A RIGOROUS DERIVATION OF A LINEAR KINETIC EQUATION OF FOKKER-PLANCK TYPE IN THE LIMIT OF GRAZING COLLISIONS

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ABSTRACT. We rigorously derive a linear kinetic equation of Fokker-Planck type for a 2–D Lorentz gas in which the obstacles are randomly distributed.

Each obstacle of the Lorentz gas generates a potential $\varepsilon^\alpha V(\frac{|x|}{\varepsilon})$, where V is a smooth radially symmetric function with compact support, and $\alpha > 0$. The density of obstacles diverges as $\varepsilon^{-\delta}$, where $\delta > 0$. We prove that when $0 < \alpha < 1/8$ and $\delta = 2\alpha + 1$, the probability density of a test particle converges as $\varepsilon \to 0$ to a solution of our kinetic equation.

1. INTRODUCTION

In this paper we address the problem of a rigorous derivation of a linear kinetic equation in the limit of grazing collisions, that is, when each collision changes only slightly the velocity of a particle.

We consider the behavior of a test particle under the action of a 2 - D random distribution of obstacles (also called scatterers). Given a small parameter $\varepsilon > 0$, the potential generated from a scatterer at a position $c \in \mathbb{R}^2$ is of the form:

$$\check{V}_\varepsilon(x-c) = \varepsilon^\alpha V(\frac{|x-c|}{\varepsilon}), \tag{1}$$

and, for the sake of simplicity, we shall assume that the unrescaled radial potential V is a smooth function with compact support.

The distribution of scatterers is a Poisson law of intensity $\mu_\varepsilon = \varepsilon^{-\delta}\mu$, where $\mu, \delta > 0$ are fixed.

The Boltzmann-Grad limit would consist in making $\delta = 1$, $\alpha = 0$ and letting $\varepsilon \to 0$. The limit would then lead to the solution of a linear Boltzmann equation (cf. [G], [Bo, Bu, Si], [De, Pu], [S1], [S2]). In order to get an equation of Fokker-Planck type, we propose a slightly different scaling, namely $\alpha > 0$, $\delta = 2\alpha + 1$. The fact that $\alpha > 0$ exactly means that we are in the limit of grazing collisions: the potential created by a scatterer being weak, the particle will not deviate very much from a straight trajectory. On the other hand, in order to get a finite effect at the end (we do not wish to get the solution of the free transport equation), the density of scatterers has to grow faster than in the Boltzmann–Grad limit when $\varepsilon \to 0$. This explains why $\delta > 1$. The extra technical assumption that $\alpha < 1/8$ allows us to rigorously prove the convergence toward the solution of a linear kinetic equation of Fokker-Planck type of the test particle probability density in the phase space.

The same problem for $\alpha = 1/2$ was studied in [Du, Go, Le], where the convergence is obtained by proving compactness of the family of measures associated to the stochastic processes describing the motion of the light particle for $\varepsilon > 0$. Here we use different techniques, related to those developed in [G] to prove the validity of the linear Boltzmann equation. Notice that we are allowed to use these techniques after choosing a value for α such that the ratio between the mean free path and the size of the obstacles diverges (for this we need in general $\alpha < 1/2$), whereas in [Du, Go, Le] this ratio is constant. We are then in a low density limit with respect to [Du, Go, Le].

As for the case of the long-range potentials considered in [De, Pu], it does not seem possible to directly apply the techniques of [G], because of the lack of a semi–explicit form of the solution of the limit equation. Therefore, we produce an explicit estimate of the non–Markovian component of the distribution density, and use a semi–explicit form of the solutions of a family of Boltzmann equations with a cross section concentrating on grazing collisions.

Note also that in a forthcoming paper (Cf. [Pou, Va]), Poupaud and Vasseur propose for closely related problems a different approach consisting in passing to the limit directly in the equation, and not in a semi–explicit form of its solution.

Note finally that for the nonlinear Fokker–Planck equation (also called Landau equation) (Cf. [Lif, Pi], [De, Vi]), no rigorous derivation from an N-particle system exists, even in the framework of local in time solutions, whereas such a result exists in the case of the Boltzmann equation (Cf. [Lanf], [Ce, Il, Pu]).

In section 2, we present our notations and our main theorem. Sections 3 and 4 are devoted to its proof. More precisely, in section 3, a single grazing collision is studied, while in section 4 the collective effect of collisions is taken into account.

The same technique can be applied in dimension d bigger than two, where $\delta = 2\alpha + d - 1$, by simply putting a little bit more effort in evaluating the bound on the probability of recollisions, due to the fact that now the trajectories don't lie in general on a plane. In this case, convergence is obtained for $\alpha < 1/4$, the upper bound for α being fixed by the requirements that the probabilty of overlappings of two obstacles met by the particle trajectory is negligeable in the limit.

2. Notations and results

In the sequel we shall denote by $B(x,R) = \{y \in \mathbb{R}^2 / \, |x-y| < R\}$ the open disk of center x and radius R, by C any positive constant (possibly depending on the fixed parameters, but independent of ε), and by $\varphi = \varphi(\varepsilon)$ any nonnegative function vanishing when $\varepsilon \to 0$.

We fix an arbitrary time $T > 0$ and consider our dynamical problem for times t such that $0 \leq t \leq T$.

We use a Poisson repartition of fixed scatterers in $\mathbb{R}^2$ of parameter $\mu_\varepsilon = \varepsilon^{-\delta}\mu$, where $\mu, \delta > 0$ are fixed and $\varepsilon \in]0,1]$. The probability distribution of finding exactly N obstacles in a bounded (or more generally of finite measure) measurable set $\Lambda \subset \mathbb{R}^2$ is given by:

$$P(d\mathbf{c}_N) = e^{-\mu_\varepsilon |\Lambda|} \frac{\mu_\varepsilon^N}{N!} dc_1 \ldots dc_N, \tag{2}$$

where $c_1 \ldots c_N = \mathbf{c}_N$ are the positions of the scatterers and $|\Lambda|$ denotes the Lebesgue measure of Λ.

The expectation with respect to the Poisson repartition of parameter μ_ε will be denoted by $\mathbb{E}^\varepsilon$.

We now introduce a radial potential V (here, V will at the same time denote the function of two variables (x_1, x_2) and the function of the radial variable $r = \sqrt{x_1^2 + x_2^2}$, since no confusion can occur) such that:

1. $V \in C^2(\mathbb{R}^2)$;
2. $V(0) > 0$ and $r \to V(r)$ is strictly decreasing in $[0,1]$;
3. $\mathrm{supp} V \subset [0,1]$.

Then, we consider the Hamiltonian flow $T^t_{\mathbf{c},\varepsilon}$ (or more simply $T^t_{\mathbf{c}}$ when no confusion can occur) generated by the distribution of obstacles $\mathbf{c}$ and associated with the potential $\check{V}_\varepsilon$ given in (1), that is $T^t_{\mathbf{c},\varepsilon}(x,v) = (x_{\mathbf{c}}(t), v_{\mathbf{c}}(t))$, where $x_{\mathbf{c}}(t), v_{\mathbf{c}}(t)$ satisfy the Newtonian law of motion:

$$\dot{x}_{\mathbf{c}}(t) = v_{\mathbf{c}}(t), \tag{3}$$

$$\dot{v}_{\mathbf{c}}(t) = -\varepsilon^{\alpha-1} \sum_{c \in \mathbf{c}} \nabla V\left(\frac{|x-c|}{\varepsilon}\right), \tag{4}$$

$$x_{\mathbf{c}}(0) = x, \qquad v_{\mathbf{c}}(0) = v. \tag{5}$$

As discussed for example in [De, Pu], the quantity $T^t_{\mathbf{c},\varepsilon}(x,v)$ is well defined for all $t \in \mathbb{R}, x \in \mathbb{R}^2, v \in S^1$, except maybe when c belongs to a negligeable set with respect to the Poisson repartition.

For a given initial datum $f_{in} \in L^1 \cap L^\infty \cap C(\mathbb{R}^2 \times \mathbb{R}^2)$, we can define the following expectation:

$$f_\varepsilon(t,x,v) = \mathbb{E}^\varepsilon[f_{in}(T^{-t}_{\mathbf{c},\varepsilon}(x,v))]. \tag{6}$$

The main result is then the following:

Theorem 1. *Let $\alpha \in]0,1/8[$ and $\delta = 2\alpha+1$, f_{in} be an initial datum belonging to $L^1 \cap W^{1,\infty}(\mathbb{R}^2 \times \mathbb{R}^2)$ and V be a potential satisfying 1., 2., 3. Then, for any $T>0$, the quantity f_ε defined by (3) – (6) converges when $\varepsilon \to 0$ to h in $C([0,T]; W^{-2,1}_{loc}(\mathbb{R}^2 \times S^1))$, where h is the (unique) weak solution of the following linear equation of Fokker-Planck type:*

$$\begin{aligned} (\partial_t + v\cdot\nabla_x)h(t,x,v) &= \zeta\,\triangle_v h(t,x,v) \\ h(0,x,v) &= f_{in}(x,v). \end{aligned} \tag{7}$$

In (7), $\triangle_v$ is the Laplace-Beltrami operator on S^1 (that is, if $\bar f(\theta) = f(\cos\theta,\sin\theta)$, then $\triangle_v f(\cos\theta,\sin\theta) = \bar f''(\theta)$), and

$$\zeta = \frac{\mu}{2}\int_{-1}^{1}\left(\int_\rho^1 \frac{\rho}{u}V'(\frac{\rho}{u})\frac{du}{\sqrt{1-u^2}}\right)^2 d\rho. \tag{8}$$

Note that since $r \to r\,V'(r)$ is bounded, we have $\zeta < +\infty$. We also obviously have $\zeta > 0$ under our assumptions on μ and V.

The remaining part of this work will be devoted to the proof of theorem 1.

3. Study of grazing collisions

This part is devoted to the proof of the following proposition, which explains the asymptotic behavior of the scattering angle as a function of the impact parameter in the limit when the potential is rescaled as $V \to \varepsilon^\alpha V$ with $\varepsilon \to 0, \alpha > 0$.

Proposition 1. *Consider the deflection angle $\theta_1(\rho,\varepsilon)$ of a particle with impact parameter ρ due to a scatterer generating a radial potential $\varepsilon^\alpha V$, where $\alpha > 0$ and V satisfies assumptions 1.,2.,3. Then, the following asymptotic formula holds:*

$$\theta_1(\rho,\varepsilon) = -2\,\varepsilon^\alpha \int_\rho^1 \frac{\rho}{w}\,V'\left(\frac{\rho}{w}\right)\frac{dw}{\sqrt{1-w^2}} + O(\varepsilon^{2\alpha}),$$

where the $O(\varepsilon^{2\alpha})$ is uniform in ρ (when $\rho \in [-1,1]$).

Proof of Proposition 1:

Note that for $\varepsilon > 0$ small enough,

$$\varepsilon^\alpha V(0) < \frac{1}{2}. \tag{9}$$

Therefore, the deflection angle is given (when $\rho > 0$) by the classical formula:

$$\theta_1(\rho,\varepsilon) = \pi - 2\int_{r_{min}(\rho,\varepsilon)}^{+\infty} \frac{\rho}{\sqrt{1-\frac{\rho^2}{r^2} - 2\,\varepsilon^\alpha\, V(r)}}\frac{dr}{r^2}$$

$$= \pi - 2\int_0^{\frac{\rho}{r_{\min}(\rho,\varepsilon)}} \frac{dw}{\sqrt{1-w^2-2\varepsilon^\alpha V(\frac{\rho}{w})}}, \tag{10}$$

where $w = \frac{\rho}{r}$ and $r_{\min}(\rho,\varepsilon)$ is implicitly defined by

$$\frac{1}{2}\frac{\rho^2}{r_{\min}^2(\rho,\varepsilon)} + \varepsilon^\alpha V(r_{\min}(\rho,\varepsilon)) = \frac{1}{2}. \tag{11}$$

We denote by K a constant related to the two first derivatives of V:

$$K = \sup_{r\in[0,1]} \Big(|V(r)| + r\,|V'(r)| + r^2\,|V''(r)| \Big),$$

and we consider only parameters $\varepsilon > 0$ which are such that

$$2\,\varepsilon^\alpha\, K < 1/2. \tag{12}$$

Then, we can perform the change of variables

$$\frac{w}{\sqrt{1-2\varepsilon^\alpha V(\frac{\rho}{w})}} = u, \tag{13}$$

so that

$$du = \frac{1}{\sqrt{1-2\varepsilon^\alpha V(\frac{\rho}{w})}}\left[1 - \frac{\varepsilon^\alpha \frac{\rho}{w} V'(\frac{\rho}{w})}{1-2\varepsilon^\alpha V(\frac{\rho}{w})}\right] dw. \tag{14}$$

We obtain for the deflection angle

$$\theta_1(\rho,\varepsilon) = \pi - 2\int_0^1 \frac{1}{1-\frac{\varepsilon^\alpha\,\frac{\rho}{w}\,V'(\frac{\rho}{w})}{1-2\,\varepsilon^\alpha\,V(\frac{\rho}{w})}}\frac{du}{\sqrt{1-u^2}}$$

$$= 2\int_\rho^1 \left(1 - \frac{1-2\varepsilon^\alpha V(\frac{\rho}{w})}{1-\varepsilon^\alpha\{2V(\frac{\rho}{w}) + \frac{\rho}{w}V'(\frac{\rho}{w})\}}\right)\frac{du}{\sqrt{1-u^2}} \tag{15}$$

(remember that $V(\frac{\rho}{w}) = 0$ for $\rho > w$ (or $\rho > u$)).

Using the identity

$$\frac{1}{1-x} = 1 + \frac{x}{1-x},$$

and (12), we see that

$$\theta_1(\rho,\varepsilon) = -2\,\varepsilon^\alpha \int_\rho^1 \frac{\rho}{w} V'(\frac{\rho}{w}) \frac{du}{\sqrt{1-u^2}} + \varepsilon^{2\alpha}\, L(\rho,\varepsilon), \tag{16}$$

with

$$|L(\rho,\varepsilon)| \le 6\,\pi\, K^2.$$

Moreover, assumption (12) also ensures that

$$|w-u| \le 2\,\varepsilon^\alpha\, V(\frac{\rho}{w})\,|w|.$$

Then, using the fact that $u > w$, we get

$$\left|\frac{\rho}{w}V'(\frac{\rho}{w}) - \frac{\rho}{u}V'(\frac{\rho}{u})\right| \le |w-u| \sup_{r\in[w,u]} \left|\frac{\rho}{r^2}\,V'(\frac{\rho}{r}) + \frac{\rho^2}{r^3}\,V''(\frac{\rho}{r})\right|$$

$$\le 2\,\varepsilon^\alpha\, K\,|w|\,K \sup_{r\in[w,u]} (1/r)$$

$$\le 2\,K^2\,\varepsilon^\alpha.$$

Finally, we can write

$$\theta_1(\rho,\varepsilon) = -2\,\varepsilon^\alpha \int_\rho^1 \frac{\rho}{u} V'(\frac{\rho}{u}) \frac{du}{\sqrt{1-u^2}} + \varepsilon^{2\alpha}\, M(\rho,\varepsilon), \tag{17}$$

with

$$|M(\rho,\varepsilon)| \le 6\,\pi\,K^2 + 4\,K^2 \int_\rho^1 \frac{du}{\sqrt{1-u^2}}$$

$$\le 8\,\pi\,K^2,$$

which ends the proof of the proposition when $\rho > 0$. We conclude by noticing that θ_1 is an even function, so that the estimate also holds when $\rho < 0$.

Corollary 1. *Let V be a radial potential satisfying assumptions 1.,2.,3. Then the scattering cross section Ψ_ε associated with $\check{V}_\varepsilon (= \varepsilon^\alpha\, V(\frac{|\cdot|}{\varepsilon}))$ lies in $L^\infty([-\pi,\pi])$ (for a given $\varepsilon > 0$) and verifies*

$$\forall \theta_0 > 0, \exists \varepsilon_0(\theta_0) > 0, \forall \varepsilon \in [0,\varepsilon_0(\theta_0)], \qquad \Psi_\varepsilon([\theta_0,\pi]) = 0, \tag{18}$$

$$\varepsilon^{-1-2\,\alpha} \lim_{\varepsilon\to 0} \frac{\mu}{2} \int_{-\pi}^{\pi} \theta^2 \Psi_\varepsilon(\theta)\,d\theta = \zeta, \tag{19}$$

with ζ defined by (8).

Proof of corollary 1: We recall that Ψ_ε is defined by the formula

$$\Psi_\varepsilon(\theta) = \frac{d\rho}{d\theta}(\theta), \qquad \text{if} \qquad |\theta| \leq \theta_{max},$$
$$0 \qquad \text{if} \qquad |\theta| > \theta_{max},$$

where the deflection angle θ corresponds to the impact parameter ρ, the potential being $\check{V}_\varepsilon$, and θ_{max} is the largest possible angle of deflection. Note that θ is a decreasing function of ρ, so that ρ is also a decreasing function of θ, and $\frac{d\rho}{d\theta}$ is well defined.

Then, it is easy to see that

$$\Psi_\varepsilon(\theta) = \varepsilon\, \Phi_\varepsilon(\theta),$$

where Φ_ε is the scattering cross section associated with the potential $\varepsilon^\alpha V$ (Cf. [De, Pu] for example).

Note first that according to proposition 1,

$$\theta_1(\rho,\varepsilon) \leq \pi\, \varepsilon^\alpha \sup_{r\in[0,1]} |r\, V'(r)| + C\, \varepsilon^{2\alpha},$$

with C independant of ρ, so that $\theta_{max} \leq C'\, \varepsilon^\alpha$, and (18) clearly holds.

Moreover,

$$\begin{aligned}\frac{\mu}{2}\int_{-\pi}^{\pi} \theta^2 \Psi_\varepsilon(\theta) d\theta &= \varepsilon\, \frac{\mu}{2}\int_{-\pi}^{\pi} \theta^2 \Phi_\varepsilon(\theta)\, d\theta \\ &= \varepsilon \frac{\mu}{2}\int_{-1}^{1} \theta_1(\rho,\varepsilon)^2\, d\rho \\ &= \varepsilon^{1+2\alpha}\zeta + O(\varepsilon^{1+3\alpha}),\end{aligned}$$

which ends the proof of corollary 1.

4. Proof of theorem 1

In order to study the asymptotic behavior of f_ε when $\varepsilon \to 0$, we are led to compare f_ε to the solution h_ε of the following Boltzmann equation:

$$(\partial_t + v\cdot\nabla_x) h_\varepsilon(t,x,v) \;=\; \mu \int_{\theta=-\pi}^{\pi} \Gamma_\varepsilon(|\theta|) \Big\{ h_\varepsilon(t,x,R_\theta(v)) - h_\varepsilon(t,x,v) \Big\}\, d\theta,$$
$$(20) \qquad h_\varepsilon(0,x,v) \;=\; f_{in}(x,v).$$

Here, R_θ denotes the rotation of angle θ and $\Gamma_\varepsilon = \varepsilon^{-1-2\alpha}\, \Psi_\varepsilon$, where Ψ_ε is defined in corollary 1.

It is clear thanks to corollary 1 that Γ_ε is a family of functions satisfying

$$(21) \qquad \forall \theta_0 > 0, \qquad \lim_{\varepsilon\to 0} \int_{\theta_0 < |\theta| < \pi} \Gamma_\varepsilon(\theta)\, d\theta = 0,$$

$$\lim_{\varepsilon \to 0} \frac{\mu}{2} \int_{-\pi}^{\pi} \theta^2 \, \Gamma_\varepsilon(\theta) \, d\theta = \zeta. \tag{22}$$

Such a family of cross sections is said (usually in a nonlinear context) to "concentrate on grazing collisions" (Cf. [Vil]).

Formally, we can easily derive (7) from (20) by observing that condition (21) allows us to consider only small rotation angles in the integral. Then we can perform a Taylor's expansion of $h_\varepsilon(t, x, R_\theta(v))$ with respect to the last argument

$$\begin{aligned} h_\varepsilon(t, x, R_\theta(v)) &= h_\varepsilon(t, x, v) + (R_\theta(v) - v) \cdot \nabla_v h_\varepsilon(t, x, v) \\ &+ \frac{1}{2}(R_\theta(v) - v) \otimes (R_\theta(v) - v) : \nabla_v \nabla_v h_\varepsilon(t, x, v) + O(\|R_\theta(v) - v\|^3) \end{aligned}$$

and, by inserting this expression in the right–hand side of (20), we obtain

$$\begin{aligned} &\mu \int_{\theta=-\pi}^{\pi} \Gamma_\varepsilon(|\theta|) \left\{ h_\varepsilon(t, x, R_\theta(v)) - h_\varepsilon(t, x, v) \right\} d\theta \\ &\quad = \mu \frac{\triangle_v h_\varepsilon}{2} \int_{\theta=-\pi}^{\pi} \Gamma_\varepsilon(|\theta|) \theta^2 d\theta + \phi(\epsilon). \end{aligned}$$

which in the limit $\epsilon \to 0$ is the right–hand side of (7).

This computation can be made rigorous without difficulty. It yields the

Proposition 2. *Suppose that f_{in} is a nonnegative initial datum lying in $L^2(\mathbb{R}^2 \times S^1)$ and that for all $\varepsilon > 0$, the cross section Γ_ε belongs to $L^\infty([0, \pi])$. Then there exists a unique weak solution h_ε to (20) in $C([0, T]; L^2(\mathbb{R}^2 \times S^1))$. If moreover the family Γ_ε satisfies (21), (22), then the sequence h_ε converges when $\varepsilon \to 0$ in (for example) $C([0, T]; W^{-2,1}_{loc}(\mathbb{R}^2 \times S^1))$ towards h weak solution of (7).*

Therefore, in order to prove our main theorem (theorem 1), it is enough to show that h_ε and f_ε are close when $\varepsilon \to 0$ (in a topology at least as strong as that of $W^{-2,1}_{loc}$). Accordingly, the remaining part of this work is devoted to the proof of the following proposition:

Proposition 3. *Assume that $\alpha \in]0, 1/8[$ and $\delta = 2\alpha + 1$. Let the initial datum f_{in} belong to $L^1 \cap W^{1,\infty}(\mathbb{R}^2 \times \mathbb{R}^2)$ and V be a potential satisfying 1., 2., 3. Then, the function f_ε defined in (6) and h_ε in (20) are asymptotically close in L^1_{loc}. More precisely, for all $R > 0$,*

$$\lim_{\varepsilon \to 0} ||f_\varepsilon - h_\varepsilon||_{L^\infty([0,T]; L^1(B(0,R) \times S^1))} = 0.$$

Proof of proposition 3: We define

$$\chi_1(\mathbf{c}_N) = \chi(\Big\{\mathbf{c}_N \in B(x)^N, \quad \forall i = 1 \dots N, \quad |c_i - x| > \varepsilon\Big\}), \tag{23}$$

that is $\chi_1 = 1$ if the particle is outside the range of all scatterers at time 0. When $\chi_1 = 1$, the conservation of energy entails that the velocity of the particle will always be less than 1, so that only the scatterers at distance less than t can influence the trajectory of the particle up to time t.

Noticing that as soon as $\alpha < 1/2$ (i.-e. $\delta < 2$),

$$\mathbb{E}^\varepsilon(\chi_1) \geq 1 - \varphi(\varepsilon),$$

(that is, we are in a situation in which, asymptotically, the particle is initially almost surely outside of the range of all the scatterers) we see that f_ε can be expanded as:

(24)
$$f_\varepsilon(t,x,v) = e^{-\mu_\varepsilon |B(x,t)|} \sum_{N \geq 0} \frac{\mu_\varepsilon^N}{N!} \int_{B(x)^N} d\mathbf{c}_N \chi_1(\mathbf{c}_N) f_{in}(T^{-t}_{\mathbf{c}_N}(x,v)) + \varphi(\varepsilon).$$

We can distinguish between *external obstacles*, $c \in \mathbf{c} \cap B(x,t)$ such that

$$\inf_{0 \leq s \leq t} |x_{\mathbf{c}}(s) - c| \geq \varepsilon, \tag{25}$$

and *internal obstacles*, $c \in \mathbf{c} \cap B(x,t)$ such that

$$\inf_{0 \leq s \leq t} |x_{\mathbf{c}}(s) - c| < \varepsilon. \tag{26}$$

A given configuration $\mathbf{c}_N$ of $B(x,t)^N$ can be decomposed as:

$$\mathbf{c}_N = \mathbf{a}_P \cup \mathbf{b}_Q,$$

where $\mathbf{a}_P$ is the set of all external obstacles and $\mathbf{b}_Q$ is the set of all internal ones.

After suitable manipulations, and recalling that the external scatterers do not influence the trajectory, we have in fact

$$f_\varepsilon(t,x,v) = \sum_{Q \geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{B(x)^Q} d\mathbf{b}_Q e^{-\mu_\varepsilon |\mathcal{T}(\mathbf{b}_Q)|} \chi_1(\mathbf{b}_Q)$$
$$\times \chi(\Big\{ \text{ the } \mathbf{b}_Q \text{ are internal} \Big\}) f_{in}(T^{-t}_{\mathbf{b}_Q}(x,v)) + \varphi(\varepsilon),$$

where $\mathcal{T}(\mathbf{b}_Q)$ is the tube (at time t) defined by

$$\mathcal{T}(\mathbf{b}_Q) = \Big\{ y \in B(x,t), \quad \exists s \in [0,t], \quad |y - x_{\mathbf{b}_Q}(s)| < \varepsilon \Big\}. \tag{27}$$

Since the velocity of the particle is always less than 1, one has

$$|\mathcal{T}(\mathbf{b}_Q)| \le 2\,t\,\varepsilon. \tag{28}$$

We then introduce the characteristic function χ_2 of distributions of scatterers for which there is no overlapping of internal scatterers, that is

(29)

$$\chi_2(\mathbf{b}_Q) = \chi(\left\{ \mathbf{b_Q} \in B(x)^Q, \quad \forall\, 1 \le i < j \le Q, \quad |b_i - b_j| > 2\,\varepsilon \right\}).$$

It is then easy to prove (Cf. [De, Pu]) that if $\alpha < 1/4$ (i.-e. $\delta < \frac{3}{2}$), one has

(30)

$$\sum_{Q \ge 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{B(x)^Q} e^{-\mu_\varepsilon |\mathcal{T}(\mathbf{b}_Q)|} \chi(\left\{ \mathbf{b}_Q \subset \mathcal{T}(\mathbf{b}_Q) \right\})\, \chi_1\chi_2(\mathbf{b}_Q)\, d\mathbf{b}_Q \ge 1 - \varphi(\varepsilon).$$

Note however that the probability of overlapping of a pair of not necessarily internal obstacles is asymptotically 1 even for $\alpha = 0$ (i.-e. $\delta = 1$).

Then,

$$f_\varepsilon(t,x,v) = \sum_{Q \ge 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{B(x)^Q} d\mathbf{b}_Q\, e^{-\mu_\varepsilon |\mathcal{T}(\mathbf{b}_Q)|} \chi_1(\mathbf{b}_Q)\chi_2(\mathbf{b}_Q)$$

$$\times \chi(\left\{ \text{ the } \mathbf{b}_Q \text{ are internal} \right\})\, f_{in}(T^{-t}_{\mathbf{b}_Q}(x,v)) + \varphi(\varepsilon).$$

From now on, we shall replace for the sake of simplicity the flow $T^{-t}_{\mathbf{b}_Q}$ by the flow $T^{t}_{\mathbf{b}_Q}$. The result will be the same thanks to the reversibility of this Hamiltonian flow.

Remark Notice that the bound $\alpha < 1/4$ doesn't depend on the dimension. As we will see, this will fix the bound on α in dimension higher than 2.

For a given configuration $\mathbf{b}_Q \in B(x)^Q$ such that $\chi_1\chi_2(\mathbf{b}_Q) = 1$ and such that the b_i's are internal for $i = 1 \dots Q$, we define the characteristic function χ_3 of the set of configurations for which there is no recollisions (up to time t) of the light particle with a given obstacle:

(31)

$$\chi_3(\mathbf{b}_Q) = \chi(\left\{ \mathbf{b}_Q, \quad \forall i = 1 \dots Q, \quad x^{-1}_{\mathbf{b}_Q}(B(b_i,\varepsilon)) \text{ is connected in } [0,t] \right\}).$$

Instead of f_ε, we first analyse $\tilde f_\varepsilon$, defined by

$$\begin{aligned}\tilde f_\varepsilon(t,x,v) &= e^{-2\,t\,\mu_\varepsilon\,\varepsilon}\sum_{Q\geq 0}\frac{\mu_\varepsilon^Q}{Q!}\int_{B(x)^Q}\chi(\Big\{\mathbf{b}_Q\subset\mathcal{T}(\mathbf{b}_Q)\Big\}) \\ &\times\quad \chi_1\chi_2\chi_3(\mathbf{b}_Q)\,f_0(T^t_{\mathbf{b}_Q}(x,v))d\mathbf{b}_Q. \qquad (32)\end{aligned}$$

Note that thanks to (28), we already know that

$$\tilde f_\varepsilon \leq f_\varepsilon + \varphi(\varepsilon). \qquad (33)$$

We now proceed as in [De, Pu].

We say that the light particle performs a collision with the scatterer b_i when it enters into its protection disk $B(b_i,\varepsilon)$. For a configuration such that $\chi_1\chi_2\chi_3 = 1$, the light particle has a straight trajectory between two separated collisions with different scatterers. During the collision with the obstacle b_i (i.-e. for the times t such that $|x_{\mathbf{b}_Q}(t) - b_i| \leq \varepsilon$), the dynamics is that of a particle moving in the potential $\check V_\varepsilon(\cdot - b_i)$.

For a trajectory corresponding to a configuration such that $\chi_1\chi_2\chi_3 = 1$, one can define, for each obstacle $b_i \in \mathbf{b}_Q$ $(i = 1\ldots Q)$, the time t_i of the first (and unique because $\chi_3 = 1$) entrance in the protection disk $B(b_i,\varepsilon)$, and the (unique) time $t_i' > t_i$ when the light particle gets out of this protection disk. We also define the impact parameter ρ_i, which is the algebraic distance between b_i and the straight line containing the straight trajectory followed by the light particle immediately before t_i.

Then we use the change of variables (which depends upon t,x,v,ε)

$$\mathcal{Z}:\mathbf{b}_Q\to\{\rho_i,t_i\}_{i=1}^Q(\mathbf{b}_Q)$$

which is well–defined on the set $\Gamma\subset B(x)^Q$ of "well–ordered" configurations $\mathbf{b}_Q$ constituted of internal scatterers satisfying the property $\chi_1\chi_2\chi_3(\mathbf{b}_Q) = 1$.

The variables $\{\rho_i,t_i\}_{i=1}^Q$ satisfy then the constraints

$$0\leq t_1 < t_2 < \cdots < t_Q \leq t, \qquad (34)$$

and

$$\forall i = 1,..,Q, \qquad |\rho_i| < \varepsilon. \qquad (35)$$

The inverse mapping $\mathcal{Z}^{-1}$ is built as follows: Let a sequence $\{\rho_i,t_i\}_{i=1}^Q$ satisfying (34) and (35) be given. We build a corresponding sequence of obstacles $\beta_{\mathbf{Q}} = \beta_1\,..\,\beta_Q$ and a trajectory $(\xi(s),v(s))$ inductively. Suppose that one has been able to define the obstacles $\beta_1\,..\,\beta_{i-1}$ and a trajectory $(\xi(s),v(s))$ up to the time t_{i-1}. We then define the trajectory between times t_{i-1} and t_i as that of the evolution of a particle moving in the potential $\check{V}_\varepsilon(\cdot - \beta_{i-1})$ with initial datum at time t_{i-1} given by $(\xi(t_{i-1}), v(t_{i-1}))$. Then, $\tau'_{i-1} > t_{i-1}$ is defined to be the first time of exit of the trajectory from the protection disk of β_{i-1}. Finally β_i is defined to be the only point at distance ε of $\xi(t_i)$ and algebraic distance ρ_i from the straight line which is tangent to the trajectory at the point $\xi(t_i)$.

Then it is easy to describe the range of $\mathcal{Z}$. The $\{\rho_i, t_i\}_{i=1}^Q$ which do not belong to this range correspond to at least one of those situations:

1. A bad beginning occurs:
$$\exists i = 1, .., Q, \qquad \xi(0) \in B(\beta_i, \varepsilon)$$
(this corresponds to $\chi_1 = 0$),
2. two scatterers overlap:
$$\exists i, j \in [1, .., Q], \qquad |\beta_i - \beta_j| \le 2\,\varepsilon$$
(this corresponds to $\chi_2 = 0$),
3. a "recollision" happens somewhere:
$$\exists i \ne j \in [1, .., Q], \qquad \beta_j \in \cup_{s \in]t_i, t_{i+1}[} B(\xi(s), 2\varepsilon)$$
(this corresponds to $\chi_3 = 0$ and is in its turn splits into the cases when $i > j$, *proper recollisions*, and when $i < j$, sometimes called *interferences*).

Performing the described change of variable, we get

$$\tilde{f}_\varepsilon(t,x,v) = e^{-2\,t\,\mu_\varepsilon\,\varepsilon} \sum_{Q\ge 0} \mu_\varepsilon^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q \int_{-\varepsilon}^{\varepsilon} d\rho_1 \int_{-\varepsilon}^{\varepsilon} d\rho_2 \cdots \int_{-\varepsilon}^{\varepsilon} d\rho_Q$$

$$\chi\Big(\{\rho_i, t_i\}_{i=1}^Q \text{ is in the range of } \mathcal{Z}\Big) f_0(\xi(t), v(t)) + \varphi(\varepsilon). \tag{36}$$

We now introduce the

Lemma 1. *As soon as $\alpha < 1/8$ (i.-e. $\delta < 5/4$), one has*

$$I_\varepsilon = e^{-2\,t\,\mu_\varepsilon\,\varepsilon} \sum_{Q\ge 0} \mu_\varepsilon^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q \int_{-\varepsilon}^{\varepsilon} d\rho_1 \int_{-\varepsilon}^{\varepsilon} d\rho_2 \cdots \int_{-\varepsilon}^{\varepsilon} d\rho_Q$$

$$\chi\Big(\{\rho_i, t_i\}_{i=1}^Q \textit{ is not in the range of } \mathcal{Z}\Big) \le \varphi(\varepsilon). \tag{37}$$

Proof of Lemma 1: We can write

$$I_\varepsilon \le I^1_\varepsilon + I^2_\varepsilon + I^3_\varepsilon,$$

where each term corresponds to the situations described earlier. Then, as in [De, Pu], we notice that

$$I^1_\varepsilon + I^2_\varepsilon + I^3_\varepsilon \le J^i_\varepsilon + J^{ii}_\varepsilon,$$

where J^i_ε estimates the probability of overlapping of two successive scatterers β_i, β_{i+1} (including the beginning of the trajectory, with the convention $t_0 = 0$, $\theta_0 = 0$, $x = \beta_0$), and J^{ii}_ε estimates the probability of other possible overlappings and recollisions.

We begin with the estimate on J^i_ε:

$$J^i_\varepsilon = e^{-2\,t\,\mu_\varepsilon\,\varepsilon} \sum_{Q\ge 1} \mu_\varepsilon^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q$$

$$\int_{-\varepsilon}^{\varepsilon} d\rho_1 \int_{-\varepsilon}^{\varepsilon} d\rho_2 \cdots \int_{-\varepsilon}^{\varepsilon} d\rho_Q \sum_{i=0}^{Q-1} \chi(\Big\{ |\beta_i - \beta_{i+1}| \le 2\,\varepsilon \Big\})$$

$$\le C\,\varepsilon^{5-2\delta}. \tag{38}$$

Then, we turn to J^{ii}_ε:

$$J^{ii}_\varepsilon = J^{ii}_{1,\varepsilon} + J^{ii}_{2,\varepsilon} = e^{-2\,t\,\mu_\varepsilon\,\varepsilon} \textstyle\sum_{Q\ge 1} \mu_\varepsilon^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q$$

$$\textstyle\int_{-\varepsilon}^{\varepsilon} d\rho_1 \int_{-\varepsilon}^{\varepsilon} d\rho_2 \cdots \int_{-\varepsilon}^{\varepsilon} d\rho_Q \Big[\sum_{i=0}^{Q-1} \sum_{j=i+2}^{Q} \chi(\Big\{ \beta_j \in \cup_{s\in]t_i,t_{i+1}[} B(\xi(s), 2\,\varepsilon) \Big\}$$

$$+ \textstyle\sum_{i=2}^{Q} \sum_{j=1}^{i-1} \chi(\Big\{ \beta_j \in \cup_{s\in]t_i,t_{i+1}[} B(\xi(s), 2\,\varepsilon) \Big\}) \Big]. \tag{39}$$

We only estimate $J^{ii}_{1,\varepsilon}$, the estimate of $J^{ii}_{2,\varepsilon}$ being completely analogous.

Note first that, denoting as usual by θ_i the scattering angle corresponding to the impact parameter ρ_i, a recollision (or overlapping of non consecutive scatterers) can occur only if the rotation angle $|\sum_{k=i+1}^{j-1} \theta_k|$ is bigger than π. Since we know moreover that for all $k \in]i+1, j-1[$, $|\theta_k| \le C\varepsilon^\alpha$, it means that we can find $h \in]i+1, j-1[$ such that

$$|\pi/2 - \sum_{k=i+1}^{h-1} \theta_k| \le \pi/4.$$

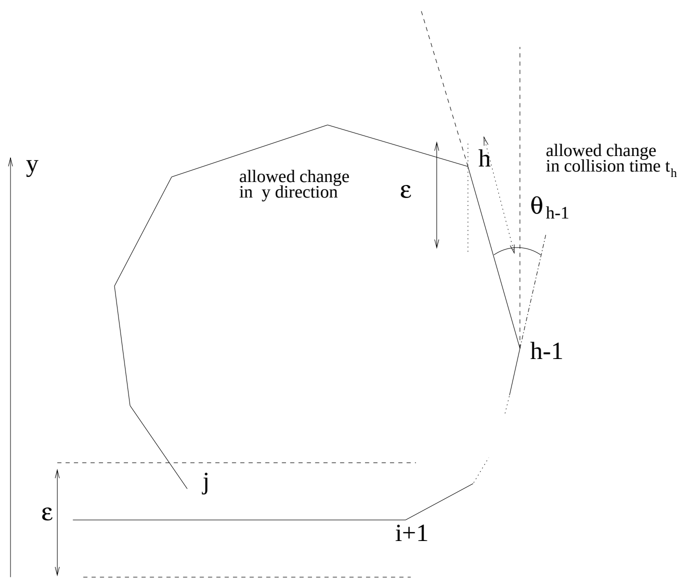

FIGURE 1

Then, we can write

$$J^{ii}_{1,\varepsilon} \le e^{-2\,t\,\mu_\varepsilon\,\varepsilon} \sum_{Q\ge 1} \mu_\varepsilon^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q \int_{-\varepsilon}^{\varepsilon} d\rho_1 \int_{-\varepsilon}^{\varepsilon} d\rho_2 \cdots \int_{-\varepsilon}^{\varepsilon} d\rho_Q$$

$$\sum_{i=0}^{Q-1} \sum_{j=i+2}^{Q} \sum_{h=i+1}^{j} \chi(\left\{ |\theta_{i+1} + \cdots + \theta_{h-1} - \pi/2| \le \pi/4 \right\})$$

$$\times \chi(\left\{ \beta_j \in \cup_{s\in]t_i,t_{i+1}[} B(\xi(s), 2\,\varepsilon) \right\}).$$

Fixing all times but t_h in the sequence $t_1, \ldots, t_Q$, and noticing that t_h can assume values in a set of measure at most $4\,\sqrt{2}\,\varepsilon$ (see fig. 1), we finally get:

$$J^{ii}_{1,\varepsilon} \le e^{-2\,t\,\mu_\varepsilon\,\varepsilon} \sum_{Q\ge 1} \frac{(2\,\mu_\varepsilon\,\varepsilon)^Q}{(Q-1)!} Q^3 t^{Q-1} \varepsilon$$

$$\le C(T)\,\varepsilon^{5-4\delta}, \tag{40}$$

so that Lemma 1 is proved.

■

Remark By applying the same technique in dimension d higher than 2, we would get from the estimate of the recollision probability $\alpha < (d-1)/8$.

The final bound for α is then given in this case by the requirement to have a negligeable probability for overlappings of internal obstacles in the limit.

Thanks to lemma 1, we now can write

$$\tilde{f}_\varepsilon(t,x,v) = e^{-2\,t\,\mu_\varepsilon\,\varepsilon} \sum_{Q\geq 0} \mu_\varepsilon^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q \int_{-\varepsilon}^{\varepsilon} d\rho_1 \int_{-\varepsilon}^{\varepsilon} d\rho_2 \cdots \int_{-\varepsilon}^{\varepsilon} d\rho_Q$$

$$\chi\Big(\{\rho_i, t_i\}_{i=1}^Q \text{ is in the range of } \mathcal{Z}\Big) f_{in}(\xi(t), v(t)) + \varphi(\varepsilon). \tag{41}$$

We make then the change of variables

$$\{\rho_i\}_{i=1,\ldots,Q} \to \{\theta_i\}_{i=1,\ldots,Q}, \tag{42}$$

where θ_i is the angle of the scattering produced by the i–th obstacle. The Jacobian determinant of this change of variables is given by $\prod_{i=1}^Q \frac{d\rho_i}{d\theta_i} = \prod_{i=1}^Q \Psi_\varepsilon(\theta_i) = \prod_{i=1}^Q \varepsilon^{1+2\alpha}\,\Gamma_\varepsilon(\theta_i)$. We now use the following estimates:

$$|\xi(t) - (x + \sum_{i=0}^{Q} R_{\psi_i}(v)\,(t_{i+1}-t_i))| \;\;\leq\;\; Q\,\varepsilon \tag{43}$$

$$|t_i - t_i^{'}| \leq 3\,\varepsilon, \tag{44}$$

$$|v(t_i') - v(t_i)| = O(\varepsilon^\alpha), \tag{45}$$

(here ψ_j is defined as $\psi_j = \sum_{i=1}^j \theta_i$, with the convention $\psi_0 = 0$ and $t_0 = 0$, $t_{Q+1} = t$). Using also the fact that f_{in} lies in $W^{1,\infty}$, we get

$$\tilde{f}_\varepsilon(t,x,v) = e^{-t\,\mu\int_{-\pi}^{\pi} d\theta \Gamma_\varepsilon(\theta)} \sum_{Q\geq 0} \mu^Q \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{Q-1}}^t dt_Q \int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\theta_2 \cdots \int_{-\pi}^{\pi} d\theta_Q$$

$$\prod_{i=1}^Q \Gamma_\varepsilon(\theta_i)\, f_0(x + \sum_{i=0}^Q R_{\psi_i}(v)\,(t_{i+1}-t_i), R_{\psi_Q}(v)) + \varphi(\varepsilon). \tag{46}$$

But the right–hando side of (46) is nothing else than h_ε in the form of the series solution to (20), so that $\tilde{f}_\varepsilon = h_\varepsilon + \varphi(\varepsilon)$.

Using now (33) and the conservation of mass:

$$\int h_\varepsilon\,dxdv = \int f_0\,dxdv,$$

we also see that

$$f_\varepsilon - h_\varepsilon \to 0$$

in $L_t^\infty(L^1_{loc,x,v})$. □

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