CONVERGENCE TO EQUILIBRIUM IN VARIOUS SITUATIONS FOR THE SOLUTION OF THE BOLTZMANN EQUATION

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October 31, 2000

Abstract

We present in this work some contributions made by various authors in the last twenty years to understand how solutions of the Boltzmann equation reach the equilibrium when the time tends to infinity. We are interested here in the form of the limit as well as in various estimates concerning the rate of convergence towards this limit.

1 Introduction

The dynamics of a rarefied gas is usually described by a density $f(t,x,v)$ of particles which at time t and point x move with a given velocity v in $\mathbb{R}^3$ (Cf. [Ce], [Tr, Mu], [Ch, Co]). The function f satisfies the Boltzmann equation,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f), \tag{1.1}$$

where Q is a quadratic collision kernel taking in account any collisions preserving momentum and kinetic energy, and acting only on the velocity variable v,

$$Q(f,f)(v) = \int_{v_1 \in \mathbb{R}^3} \int_{\omega \in S^2} \{ f(v - (\omega \cdot (v - v_1))\, \omega) f(v_1 + (\omega \cdot (v - v_1))\, \omega)$$

$$-f(v)\, f(v_1)\}\, B(|v-v_1|,\theta)\, d\omega dv_1, \tag{1.2}$$

and

$$\cos\theta = |\omega \cdot \frac{v-v_1}{|v-v_1|}|. \tag{1.3}$$

When the colliding molecules are modelled by hard spheres, the cross section B writes

$$B(|v-v_1|,\theta) = |v-v_1| \cos\theta, \tag{1.4}$$

but when the interaction between the particles is due to a force proportional to the inverse s^{th} power of the distance, it becomes

$$B(|v-v_1|,\theta) = |v-v_1|^{\frac{s-5}{s-1}} |\cos\theta|^{-\frac{s+1}{s-1}} \beta(\theta), \tag{1.5}$$

where β is a bounded function (different from 0 at point $\frac{\pi}{2}$) (Cf. [Ce]).

Since this cross section is very singular when $\theta \longrightarrow \frac{\pi}{2}$, the traditional angular cut–off assumption of Grad that

$$B(|v-v_1|,\theta) = |v-v_1|^{\frac{s-5}{s-1}} \beta'(\theta), \tag{1.6}$$

where β' is a bounded function, is often made (Cf. [Gr]).

It is easy to prove that, at least at the formal level,

$$\int_{v\in \mathbb{R}^3} Q(f,f)(v)\, \psi(v)\, dv = 0, \tag{1.7}$$

where

$$\psi(v) = (1, v_1, v_2, v_3, |v|^2). \tag{1.8}$$

We denote from now on $\psi_0(v) = 1$, $\psi_i(v) = v_i$ for $i = 1$ to 3 and $\psi_4(v) = |v|^2$. Eq. (1.7) simply shows that the mass, momentum and kinetic energy are conserved by the collision term.

Moreover, according to Boltzmann's H–theorem, the dissipation of entropy is nonpositive:

$$\int_{v\in \mathbb{R}^3} Q(f,f)(v)\, \log f(v)\, dv \le 0, \tag{1.9}$$

and becomes 0 if and only if f is a Maxwellian function of v:

$$f(v) = \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|v-u|^2}{2T}}, \tag{1.10}$$

where ρ is the density, u the mean velocity and T the temperature. A Maxwellian is said to be absolute when its parameters ρ, u and T do not depend on the variables t and x.

According to these properties of Q, it is conjectured that, at least in some situations (linearized or weakly nonlinear equation near a given absolute Maxwellian, spatially homogeneous equation, full equation with periodic boundary conditions), the solutions of eq. (1.1) should converge in some sense to the absolute Maxwellian whose mass, momentum and energy are the same.

In section 2, we recall the known facts about the linearized equation, then we consider in section 3 the spatially homogeneous case. Sections 4 and 5 are devoted respectively to the study of the full nonlinear equation near an absolute Maxwellian, and near vacuum. Finally, we conclude in section 6 by some remarks on renormalized solutions of the full nonlinear equation.

2 The Linearized, Spatially Homogeneous Equation

When eq. (1.1) is linearized around a given absolute Maxwellian $\Omega(v)$, for example $e^{-\frac{v^2}{2}}$, the fluctuation g around the Maxwellian defined by

$$f(t,v) = \Omega(v) + \Omega(v)^{1/2} g(t,v) \tag{2.1}$$

(Cf. [Caf 1]), satisfies in the spatially homogeneous case

$$\frac{\partial g}{\partial t} = Lg, \tag{2.2}$$

where

$$\begin{aligned} Lg(t,v) = \int_{v_1 \in I\!\!R^3} \int_{\omega \in S^2} \{ & g(t, v - (\omega \cdot (v - v_1))\, \omega)\, \Omega^{1/2}(v_1 + (\omega \cdot (v - v_1))\, \omega) \\ & + g(t, v_1 + (\omega \cdot (v - v_1))\, \omega)\, \Omega^{1/2}(v - (\omega \cdot (v - v_1))\, \omega) \\ & - g(t,v)\, \Omega^{1/2}(v_1) - g(t, v_1)\, \Omega^{1/2}(v) \}\, \Omega^{1/2}(v_1)\, B(|v - v_1|, \theta)\, d\omega dv_1. \end{aligned} \tag{2.3}$$

The conservations of mass, momentum and energy (1.7) still hold, but the H–theorem now becomes

$$\int_{v \in I\!\!R^3} g(t,v)\, Lg(t,v)\, dv \le 0, \tag{2.4}$$

and there is equality in (2.4) if and only if g is in the subspace spanned by the functions ψ_i, for $i = 0$ to 4.

Since the operator L is self–adjoint in $L^2(dv)$, the corresponding information on its spectrum in this space is that it is included in $\mathbb{R}_-$ and that 0 is an eigenvalue, the corresponding eigenspace being spanned by the functions ψ_i, for $i = 0$ to 4.

It is now classical that (under the cut–off assumption), eq. (2.1) can be put under the form

$$\frac{\partial g}{\partial t} = Kg - \nu(v)\, g, \tag{2.5}$$

where K is a compact operator of $L^2(dv)$ (Cf. [Gr] and [Df]) and

$$\nu(v) = \int_{v_1 \in \mathbb{R}^3} \int_{\omega \in S^2} e^{-\frac{v_1^2}{2}} B(|v - v_1|, \theta)\, d\omega dv_1. \tag{2.6}$$

Therefore, in the case of hard spheres, cut–off hard potentials and Maxwellian molecules,

$$\forall v \in \mathbb{R}^3, \quad 0 < \nu(0) \leq \nu(v), \tag{2.7}$$

and according to a classical theorem (Cf. [Sc] for example), the operator K does not change the essential spectrum of $-\nu(v)$, which implies that near zero, only pointwise spectrum can exist.

The corresponding information about the decay of the solution of eq. (2.1) is that there exists a strictly positive α_0 such that if $g(0, v)$ is in $L^2(dv)$ and

$$\int_{v \in \mathbb{R}^3} g(0, v)\, \psi_i(v)\, dv = 0 \quad \text{ for } \quad i = 0, .., 4, \tag{2.8}$$

then

$$||g(t, v)||_{L^2(dv)} \leq e^{-\alpha_0 t}\, ||g(0, v)||_{L^2(dv)}. \tag{2.9}$$

Note that in the case of Maxwellian molecules, the constant α_0 is known explicitly, as is in fact the whole spectrum of L (Cf. [WCh, Uh]).

The case of soft potentials is more intricate, since

$$\forall v \in \mathbb{R}^3, \quad 0 < \nu(v) \leq \nu(0). \tag{2.10}$$

According to (2.10), the continuous spectrum of L touches 0 and therefore one cannot hope to find an exponential decay for the solutions of (2.1). However, Caflish was able to prove in 1980 (Cf. [Caf 1]) the following estimate:

If $3 < s < 5, 0 < \alpha < 1/4$, $g(0,v)$ is in $L^\infty(e^{\alpha v^2} dv)$ and (2.8) holds, then there exist $C, \lambda > 0$ (λ being known explicitly) such that

$$||g(t,v)||_{L^2(dv)} \leq C e^{-\lambda t^{\frac{2s-2}{s+3}}} ||g(0,v)||_{L^\infty(e^{\alpha \frac{v^2}{2}} dv)}. \tag{2.11}$$

This estimate is based on the fact that the operator K does not change in an essential way the long time behavior of the solution h of the equation

$$\frac{\partial h}{\partial t} = -\nu(v)\, h. \tag{2.12}$$

The same kind of results can be obtained with another method (Cf. [De 1]). The reader can also refer to [Pa], [Kl] and [Cz, Pal] to obtain information when a radial cut–off or no cut–off at all is assumed.

3 The Nonlinear, Spatially Homogeneous Equation

3.1 Position of the problem

The spatially homogeneous Boltzmann equation writes:

$$\frac{\partial f}{\partial t} = Q(f,f), \tag{3.1}$$

where

$$Q(f,f)(t,v) = \int_{v_1 \in \mathbb{R}^3} \int_{\omega \in S^2} \{f(t, v - (\omega \cdot (v - v_1))\, \omega)\, f(t, v_1 + (\omega \cdot (v - v_1))\, \omega)$$
$$- f(t,v)\, f(t,v_1)\}\, B(|v - v_1|, \theta) d\omega dv_1. \tag{3.2}$$

Estimates (1.7) implies the following formal conservations,

$$\int_{v \in \mathbb{R}^3} g(t,v)\, \psi_i(v)\, dv = \int_{v \in \mathbb{R}^3} g(0,v)\, \psi_i(v)\, dv \quad \text{for} \quad i = 0,..,4. \tag{3.3}$$

Moreover, the entropy inequality becomes

$$\int_{v \in \mathbb{R}^3} g(t,v) \log g(t,v)\, dv \leq \int_{v \in \mathbb{R}^3} g(s,v) \log g(s,v)\, dv \quad \text{for} \quad s \leq t. \tag{3.4}$$

The usual conjecture is that solutions of eq. (3.1) of finite mass, energy and entropy should converge to the absolute Maxwellian $m(v)$ of same mass, momentum and energy. Of course, in order to prove such a conjecture, one needs to know the existence of a solution of eq. (3.1). We give here the results obtained in this direction for various types of cross sections and in different spaces.

3.2 Hard spheres

For hard spheres, Carleman proved in 1933 (Cf. [Ca]) the existence of a solution $f(t,v)$ to eq. (3.1) when the initial datum $f(0,v)$ is continuous and satisfies

$$\sup_{v\in\mathbb{R}^3} (1+|v|^{6+\epsilon})\, f(0,v) < +\infty \tag{3.5}$$

for some $\epsilon > 0$. This solution converges to $m(v)$ in $L^\infty(dv)$ when t tends to infinity.

3.3 Hard potentials with cut–off

For hard potentials with cut–off, there is a lot of results in various spaces, among which the followings:

In an L^∞ setting, Maslova and Tchubenko extended in 1976 (Cf. [Ma, Tc]) the result of Carleman in the following way:
If the initial datum $f(0,v)$ satisfies

$$\sup_{v\in\mathbb{R}^3} (1+|v|^{\chi})\, f(0,v) < +\infty, \tag{3.6}$$

where χ depends on the cross section B, then the same result as in the case of hard spheres holds.

In an L^1 setting, Arkeryd proved in 1986 (Cf. [Ar 3]) that if the initial datum $f(0,v)$ satisfies

$$||f\log f(0,v)||_{L^1(dv)} + ||f(0,v)||_{L^1((1+|v|^k)dv)} < +\infty \tag{3.7}$$

for some $k > 2$, then a solution of eq. (3.1) exists and tends to $m(v)$ strongly in $L^1((1+|v|^{k'})dv)$ when $k' < k$ (The weak convergence was proved in earlier papers (Cf. [Ar 1], [Ar 2])).

Moreover, this result was improved by Arkeryd in 1988 when $f(0,v)$ possesses moments of order high enough (Cf. [Ar 4]).

More precisely, if the mass, momentum and energy of $f(0,v)$ are fixed, there exists $\mu > 0$ and $k_0 \geq \frac{3s-7}{s-1}$ such that if $k \geq k_0$ and $f(0,v)$ satisfies

$$||f\log f(0,v)||_{L^1(dv)} + ||f(0,v)||_{L^1((1+|v|^k)dv)} < +\infty, \tag{3.8}$$

then for any $\gamma < \nu$, $k' < k$,

$$||f(t,v) - m(v)||_{L^1((1+|v|^{k'})dv)}$$

$$\leq C(k', \gamma, f(0,.))\, e^{-\gamma t}\, ||f(0,v) - m(v)||_{L^1((1+|v|^{k'})dv)}. \tag{3.9}$$

This improvement is based on the fact that when t is large enough, $f(t,v)$ approximates $m(v)$ in L^1 norm (because of the result mentionned above), and therefore one can apply the theory of the linearized equation (although with a lot of new difficulties).

This theorem is now improved by the results on moments obtained in [De 4]. Namely, the conclusion (3.9) holds for any $k' > 0$ as soon as estimate (3.7) on the initial datum holds.

The same kind of results is now available in the $L^1 \cap L^p$ setting. Gustaffson proved in 1988 (Cf. [Gu]) that for $1 < p \leq +\infty, k_1 \geq 2$, and $\frac{s-5}{s-1} \leq k \leq k_1 - \frac{s-5}{p(s-1)}$, if

$$||f \log f(0,v)||_{L^1(dv)} + ||f(0,v)||_{L^1((1+|v|^{k_1})dv)} + ||f(0,v)||_{L^p((1+|v|^{kp})dv)} < +\infty, \tag{3.10}$$

then there exists a solution of eq. (3.1) such that $f(t,v)$ tends to $m(v)$ in $L^1((1+|v|^{k_1'})dv) \cap L^p((1+|v|^{k'p})dv)$ for every k', k_1' such that $0 \leq k_1' < k_1$ and $0 \leq k' < k$ (for $p = +\infty$, the initial datum $f(0,v)$ must also be continuous).

This result was improved by Wennberg in 1992 (Cf. [We 1]) in order to obtain the exponential decay in that kind of norms.

More precisely, if the mass, momentum and energy of $f(0,v)$ are fixed, if $1 < p < +\infty$, and $\beta > 0$, then there exists $\mu > 0$ and $k_0 \geq 5 - 3/p$ such that if $k_0 < k < k_1 - \frac{\beta}{p}$ and $f(0,v)$ satisfies

$$\begin{aligned} &||f \log f(0,v)||_{L^1(dv)} + ||f(0,v)||_{L^1((1+|v|^{k_1+\beta})dv)} \\ &\qquad + ||f(0,v)||_{L^p((1+|v|^{p(k+\beta)})dv)} < +\infty, \end{aligned} \tag{3.11}$$

then for any $\gamma < \nu$, $k' < k$, $k_1' < k_1$,

$$\begin{aligned} &||f(t,v) - m(v)||_{L^1((1+|v|^{k_1'})dv) \cap L^p((1+|v|^{pk'})dv)} \\ &\leq C(k', k_1', \gamma, f(0,.))\, e^{-\gamma t}\, ||f(0,v) - m(v)||_{L^1((1+|v|^{k_1})dv) \cap L^p((1+|v|^{pk})dv)}. \end{aligned} \tag{3.12}$$

Wennberg has also extended in 1993 the results of [De 4] in this context, the assumption (3.11) can therefore be relaxed (Cf. [We 3]).

Finally, results about the uniformity of the decay were found by Carlen (Cf. [Car]).

3.4 Maxwellian molecules

The case of Maxwellian molecules is well–adapted to the search of explicit solutions. The exponential convergence of every moments of the solution of (3.1), first discovered by Truesdell, is now well–known (Cf. [Tr]). Moreover, we recall the BKW mode,

$$f(t,v) = (2\pi\tau)^{-3/2} e^{-\frac{v^2}{2\tau}} \{1 + (\frac{1}{\tau} - 1)(\frac{v^2}{2\tau} - \frac{3}{2})\} \tag{3.13}$$

(Cf. [Bo 1] and [Kr, Wu]), where $\tau = 1 - e^{-\lambda t}$, and in which the exponential convergence towards equilibrium is clear. Note also the following result by Bobylev (Cf. [Bo 2]), based on explicit computations:
We consider an isotropic initial datum $f(0, |v|)$ for equation (3.1) such that

$$\int_{v\in I\!\!R^3} f(0,v)\, \psi(v)\, dv = (1,0,0,0,3). \tag{3.14}$$

We write then the Taylor expansion of its Fourier transform with respect to the variable v:

$$\hat{f}(0,\xi) = e^{-\frac{\xi^2}{2}} \sum_{n=0}^{+\infty} u_n(0) \frac{\xi^{2n}}{n!}, \tag{3.15}$$

and we assume that

$$\sup_{n\in I\!\!N} |u_n(0)|^{1/n} \leq \sqrt{3/7}. \tag{3.16}$$

Then there exists $K > 0$ (explicitely known), such that

$$|f(t,|v|) - (2\pi)^{-3/2} e^{-\frac{v^2}{2}}| \leq C e^{-Kt}. \tag{3.17}$$

Results in this direction are also proved by Toscani (Cf. [To 1]).

Moreover, Bobylev proved also the following result (Cf. [Bo 2]):

For every $\delta \in [0, K[$, one can find an initial datum $f(0,|v|)$ (with algebraic decay when $|v| \to +\infty$) and $C_1, C_2 > 0$ such that

$$C_1 e^{-\delta t} \leq |\hat{f}(t,\xi) - e^{-\frac{\xi^2}{2}}| \leq C_2 e^{-\delta t}. \tag{3.18}$$

This seems to imply that no uniformity can be obtained for the rate of exponential convergence towards equilibrium unless some properties of nice decay when $|v| \to +\infty$ on the initial datum are assumed.

Finally, note also the results of Petrina and Mischenko in 1988 (Cf. [Pe, Mi]) about exponential convergence towards equilibrium for cross sections of the type

$$B(|v - v_1|, \theta) = \sigma_1(\theta) + |v - v_1|^2 \sigma_2(\theta). \tag{3.19}$$

3.5 Soft potentials

Solutions $f(t,v)$ of eq. (3.1) with that kind of potentials (with $s > 3$) are known to exist (Cf. [Ar 5]), but no result of convergence of the complete function $f(t,v)$ towards $m(v)$ is proved. This is mainly due to the lack of uniform estimates on the moments of order higher than 2 of f. One can however prove that there exists a sequence t_n going to infinity, such that $f_n(t) = f(t+t_n)$ converges weakly to $m(v)$ as soon as the moment of order 3 is initially bounded (Cf. [De 4]).

We have also the following estimate when $f(t,v) \geq C_1 e^{-C_2|v|^2}$ for some $C_1, C_2 > 0$, with or without the cut–off assumption (Cf. [De 2]):

$$\inf_{m \text{ Maxwellian}} \int_t^{2t} \int_{v \in \mathbb{R}^3} |\log f(s,v) - m(v)|\, e^{-(C_2+\epsilon)|v|^2}\, dv \frac{ds}{t} \leq \frac{K_\epsilon}{\sqrt{t}} \tag{3.20}$$

for every $\epsilon > 0$, which seems to imply at least an algebraic decay (recall that in the linearized case, the decay is proved to be superalgebraic).

Note also that Wennberg extended this kind of estimates in the case of hard–spheres and hard potentials (Cf. [We 2]).

3.6 Hard potentials without cut–off

In that case, Arkeryd proved in 1982 (Cf. [Ar 2]), using non–standard arguments, the existence of a solution of eq. (3.1) converging weakly in L^1 to $m(v)$ when $t \to +\infty$ as soon as hypothesis (3.7) holds.

Elmroth gave also a standard proof in 1984 of the same result (Cf. [El]).

4 The Full Nonlinear Boltzmann Equation near a Maxwellian

4.1 Position of the problem

Making in eq. (1.1) the change of variables (2.1), one gets

$$\frac{\partial g}{\partial t} + v \cdot \nabla_x g = Lg + \Gamma(g,g), \tag{4.1}$$

where L is defined in (2.3) and

$$\Gamma(g,g)(v) = \Omega^{-1/2}(v)\, Q(\Omega^{1/2} g, \Omega^{1/2} g)(v). \tag{4.2}$$

In order to study eq. (4.1), one has to do a thorough investigation of the linearized equation

$$\frac{\partial h}{\partial t} + v \cdot \nabla_x h = Lh. \tag{4.3}$$

When a good decay to 0 in some norm of the solution h of eq. (4.3) is obtained when the time goes to infinity, it is sometimes possible to prove the existence of a solution to eq. (4.1) when the corresponding norm of the initial datum is small enough. Such a solution will generally have the same behavior for large times as the solution of eq. (4.3).

We give various examples in which this program is carried out. The following assumption on the initial datum is always made:

$$\int_{x\in\mathcal{O}} \int_{v\in\mathbb{R}^3} g(0,x,v)\, \psi_i(v)\, dvdx = 0, \tag{4.4}$$

where $\mathcal{O}$ is the domain where the particles evolve.

4.2 The equation in a periodic box

The case of hard potentials with cut–off was studied by Ukaï in 1974 (Cf. [Uk]). He proved that if $\beta \ge 3/2$, $l \ge 1/2$, $\epsilon > 0$, there exists $c_0 > 0$ such that if the initial datum $g(0,x,v)$ satisfies

$$||g(0,x,v)||_{L^\infty((1+|v|^{\beta+1+\epsilon})dv;H^{l+1+\epsilon}(dx))} \le c_0, \tag{4.5}$$

then $\alpha, \mu > 0$ and a solution $g(t,x,v)$ to eq. (4.1) exist such that

$$\forall \gamma < \mu, \quad ||g(t,x,v)||_{L^\infty((1+|v|^{\beta+1+\epsilon})dv;H^{l+1+\epsilon}(dx))} \le \alpha e^{-\gamma t}. \tag{4.6}$$

Note that Shizuta proved a similar result in a paper appeared in 1983, but with a different norm (Cf. [Sh]).

Caflisch extended this result in 1980 (Cf. [Caf 2]) in the case of cut–off soft potentials with $3 < s < 5$ in the following way:

Suppose that $0 < \alpha < 1/4$. Then, there exists $c_0 > 0$ such that if the initial datum $g(0,x,v)$ satisfies

$$||g(0,x,v)||_{L^\infty(e^{\alpha v^2}dv;H^4(dx))} \le c_0, \tag{4.7}$$

then $C, \lambda > 0$ and a solution $g(t,x,v)$ to eq. (4.1) exist such that

$$||g(t,x,v)||_{L^2(dv;H^4(dx))} \le C\, e^{-\lambda t^{\frac{2s-2}{s-3}}}. \tag{4.8}$$

Note that in both cases, the same kind of decay is observed as in the linearized, homogeneous case.

Note also that Asano and Shizuta have obtained in 1977 (Cf. [As, Sh]) the same kind of exponential decay as Ukaï (but not with the same norm) with cut–off hard potentials and in a bounded convex C^3 domain, with positive principal curvatures, together with specular reflexion boundary conditions.

4.3 The equation in the whole space

The case of hard potentials with cut–off was treated by Imaï and Nishida in 1976 (Cf. [Im, Ni]).

They proved that if $l, m \geq 3$, there exists c_0 such that if the initial datum $g(0, x, v)$ satisfies

$$||g(0,x,v)||_{L^1(dx;L^2(dv))} + ||g(0,x,v)||_{L^\infty((1+|v|^m)dv;H^l(dx))} \leq c_0, \tag{4.9}$$

then $C > 0$ and a solution $g(t, x, v)$ to eq. (4.1) exist such that

$$||g(t,x,v)||_{L^\infty((1+|v|^m)dv;H^l(dx))} \leq \frac{C}{(1+t)^{3/4}}. \tag{4.10}$$

If moreover

$$||g(0,x,v)||_{L^1(xdx;L^2(dv))} < +\infty, \tag{4.11}$$

then there exists $D > 0$ such that

$$||g(t,x,v)||_{L^\infty((1+|v|^m)dv;H^l(dx))} \leq \frac{D}{(1+t)^{5/4}}. \tag{4.12}$$

Note also that assumption (4.4) is not necessary to obtain estimate (4.10). Finally, Ukaï and Asano considered in 1982 (Cf. [Uk, As]) the case of cut–off soft potentials with $7/3 < s < 5$.

Assume that $\delta = -\frac{s-5}{s-1}$, $n \geq 2$, $\alpha = \min\left(\frac{n}{2}(\frac{1}{p} - \frac{1}{2}); 1\right)$, $l \geq n/2$ and $\beta \geq n/2 - \delta$. Then there exists c_0 such that if the initial datum $g(0, x, v)$ satisfies

$$||g(0,x,v)||_{L^2(dv;L^p(dx))} + ||g(0,x,v)||_{L^\infty((1+|v|^{(\beta+\alpha\delta)})dv;H^l(dx))} \leq c_0, \tag{4.13}$$

then $C > 0$ and a solution $g(t, x, v)$ to equation (4.1) exist such that

$$||g(t,x,v)||_{L^\infty((1+|v|^{(\beta+\alpha\delta)})dv;H^l(dx))} \leq \frac{C}{(1+t)^{\alpha}}. \tag{4.14}$$

We can see that when x varies in the whole space $\mathbb{R}^3$, the decays are not as fast as in the linearized homogeneous case.

5 The Full Nonlinear Boltzmann Equation near the Vacuum in the Whole Space

A lot of work has been done on this subject (Cf. [Ka, Shi], [Il, Shi], [Be, To], [Ha 1]), and note that in this situation, the solution f of eq. (1.1) is known to go to 0 when the time becomes large. This is due to the fact that the particles leave any bounded domain after some time. However, it is also known that the function $\tilde{f}(t,x,v) = f(t, x+vt, v)$ converges to a finite limit $f_\infty(x,v)$. It was wondered wether or not this limit was necessarily a Maxwellian, since its associated entropy was proved to be nonincreasing. Toscani gave a negative answer to this question in 1988 (Cf. [To 2]), its proof is based on the fact that the limit f_∞ may not have the same decay when $v \to +\infty$ as a Maxwellian. Note also the computations by Pitteri (Cf. [Pi]).

6 Renormalized Solutions

We recall that DiPerna and Lions proved in 1989 (Cf. [DP, L 1]) the existence of a renormalized solution to eq. (1.1) for a large class of cross sections (including cut–off hard and soft potentials for $s > 2$), as soon as the initial datum has finite mass, energy, entropy and second moment in the x variable.

Hamdache extended this result to the case of a bounded domain with various boundary conditions (including specular and reverse reflexion) in 1990 (Cf. [Ha 2]).

Mass and momentum are known to be preserved for renormalized solutions, and the entropy is also known to be nonincreasing (Cf. [DP, L 2]), but the energy might decrease. Therefore one cannot hope to identify the Maxwellian limit (when it exists) of f. However, the same kind of theorems as in the case of the homogeneous equation with soft potentials can be given.

In the case when x varies in the whole space $\mathbb{R}^3$, the same kind of pointwise decay to 0 of f will be observed as in the case of the equation near vacuum. Therefore we shall concentrate on the equation in a bounded domain.

Arkeryd proved in 1988, using non–standard analysis (Cf. [Ar 6]), that in a periodic box, for every sequence t_n going to infinity, there exists a subsequence t_{n_k} and a global Maxwellian $m(v)$ such that $f_{n_k}(t,x,v) = f(t+t_{n_k},x,v)$ converges to $m(v)$ in $L^1([0,T]\times \mathcal{O} \times I\!R^3)$ weak.

This result was extended by Desvillettes in 1990 (Cf. [De 3]) by standard arguments in the case of a domain with reverse or specular reflexion boundary condition.

The reverse reflexion case is similar to the case of a periodic box, but in the case of specular reflexion, the Maxwellian m may depend on x when the domain has a symmetry of revolution. This is due to the existence in this case of another conservation, the conservation of the kinetic momentum.

Note that in these works, the dependance of $m(v)$ with respect to the initial data would be completely known if the conservation of energy was known to hold.

Finally, Arkeryd proved in 1991 that the above convergences held in fact in L^1 strong, using non–standard arguments (Cf [Ar. 7]). This result was also proved by standard techniques by Lions (Cf. [L]) in 1993, as a consequence of compactness properties of the positive part of the Boltzmann kernel Q.

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