{T}}\int_{-1/2}^{1/2} |\bar f - f|^2\, dv dx - 4 \int_{\mathbb{T}}\int_{-1/2}^{1/2} |\partial_x(\overline{v\,f})|^2\, dv dx$$

$$-2 \bigg| \int_{\mathbb{T}}\int_{-1/2}^{1/2} (f-\bar f)\, v^2\,\partial_{xx} f\, dv dx \bigg| - 2 \bigg| \int_{\mathbb{T}}\int_{-1/2}^{1/2} (f-\bar f)\, v\,\partial_x \bar f\, dv dx \bigg|$$

$$-4 \bigg| \int_{\mathbb{T}}\int_{-1/2}^{1/2} (f-\bar f)\, v\,\partial_x f\, dv dx \bigg| - 4 \bigg| \int_{\mathbb{T}}\int_{-1/2}^{1/2} (f-\bar f)\, \partial_x(\overline{v\,f})\, dv dx \bigg|$$

$$-2 \bigg| \int_{\mathbb{T}}\int_{-1/2}^{1/2} (f-\bar f)^2\, dv dx \bigg| - 2 \bigg| \int_{\mathbb{T}}\int_{-1/2}^{1/2} (f-\bar f)\, \partial_{xx}(\overline{v^2\,f})\, dv dx \bigg|.$$

We denote by $I$ the sum of the second, third, seventh and eighth term in the previous inequality, and by $J$ the sum of the fourth, fifth, sixth and ninth term.

Then, after having used $ab \leq \frac{1}{2}\,(a^2+b^2)$ for the seventh term,

$$I \leq 8 \int_{\mathbb{T}}\int_{-1/2}^{1/2} |\bar f - f|^2\, dv dx + 6 \int_{\mathbb{T}}\int_{-1/2}^{1/2} |\partial_x(\overline{v\,(f-\bar f)})|^2\, dv dx$$

$$\leq 8 \int_{\mathbb{T}}\int_{-1/2}^{1/2} |\bar f - f|^2\, dv dx + 6 \int_{\mathbb{T}}\int_{-1/2}^{1/2} |v|^2\, |\partial_x(f-\bar f)|^2\, dv dx$$

$$\leq 8 \int_{\mathbb{T}}\int_{-1/2}^{1/2} |\bar f - f|^2\, dv dx + \frac{3}{2} \int_{\mathbb{T}}\int_{-1/2}^{1/2} |\partial_x(f-\bar f)|^2\, dv dx$$

$$\leq 8\, ||f-\bar f||^2_{L^2_x(L^2_v)} + \frac{3}{2}\, ||\partial_x(f-\bar f)||^2_{L^2_x(L^2_v)},$$

and

$$J \le 2 \left| \int_{\mathbb{T}} \int_{-1/2}^{1/2} (f-\bar f)\, v^2\, \partial_{xx}(f-1)\, dv dx \right|$$

$$+ 2 \left| \int_{\mathbb{T}} \int_{-1/2}^{1/2} (f-\bar f)\, v\, \partial_x(\bar f - 1)\, dv dx \right| + 4 \left| \int_{\mathbb{T}} \int_{-1/2}^{1/2} (f-\bar f)\, v\, \partial_x(f-1)\, dv dx \right|$$

$$+ 2 \left| \int_{\mathbb{T}} \int_{-1/2}^{1/2} (f-\bar f)\, \partial_{xx}(\overline{v^2\,(f-1)})\, dv dx \right|$$

$$\le 2 \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |f-\bar f|^2\, dv dx \right)^{1/2} \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} v^4\, |\partial_{xx}(f-1)|^2\, dv dx \right)^{1/2}$$

$$+ 2 \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |f-\bar f|^2\, dv dx \right)^{1/2} \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} v^2\, |\partial_x(\bar f-1)|^2\, dv dx \right)^{1/2}$$

$$+ 4 \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |f-\bar f|^2\, dv dx \right)^{1/2} \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} v^2\, |\partial_x(f-1)|^2\, dv dx \right)^{1/2}$$

$$+ 2 \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |f-\bar f|^2\, dv dx \right)^{1/2} \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |\partial_{xx}(\overline{v^2\,(f-1)})|^2\, dv dx \right)^{1/2}$$

$$\le \frac{1}{2} \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |f-\bar f|^2\, dv dx \right)^{1/2} \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |\partial_{xx}(f-1)|^2\, dv dx \right)^{1/2}$$

$$+ \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |f-\bar f|^2\, dv dx \right)^{1/2} \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |\partial_x(\overline{f-1})|^2\, dv dx \right)^{1/2}$$

$$+ 2 \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |f-\bar f|^2\, dv dx \right)^{1/2} \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |\partial_x(f-1)|^2\, dv dx \right)^{1/2}$$

$$+ 2 \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |f-\bar f|^2\, dv dx \right)^{1/2} \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |\partial_{xx}(v^2\,(f-1))|^2\, dv dx \right)^{1/2}$$

$$\le \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |f-\bar f|^2\, dv dx \right)^{1/2} \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |\partial_{xx}(f-1)|^2\, dv dx \right)^{1/2}$$

$$+ 3 \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |f-\bar f|^2\, dv dx \right)^{1/2} \left( \int_{\mathbb{T}} \int_{-1/2}^{1/2} |\partial_x(f-1)|^2\, dv dx \right)^{1/2}$$

$$\le 3\, K(f)^{1/2}\, ||\partial_x(f-1)||_{L^2_x(L^2_v)} + K(f)^{1/2}\, ||\partial_{xx}(f-1)||_{L^2_x(L^2_v)}.$$

Then, we observe that (thanks to (27))

$$\int_{\mathbb{T}} \int_{-1/2}^{1/2} |v\, \partial_x f|^2\, dv dx$$

$$\ge \frac{1}{2} \int_{\mathbb{T}} \int_{-1/2}^{1/2} |v\, \partial_x \bar f|^2\, dv dx - \int_{\mathbb{T}} \int_{-1/2}^{1/2} |v\, \partial_x (f-\bar f)|^2\, dv dx$$

$$\geq \frac{1}{24}\int_{\mathbb{T}}\int_{-1/2}^{1/2}|\bar{f}-1|^2\,dvdx-\frac{1}{4}\int_{\mathbb{T}}\int_{-1/2}^{1/2}|\partial_x(f-\bar{f})|^2\,dvdx$$

$$\geq \frac{1}{24}\int_{\mathbb{T}}\int_{-1/2}^{1/2}|f-1|^2\,dvdx-\frac{1}{24}\int_{\mathbb{T}}\int_{-1/2}^{1/2}|f-\bar{f}|^2\,dvdx-\frac{1}{4}\int_{\mathbb{T}}\int_{-1/2}^{1/2}|\partial_x(f-\bar{f})|^2\,dvdx.$$

$$\geq \frac{1}{24}\int_{\mathbb{T}}\int_{-1/2}^{1/2}|f-1|^2\,dvdx-\frac{1}{24}\,||f-\bar{f}||^2_{L^2_x(L^2_v)}-\frac{1}{4}\,||\partial_x(f-\bar{f})||^2_{L^2_x(L^2_v)}.$$

Finally, we get the estimate :

$$\frac{d^2}{dt^2}K(f)\geq \frac{1}{24}\int_{\mathbb{T}}\int_{-1/2}^{1/2}|f-1|^2\,dvdx$$
$$-(8+\frac{1}{24})\,||f-\bar{f}||^2_{L^2_x(L^2_v)}-\frac{7}{4}\,||\partial_x(f-\bar{f})||^2_{L^2_x(L^2_v)}$$
$$-3\,K(f)^{1/2}\,||\partial_x(f-1)||_{L^2_x(L^2_v)}-K(f)^{1/2}\,||\partial_{xx}(f-1)||_{L^2_x(L^2_v)}.$$

We now introduce the homogeneous $H^l$ norm (with respect to the variable $x$) of $f$ :

$$||f||^2_{\dot{H}^l_x(L^2_v)}=\int_{\mathbb{T}}\int_{-1/2}^{1/2}\left|\frac{\partial^l f}{\partial x^l}\right|^2\,dvdx.$$

Thanks to a standard interpolation, we obtain for any $l>1$,

$$||g||^2_{\dot{H}^1_x(L^2_v)}\leq ||g||^{2\,(1-1/l)}_{L^2_x(L^2_v)}\,||g||^{2/l}_{\dot{H}^l_x(L^2_v)},$$
$$||g||^2_{\dot{H}^2_x(L^2_v)}\leq ||g||^{2\,(1-1/l)}_{L^2_x(L^2_v)}\,||g||^{2/l}_{\dot{H}^{2l}_x(L^2_v)}.$$

We get

$$\frac{d^2}{dt^2}K(f)\geq \frac{1}{24}\int_{\mathbb{T}}\int_{-1/2}^{1/2}|f-1|^2\,dvdx$$
$$-(8+\frac{1}{24})\,||f-\bar{f}||^2_{L^2_x(L^2_v)}-\frac{7}{4}\,||f-\bar{f}||^{2-2/l}_{L^2_x(L^2_v)}\,||f-\bar{f}||^{2/l}_{\dot{H}^l_x(L^2_v)}$$
$$-3\,K(f)^{1/2}\,||f-1||^{1-1/l}_{L^2_x(L^2_v)}\,||f-1||^{1/l}_{\dot{H}^l_x(L^2_v)}-K(f)^{1/2}\,||f-1||^{1-1/l}_{L^2_x(L^2_v)}\,||f-1||^{1/l}_{\dot{H}^{2l}_x(L^2_v)}.$$

We now use Young's inequality

$$y^{1/2}\,x^{1/2-1/(2l)}\leq\left(\frac{1}{2}-\frac{1}{2l}\right)\frac{x}{\eta^{2/(1-1/l)}}+\left(\frac{1}{2}+\frac{1}{2l}\right)\eta^{2/(1+1/l)}\,y^{1/(1+1/l)},$$

with $y=K(f)$, $x=||f-1||^2_{L^2_x(L^2_v)}$, and $\eta>0$ to be determined later. We obtain

$$\frac{d^2}{dt^2}K(f)\geq\left(\frac{1}{24}-3\left(\frac{1}{2}-\frac{1}{2l}\right)||f-1||^{1/l}_{\dot{H}^l_x(L^2_v)}\,\eta^{-2/(1-1/l)}\right.$$
$$\left.-\left(\frac{1}{2}-\frac{1}{2l}\right)||f-1||^{1/l}_{\dot{H}^{2l}_x(L^2_v)}\,\eta^{-2/(1-1/l)}\right)||f-1||^2_{L^2_x(L^2_v)}$$

$$- (8+\frac{1}{24})\,||f-\bar f||^2_{L^2_x(L^2_v)} - \frac{7}{4}\,||f-\bar f||^{2-2/l}_{L^2_x(L^2_v)}\,||f-\bar f||^{2/l}_{\dot H^l_x(L^2_v)}$$

$$-3\left(\frac{1}{2}+\frac{1}{2l}\right)\eta^{2/(1+1/l)}\,K(f)^{1/(1+1/l)}\,||f-1||^{1/l}_{\dot H^l_x(L^2_v)}$$

$$-\left(\frac{1}{2}+\frac{1}{2l}\right)\eta^{2/(1+1/l)}\,K(f)^{1/(1+1/l)}||f-1||^{1/l}_{\dot H^{2l}_x(L^2_v)}$$

$$\geq \left(\frac{1}{12} - 3\left(1-\frac{1}{l}\right)||f-1||^{1/l}_{\dot H^l_x(L^2_v)}\,\eta^{-2/(1-1/l)}\right.$$

$$\left. -\left(1-\frac{1}{l}\right)||f-1||^{1/l}_{\dot H^{2l}_x(L^2_v)}\,\eta^{-2/(1-1/l)}\right)(H(f)-H(f_{eq}))$$

$$-K(f)^{1-1/l}\left((8+\frac{1}{24})\,||f-\bar f||^{2/l}_{L^2_x(L^2_v)} + \frac{7}{4}\,||f-\bar f||^{2/l}_{\dot H^l_x(L^2_v)}\right)$$

$$-K(f)^{1-1/l}\left(\frac{1}{2}+\frac{1}{2l}\right)\eta^{2/(1+1/l)}\left(3\,||f-1||^{1/l}_{\dot H^l_x(L^2_v)}+||f-1||^{1/l}_{\dot H^{2l}_x(L^2_v)}\right)||f-\bar f||^{2/(l^2+l)}_{L^2_x(L^2_v)}.$$

We now choose

$$\eta = \left(24\,(1-\frac{1}{l})\left(3\,||f-1||^{1/l}_{\dot H^l_x(L^2_v)} + ||f-1||^{1/l}_{\dot H^{2l}_x(L^2_v)}\right)\right)^{\frac{1}{2}(1-1/l)},$$

so that

$$3\left(1-\frac{1}{l}\right)||f-1||^{1/l}_{\dot H^l_x(L^2_v)}\,\eta^{-2/(1-1/l)}+\left(1-\frac{1}{l}\right)||f-1||^{1/l}_{\dot H^{2l}_x(L^2_v)}\,\eta^{-2/(1-1/l)} = \frac{1}{24}.$$

We end up with the estimate

$$\frac{d^2}{dt^2}K(f) \geq \frac{1}{24}\,(H(f)-H(f_{eq}))-K(f)^{1-1/l}\left\{(8+\frac{1}{24})\,||f-\bar f||^{2/l}_{L^2_x(L^2_v)}+\frac{7}{4}\,||f-\bar f||^{2/l}_{\dot H^l_x(L^2_v)}\right.$$

$$\left.+(\frac{1}{2}+\frac{1}{2l})\,||f-\bar f||^{2/(l^2+l)}_{L^2_x(L^2_v)}\left(24\,(1-\frac{1}{l})\right)^{\frac{l-1}{l+1}}\left(3\,||f-1||^{1/l}_{\dot H^l_x(L^2_v)}+||f-1||^{1/l}_{\dot H^{2l}_x(L^2_v)}\right)^{\frac{2}{1+1/l}}\right\}.$$

We have the structure of eq. (22) provided that the quantities

$$||f-\bar f||_{L^2_x(L^2_v)},\qquad ||f-\bar f||_{\dot H^l_x(L^2_v)},\qquad ||f-1||_{\dot H^l_x(L^2_v)},\qquad ||f-1||_{\dot H^{2l}_x(L^2_v)}$$

can be estimated from above (uniformly when $t\to+\infty$).

We now note that since equation (25) is linear and has coefficients which do not depend on $x$, it is also satisfied by any spatial derivative $\frac{\partial^l f}{\partial^l_x}$ of $f$. But we already know that $t\mapsto H(f)(t)$ is decreasing, so that all the norms $t\to||f(t)||_{\dot H^l_x(L^2_v)}$ (for all $l\in\mathbb{N}$) also decrease.

We now define

$$C_l = ||f(0)||_{\dot H^l_x(L^2_v)}.$$

Then, for all $l \in \mathbb{N}$,

$$||(f-\bar f)(t)||_{\dot H^l_x(L^2_v)} \le 2\,C_l,$$

and for all $l \in \mathbb{N}^*$,

$$||f(t)-1||_{\dot H^l_x(L^2_v)} \le C_l.$$

Finally, we obtain

$$\frac{d^2}{dt^2}K(f) \ge \frac{1}{24}\,(H(f)-H(f_{eq})) - K(f)^{1-1/l}\left\{(8+\frac{1}{24})\,C_0^{2/l} + \frac{7}{4}\,C_l^{2/l}\right.$$

$$\left. + (\frac{1}{2}+\frac{1}{2l})\,(2\,C_0)^{2/(l^2+l)}\left(24\,(1-\frac{1}{l})\right)^{\frac{l-1}{l+1}}\left(3\,C_l^{1/l}+C_{2l}^{1/l}\right)^{\frac{2}{1+1/l}}\right\}. \tag{28}$$

If we suppose that $C_l < +\infty$ for all $l \in \mathbb{N}$, then thanks to theorem 11 of [9], we see that

$$H(f(t)) - H(f_{eq}) \le Cst(s)\,t^{-s} \tag{29}$$

for all $s > 0$.

We now compute the typical order of magnitude of the constant in (29). We suppose that $l = 2$ and $C_0 = C_2 = C_4 = 2$. That corresponds to an initial datum having derivatives (in $x$) of order up to 4 which have $L^2$ norms of order 1. Then, estimate (28) becomes

$$\frac{d^2}{dt^2}K(f) \ge \frac{1}{24}\,(H(f)-H(f_{eq})) - K(f)^{1/2}\left\{(16+\frac{1}{12}) + \frac{7}{2}\right.$$

$$\left. + \frac{3}{4}\,4^{1/3}\left(12\right)^{1/3}\left(4\,\sqrt{2}\right)^{\frac{4}{3}}\right\}$$

$$\ge \frac{1}{24}\,(H(f)-H(f_{eq})) - 48\,K(f)^{1/2}.$$

We end up with the system of two differential inequalities :

$$-\frac{d}{dt}(H(f)-H(f_{eq})) = K(f), \tag{30}$$

$$\frac{d^2}{dt^2}K(f) \ge \frac{1}{24}\,(H(f)-H(f_{eq})) - 48\,K(f)^{1/2}. \tag{31}$$

We assume (for the sake of simplicity) that $H(f)(0) - H(f_{eq}) \le 1$. We define $t_0$ and $t_0 + \mathcal{T}_0$ in such a way that $H(f)(t_0) - H(f_{eq}) = a_0$, $H(f)(t_0+\mathcal{T}_0) = a_0/2$, where $a_0 \in ]0,1[$ is given (note that $t \mapsto H(f(t))$ decreases and that (at this level) one can have $\mathcal{T}_0 = +\infty$).

Thanks to (31), we see that

$$\frac{d^2}{dt^2}K(f) + 48\,K(f)^{1/2} \ge \frac{a_0}{48}.$$

We recall here a lemma from [10], in which we precise the constants a little more.

**Lemma** : Let $y$ be a $C^2$ function on $[T_1, T_2]$ (for some $0 \le T_1 < T_2$) satisfying the differential inequality

$$y'' + u\, y^{1-\varepsilon} \ge k,$$

for some $u, k > 0$, $\varepsilon \in ]0,1[$.

Then

$$T_2 - T_1 \le 48\, u^{-1/(2\,(1-\varepsilon))}\, k^{\varepsilon/(2\,(1-\varepsilon))},$$

or

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} y(s)\, ds \ge \frac{1}{86}\, k^{1/(1-\varepsilon)}\, \inf \left( \frac{1}{u}, \left(\frac{1}{u}\right)^{\frac{3}{2}+3\,\varepsilon} \right).$$

Applying this lemma in our case, we see that

$$\mathcal{T}_0 \le 48\,(48)^{-1} \left(\frac{a_0}{48}\right)^{1/2},$$

or

$$\frac{1}{\mathcal{T}_0} \int_{t_0}^{t_0+\mathcal{T}_0} K(f)(t)\, dt \ge \frac{1}{86} \left(\frac{a_0}{48}\right)^2 \inf \left( \frac{1}{48}, (\frac{1}{48})^3 \right).$$

Then we notice that (thanks to (30))

$$\frac{a_0}{2} = \int_{t_0}^{t_0+\mathcal{T}_0} K(f)(t)\, dt,$$

so that

$$\mathcal{T}_0 \le 48^{-1/2}\, a_0^{1/2},$$

or

$$\mathcal{T}_0 \le 1.1\, 10^{10}\, a_0^{-1}. \tag{32}$$

Remembering that $a_0 < 1$, we see that (32) holds.

We now define the times $T_i$ ($i \in \mathbb{N}$) by the formula $T_0 = 0$ and (for $i \ge 1$) $H(f)(T_i) - H(f_{eq}) = 2^{-i}$. Thanks to (32), we get the estimate

$$T_{i+1} - T_i \le 1.1\, 10^{10}\, 2^i.$$

Summing those estimates for $i = 0, .., n-1$, we obtain

$$T_n \le 1.1\, 10^{10}\, 2^n.$$

Then (taking into account the decay of $t \mapsto H(f(t))$),

$$\forall t \in [1.1\, 10^{10}\, 2^n, 1.1\, 10^{10}\, 2^{n+1}], \qquad H(f)(t) - H(f_{eq}) \le 2^{-n}.$$

This ensures that for all $t > 0$,

$$H(f)(t) - H(f_{eq}) \leq \frac{2.2\,10^{10}}{t}.$$

Of course, this estimate is not optimized at all, but it gives the order of magnitude of the explicit estimate that one can obtain. In this method, it is the use of the estimate in [10] which entails an enormous increase of the final constant.

C. Mouhot, C. Villani and L. Neumann have recently proposed another approach (which does not use in the same way second order derivatives in time) which will certainly help to obtain better constants at the end (Cf. [28] and [20]).

## 3. A review on the existing results of hypocoercivity

In section 2, we applied the method for hypocoercive situations to the simplest possible problem in which it makes sense to use it. We now rapidly describe the existing applications of this method for more realistic models.

### 3.1. The Fokker-Planck equation in a confining potential.

We consider here the operator

$$Af = -v \cdot \nabla_x f + \nabla_x V \cdot \nabla_v f + \nabla_v \cdot (\nabla_v f + v\, f), \tag{33}$$

where $x, v \in \mathbb{R}^N$, $f \equiv f(t,x,v) \geq 0$, and $V \equiv V(x)$ is a confining potential satisfying the normalization property $\int_{\mathbb{R}^N} e^{-V(x)}\, dx = 1$.

One quantity is conserved along the flow of eq. (1), (33), namely the total mass :

$$\frac{d}{dt} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(t,x,v)\, dvdx = 0. \tag{34}$$

As a consequence, we suppose in the sequel that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(t,x,v)\, dvdx = 1. \tag{35}$$

Then, one introduces the (relative) entropy (or free energy)

$$H(f) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x,v) \log\left(\frac{f(x,v)}{e^{-V(x)}\, M(v)}\right) dvdx,$$

where $M$ is given by formula (10). Eq. (2) is satisfied with the entropy dissipation $D$ defined by

$$D(f) = \int_{\mathbb{R}^N} f \left|\nabla \log \frac{f}{M}\right|^2 dv. \tag{36}$$

It is easy to see that (18) is verified with

$$\mathcal{M} = \{(x,v) \mapsto \rho(x)\, M(v), \quad \rho \geq 0, \int \rho\, dx = 1\}. \tag{37}$$

That is, the set $\mathcal{M}$ of local equilibria is formed by the functions which have a (centered, normalized) Maxwellian profile with respect to the $v$ variable, and are arbitrary with respect to the $x$ variable.

Then, (19) holds with $f_{eq}(x,v) = e^{-V(x)}\, M(v)$ since $f \in \mathcal{M}$ and $Af = 0$ implies that $v \cdot \nabla_x(\rho(x)\, M(v)) - \nabla_x V \cdot \nabla_v(\rho(x)\, M(v)) = 0$, so that $\nabla_x(\rho + \rho\, V) \cdot (v\, M(v)) = 0$, and finally $\rho(x) = e^{-V(x)}$.

The intermediate functional $K(f)$ that we introduce here is the relative entropy of $f$ with respect to the local Maxwellian $\rho_f(x)\, M(v)$ (where $\rho_f(x) = \int f(x,w)\, dw$), that is

$$K(f) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x,v) \log\left(\frac{f(x,v)}{\rho_f(x)\, M(v)}\right) dxdv.$$

It is clear that (21) holds. Estimate (20) with $\Phi(x) = 2\,x$ is then a simple consequence of Gross's Sobolev logarithmic inequality (Cf. [14]).

Estimate (22) holds with $\Psi_1(x) = Cst\, x$ and $\Psi_2(x) = Cst_\varepsilon\, x^{1-\varepsilon}$ as soon as $f \in S$, where $S$ is the set consisting of functions $f$ having bounded $H^k$ derivatives (for all $k \in \mathbb{N}$) and such that $a\, f_{eq} \leq f \leq b\, f_{eq}$ for some $a, b > 0$. Noticing that $S$ is stable by the flow of eq. (1), (33) when $V$ has a behavior at infinity close to that of the quadratic potential, we obtain

$$H(f)(t) - H(f_{eq}) \leq Cst_\varepsilon\, t^{1/\varepsilon - 1},$$

and it is possible to conclude thanks to the Csiszár-Kullback-Pinsker inequality.

We summarize the above result in the following theorem, first proven in [9] :

**Theorem 1** (L. Desvillettes, C. Villani) : *Let $f_0 \equiv f_0(x,v) \geq 0$ be an initial datum of mass 1 for eq. (1), (33), such that $a\, f_{eq} \leq f_0 \leq b\, f_{eq}$ for some $a, b > 0$, and $V$ be a potential of the form $V(x) = \frac{|x|^2}{2} + \Phi(x) + V_0$ with $V_0 \in \mathbb{R}$, $\int e^{-V(x)}\, dx = 1$ and $\Phi \in H^\infty(\mathbb{R}^N)$. Then, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ explicitly computable such that*

$$||f(t) - f_{eq}||_{L^1(\mathbb{R}^N)} \leq C_\varepsilon\, t^{-1/\varepsilon}.$$

One can compare this result with more recent theorems obtained in [17] and [27], where the exponential decay is proven.

3.2. **The linear Boltzmann equation in a confining potential.** One defines the operator

$$Af = -v \cdot \nabla_x f + x \cdot \nabla_v f - \rho_f \, M + f, \tag{38}$$

where $x, v \in \mathbb{R}^N$, $f \equiv f(t,x,v) \geq 0$, $\rho_f = \int f \, dv$, and $M$ is given by formula (10).

This operator describes the interaction between particles and a fixed medium (which has $M$ as velocity distribution) in the presence of a quadratic confining potential. Note that the collision operator is more complicated than in the transport equation of section 2, in particular because $v$ does not belong to a compact set.

The total mass is still conserved along the flow of eq. (1), (38), that is, (34) holds. As a consequence, we still assume that (35) holds.

Then, one introduces the (relative) quadratic entropy

$$H(f) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| f(x,v) - \frac{e^{-x^2/2}}{(2\pi)^{N/2}} \, M(v) \right|^2 e^{x^2/2} \, (M(v))^{-1} \, dv dx.$$

Eq. (2) is satisfied with the entropy dissipation $D$ defined by

(39)

$$D(f) = 2 \, (2\pi)^{-N/2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| f(x,v) - \rho_f(x) \, M(v) \right|^2 e^{x^2/2} \, (M(v))^{-1} \, dv dx.$$

It is clear that, as in subsection 3.1, (18) is verified with $\mathcal{M}$ given by (37).

Note also that (19) holds with $f_{eq}(x,v) = \frac{e^{-x^2/2}}{(2\pi)^{N/2}} \, M(v)$ since $f \in \mathcal{M}$ and $Af = 0$ implies that $v \cdot \nabla_x(\rho(x) \, M(v)) - x \cdot \nabla_v(\rho(x) \, M(v)) = 0$, so that $(\nabla_x \rho - \rho \, x) \cdot (v \, M(v)) = 0$, and finally $\rho(x) = \frac{e^{-x^2/2}}{(2\pi)^{N/2}}$.

The intermediate functional $K(f)$ that one can take here is (as in section 2) nothing but the entropy dissipation $D(f)$. As a consequence, estimates (20) and (21) are immediately obtained.

As in the previous subsection, estimate (22) holds with $\Psi_1(x) = Cst \, x$ and $\Psi_2(x) = Cst_\varepsilon \, x^{1-\varepsilon}$ as soon as $f$ is smooth enough. Then, it is possible to show that the required smoothness is propagated, so that

$$H(f)(t) - H(f_{eq}) \leq Cst_\varepsilon \, t^{1/\varepsilon - 1},$$

and

$$||f(t) - f_{eq}||_{L^1(\mathbb{R}^N)} \leq C_\varepsilon \, t^{-1/\varepsilon}$$

thanks to Cauchy-Schwartz inequality. We summarize the above result in the following theorem, proven in [3] :

**Theorem 2** (M. Caceres, J. Carrillo, T. Goudon) : *Let* $f_0 \equiv f_0(x,v) \geq 0$ *be an initial datum of mass* 1 *for eq. (1), (38), such that*

*$a\,f_{eq} \le f_0 \le b\,f_{eq}$ for some $a, b > 0$, and $f_0 \in W^{\infty,\infty}(\mathbb{R}^N \times \mathbb{R}^N)$. Then, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ explicitly computable such that*

$$||f(t) - f_{eq}||_{L^1(\mathbb{R}^N)} \le C_\varepsilon\, t^{-1/\varepsilon}.$$

We also refer to the very recent work by F. Hérau for improvements of the results given here (Cf. [16]).

3.3. **The linear Boltzmann equation in a periodic box.** One considers the operator

$$Af = -v(k) \cdot \nabla_x f - \int_{k \in B} \Big( S(k', k)\, f(k') - S(k, k')\, f(k) \Big)\, dk' \tag{40}$$

where $x \in \mathbb{T}^N$, $k \in B \subset \mathbb{R}^N$ and $B$ is equipped with the measure $dk$, $f \equiv f(t, x, k) \ge 0$. We assume that $v, S$ satisfy the following assumptions : $k \mapsto v(k)$ is odd and

$$\forall z \ne 0, \exists k \in B, \qquad v(k) \cdot z \ne 0; \tag{41}$$

the kernel $(k, k') \mapsto S(k, k')$ is nonnegative, and there exists $M_1 \ge 0$ such that $\int_B M_1(k)\, dk = 1$, $\int_B v(k)\, M_1(k)\, dk = 0$, $\int_B |v(k)|^4\, M_1(k)\, dk < +\infty$, $Q(M_1) = 0$. Moreover, we suppose that

$$\forall k, k' \in B, \qquad \gamma \le \frac{S(k, k')}{M_1(k)} \le \Gamma$$

for some $\gamma, \Gamma > 0$.

This corresponds to a more general (with respect to subsection 3.2) linear interaction between the particles and a background (the fact that $v \equiv v(k)$ enables to treat semiconductors models), but to a simpler assumption on the domain (the torus $\mathbb{T}^N$, normalized in such a way that its volume is 1) in which the space variable $x$ lives.

Once again, the total mass is conserved along the flow of eq. (1), (40). This means that (34) holds and that we can assume that (35) also holds ($v$ has to be replaced by $k$ in those two formulas).

Then, one introduces the (relative) quadratic entropy

$$H(f) = \int_{\mathbb{T}^N} \int_B \Big| f(x, k) - M_1(k) \Big|^2 (M_1(k))^{-1}\, dk dx.$$

Eq. (2) is satisfied with the entropy dissipation $D$ defined by

$$D(f) = \frac{1}{4} \int_{\mathbb{T}^N} \int_B \int_B \Big( S(k', k)\, M_1(k') + S(k, k')\, M_1(k) \Big) \times \Big( \frac{f}{M_1}(t, x, k) - \frac{f}{M_1}(t, x, k') \Big)\, dk' dk dx.$$

Then, (18) is verified with

$$\mathcal{M} = \left\{ \rho(x)\, M_1(k), \quad \int_{\mathbb{T}^N} \rho(x)\, dx = 1 \right\}.$$

Note also that (19) holds with $f_{eq}(x,v) = M_1(k)$ since $f \in \mathcal{M}$ and $Af = 0$ implies that $v(k) \cdot \nabla_x(\rho(x)\, M_1(k)) = 0$, and assumption (41) can be used to prove that $\rho$ is a constant.

The intermediate functional $K(f)$ that one can take here is the relative (quadratic) entropy with respect to the local equilibrium, namely

$$K(f) = \int_{x\in\mathbb{T}^N} \int_{k\in B} |f - \rho_f\, M_1|^2\, M_1^{-1}\, dk dx,$$

where $\rho_f = \int_{k\in B} f\, dk$. Then, (21) is obvious, and estimate (20) is a consequence of standard computations.

As in subsection 3.2, estimate (22) holds with $\Psi_1(x) = Cst\, x$ and $\Psi_2(x) = Cst_\varepsilon\, x^{1-\varepsilon}$ as soon as $f$ is smooth enough. Finally, it is also possible to show that the required smoothness is propagated, so that

$$H(f)(t) - H(M_1) \le Cst_\varepsilon\, t^{1/\varepsilon - 1}.$$

The following theorem summarizes the results decribed above (Cf. [12]) :

**Theorem 3** (K. Fellner, L. Neumann and C. Schmeiser) : *Let $f_0 \equiv f_0(x,k) \ge 0$ be an initial datum for eq. (1), (40), such that $f_0 \in L^1(\mathbb{T}^N \times B) \cap L^2(B, M_1^{-1}\, dk; H^n(\mathbb{T}^N))$ (for some $n \ge 2$).*

*Then, there exists $C_n > 0$ explicitly computable such that*

$$||f(t) - M_1||_{L^2(B, M_1^{-1}\, dk; L^2(\mathbb{T}^N))} \le C_n\, t^{(1-n)/2}.$$

3.4. **The nonlinear Boltzmann equation.** We now consider the spatially inhomogeneous Boltzmann operator

$$Af = -v \cdot \nabla_x f + Q(f) \tag{42}$$

with $Q$ defined by (14), and we suppose that $x$ varies in a bounded domain $\Omega$ (taken without loss of generality of volumee 1) which is supposed to have no axis of symmetry. We supplement (42) with the boundary condition

(43)

$$\forall t \in \mathbb{R}, x \in \partial\Omega, v \in \mathbb{R}^N, \qquad f(t,x,v) = f(t,x,v - 2\,(v\cdot n(x))\, n(x)),$$

where $n(x)$ is the exterior normal vector to $\partial\Omega$. We refer to [5] for details on the underlying modeling : it describes a rarefied gas in a box.

The conserved quantites for eq. (1), (42), (43) are the total mass and energy :

$$\frac{d}{dt}\int_{\Omega}\int_{\mathbb{R}^N} f(t,x,v) \begin{pmatrix} 1 \\ \frac{|v|^2}{2} \end{pmatrix} dv dx = 0, \tag{44}$$

since the momentum is not conserved during the rebounds at the boundary.

We shall suppose in the sequel that $f$ is normalized in such a way that its total mass and energy are respectively equal to 1 and $N/2$.

We define the (relative) entropy $H$ by

$$H(f) = \int_{\Omega}\int_{\mathbb{R}^N} f(x,v)\,\log\left(\frac{f(x,v)}{M(v)}\right) dv dx.$$

Then, (2) holds with

$$D(f) = \frac{1}{4}\int_{\Omega}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{S^{N-1}} \Big\{ f(x,v')\,f(x,v'_*) - f(x,v)\,f(x,v_*)\Big\}$$

$$\times \log\left(\frac{f(x,v')\,f(x,v'_*)}{f(x,v)\,f(x,v_*)}\right) B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) d\sigma dv_* dv dx.$$

According to [5] for example, we see that eq. (18) holds with

$$\mathcal{M} = \Big\{ (x,v) \mapsto \frac{\rho(x)}{(2\pi\, T(x))^{N/2}} e^{-\frac{|v-u(x)|^2}{2T(x)}}, \quad \rho(x)\geq 0, u(x)\in\mathbb{R}^N, T(x)>0,$$

$$\int_{\Omega}\rho\, dx = 1, \int_{\Omega}(\rho\,\frac{|u|^2}{2} + \frac{N}{2}\,T)\, dx = \frac{N}{2}\Big\}.$$

Note that $\mathcal{M}$ can also be defined by

$$\mathcal{M} = \Big\{ g : (x,v)\mapsto \exp\,(a(x)+b(x)\cdot v - c(x)\,|v|^2), \quad a(x)\in\mathbb{R}, b(x)\in\mathbb{R}^N, c(x)>0,$$

$$\int_{\Omega}\int_{\mathbb{R}^N} g \begin{pmatrix} 1 \\ \frac{|v|^2}{2} \end{pmatrix} dv dx = \begin{pmatrix} 1 \\ \frac{N}{2} \end{pmatrix}\Big\}.$$

Then, property (19) is a consequence of the following computation of Grad (in dimension 3) (Cf. [13]) :

$$v\cdot\nabla_x(\exp\,(a(x)+b(x)\cdot v - c(x)\,|v|^2)) = 0$$

$$\Rightarrow \qquad a(x) = a_0, \quad b(x) = b_0 + b_1\times x, \quad c(x) = c_0,$$

for some constants $a_0\in\mathbb{R}$, $b_0, b_1\in\mathbb{R}^3$, $c_0\in\mathbb{R}$. Using the boundary condition (and the fact that $\Omega$ has no axis of symmetry), we get $b_0 = b_1 = 0$, so that

$$f_{eq}(x,v) = \exp(a_0 - c_0\,|v|^2),$$

and $a_0, c_0$ are computable thanks to the conservations (44). Finally, $f_{eq} = M$, where $M$ is given by (16).

Integrating with respect to $x$ estimate (4) (with $D, \Phi$ as defined in the last part of subsection 1.2), it is easy to see that (20) holds with $\Phi(x) = Cst_\varepsilon\, x^{1+\varepsilon}$, and $K$ defined as the relative entropy of $f$ and $M_f$ :

$$K(f) = \int_\Omega \int_{\mathbb{R}^N} f(x,v)\, \log\left(\frac{f(x,v)}{M_f(x,v)}\right) dvdx,$$

where $M_f$ is defined by (16). In this case, unfortunately, eq. (22) has to be replaced by a variant for many reasons. First, one needs to consider $K'(f) = ||f - M_f||^2_{L^2}$ instead of $K(f)$ in order to avoid the assumption that $f$ decreases when $|v| \to +\infty$ like a Gaussian (such a decay is in general not propagated by the flow of eq. (1), (42), (43)). Then, one has to replace $\Psi_2(K(f))$ (or rather $\Psi_2(K'(f))$) by $\Psi_2(K'(f)/\delta) + \delta\, H(f)$ (for any $\delta > 0$). Those two changes make the proof more complex but do not lead to major conceptual difficulties.

Finally, the most important change with respect to the model differential inequality (22) is related to the fact that it is not possible to reconstitute the term $\Psi_1(H(f) - H(f_{eq}))$ in (22) by using directly $K(f)$ or $K'(f)$. One in fact needs to introduce other intermediate functionals, so that one ends up with a differential system of more than 2 inequalities (actually : 4). This is realated to the fact that the dissipative term in the compressible Navier-Stokes equation acts only on $u$ and $T$ but not on $\rho$ (Cf. [10] for more details).

When all those modifications have been performed, one can prove the following result (Cf. [10]) :

**Theorem 4** (L. Desvillettes, C. Villani) : *Let $f$ be a (strong) solution of the Boltzmann equation (1), (42) (with a cross section $B$ satisfying assumptions detailed in* [10]*) on a bounded regular open set $\Omega$ of volume* 1 *with no axis of symmetry and with the specular reflexion boundary condition (43). We assume that its total mass is* 1 *and its total energy is $N/2$. We also assume that for all $k, l \in \mathbb{N}$,*

$$\sup_{t\geq 0} \int\int_{x\in\Omega, v\in\mathbb{R}^N} |\nabla^k_{x,v} f(t,x,v)|^2\, (1+|v|^2)^l\, dvdx < +\infty,$$

*and that there exists $C_1, C_2 > 0$, $q \geq 2$, such that $f(t,x,v) \geq C_1\, \exp(-C_2\, |v|^q)$ (this last assumption can be somehow relaxed, Cf.* [21]*).*

*Then for all $\varepsilon > 0$,*

$$\left|\left| f(t,x,v) - M(v) \right|\right|_{L^1(\Omega\times\mathbb{R}^N)} \leq C_\varepsilon\, t^{-1/\varepsilon},$$

*where $C_\varepsilon$ is an explicitly computable constant.*

We notice that the assumptions of this theorem are not empty, since solutions close to equilibrium satisfying them can be built (Cf. [15]). We refer to [2] and [26] for a comparison with theorems obtained by spectral theory and linearization.

In this subsection, we also quote a recent paper by L. Neumann and C. Schmeiser, devoted to another type of nonlinear Boltzmann equation, namely

(45)

$$\partial_t f + v \cdot \nabla_x f = \int_{\mathbb{R}^N} \Big( M(v)\,(1 - f(v))\, f(v') - M(v')\,(1 - f(v'))\, f(v) \Big)\, dv',$$

where $M$ is (as usual) the centered reduced Maxwellian (given by (16)). This kinetic equation corresponds to the interaction between particles of fermion type. One can prove the following result :

**Theorem 5 (L. Neumann, C. Schmeiser) :** *We suppose that $f(0,x,v)$ is an initial datum satisfying*

$$f_-(v) \le f(0,x,v) \le f_+(v),$$

*where*

$$f_\pm(v) = \frac{k_\pm\, M(v)}{1 + k_\pm\, M(v)},$$

*for some $k_\pm > 0$. Let $f$ be a (strong) solution of eq. (45) such that*

$$\sup_{t \ge 0} \int \int_{x \in \mathbb{T}^N, v \in \mathbb{R}^N} |\nabla_x^k f(t,x,v)|^2\, M^{-1}(v)\, dv dx < +\infty,$$

*Then for all $\varepsilon > 0$,*

$$\Big|\Big| f(t,x,v) - f_\infty(v) \Big|\Big|_{L^1(\mathbb{T}^N \times \mathbb{R}^N)} \le C_\varepsilon\, t^{-1/\varepsilon},$$

*where $C_\varepsilon$ is an explicitly computable constant, and*

$$f_\infty(v) = \frac{k_\infty\, M(v)}{1 + k_\infty\, M(v)},$$

*with $k_\infty$ fixed by the conservation of global mass.*

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