Onsager Conjecture, the Kolmogorv $1/3$ law and the 1984 Kato Criteria in bounded domains with boundaries: In progress with Edriss Titi and E. Wiedemann. Workshop on kinetic and fluid Partial Differential Equations

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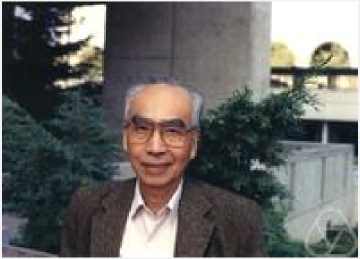
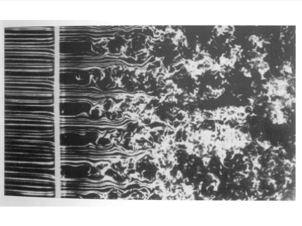




Several of my recent contributions , with Edriss Titi, Emile Wiedemann and others were motivated by the following issues:
The role of boundary effect in mathematical theory of fluids mechanic and the similarity , in presence of these effects, of the weak convergence in the zero viscosity limit and the statistical theory of turbulence.
As a consequence.

- I will recall the Onsager conjecture and compare it to the issue of anomalous energy dissipation.
- Give a proof of the local conservation of energy under convenient hypothesis in a domain with boundary.
- Give sufficient condition for the global conservation of energy in a domain with boundary and show how this imply the absence of anomalous energy dissipation.
- Give several forms of a basic theorem of Kato in the presence of a Lipschitz solution of the Euler equations. Insisting that in such case the absence of anomalous energy dissipation is **equivalent** to the persistence of regularity in the zero viscosity limit.

$u(x,t) \in C_{weak}([0,T]; L^2(\Omega))$ weak solution of the incompressible Euler equations $\Omega \subset \mathbb{R}^d$ with C^2 boundary $\partial\Omega$ and exterior normal $\vec{n}$

$$\begin{aligned} &\text{In } \mathcal{D}'((0,T)\times\Omega) \quad \partial_t u + \nabla_x \cdot (u\otimes u + pI) = 0 \quad \text{and} \quad \nabla\cdot u = 0, \\ &\text{On } \partial\Omega\times[0,T] \quad u\cdot\vec{n} = 0\,. \end{aligned} \tag{1}$$

If u is a smooth solution say Lipschitz one has:

$$\begin{aligned} &\int_\Omega \nabla p \cdot u dx = -\int_\Omega p\nabla\cdot u dx = 0\,, \\ &\int_\Omega \nabla(u\otimes u) : u dx = \int_\Omega \sum_j u_j \partial_{x_j} \frac{|u|^2}{2} dx = \int_\Omega \nabla\cdot(u\frac{|u|^2}{2}) dx = 0 \end{aligned}$$

And this implies the conservation of energy

$$\frac{d}{dt}\int_\Omega \frac{|u(t,x)|^2}{2} dx = 0\,.$$

Onsager gave a semi formal proof of the conjecture that by now carries its name:
Any weak solution which belongs to the space $C^{0,\alpha}$ with $\alpha > \frac{1}{3}$ conserves the energy.
The "formal " proof goes as follow: The term to control is

$$\langle \nabla(u \otimes u)u \rangle \simeq ((\nabla^{\frac{1}{3}} u \otimes \nabla^{\frac{1}{3}} u) : \nabla^{\frac{1}{3}} u)$$

hence appears the quantity $\|\nabla^{\frac{1}{3}} u\|_{L^3(\Omega\times[0,T])}$. This observation is the origin of serious proofs: Eyink , Constantin , E, Titi. in 1994.
Recent papers Buckmaster and als.. have shown, for $\alpha < \frac{1}{3}$, the existence of wild solutions in $C^{0,\alpha}((0,T) \times \mathbb{T}^3)$.

Under weak convergence or in statistical theory, with Navier-Stokes equation loss of regularity anomalous energy dissipation are related:

$$\|u_\nu(t)\|^2_{L^2(\Omega)} + 2\nu \int_0^t \int_\Omega |\nabla u_\nu(x,s)|^2 ds = \|u_\nu(0)\|^2_{L^2(\Omega)} \tag{2}$$

If u_ν converge weakly to a solution of the Euler equation which conserves the energy there is no anomalous energy dissipation. In statistical theory one has the Kolmogorov law:

$$\langle \frac{u_\nu(.+l,t) - u_\nu(.,t)}{|l|^{\frac{1}{3}}} \rangle \simeq (\langle \nu |\nabla u_\nu(.,t)|^2 \rangle)^{\frac{1}{3}} \,.$$

Theorem I

Let $(u,p)\in L^\infty(0,T;L^2(\Omega))\times\mathcal{D}'((0,T)\times\Omega)$ a weak solution:

$$\partial_t u+\nabla:(u\otimes u)+\nabla p=0\,,\nabla\cdot u=0 \qquad (3)$$

which in a subopen set $U=(t_1,t_2)\times\tilde\Omega$ satisfies:
For any small enough $\gamma>0$ and $V_\gamma=\{x,d(x,\partial\tilde\Omega)<\gamma\}$ there exists a $\beta(V)>0$ such that :

1. $$p\in C((t_1,t_2);H^{-\beta(V_\gamma)}(V_\gamma)\le M_0(V)<\infty \qquad (4a)$$
2. $$\int_{t_1}^{t_2}\|u(.,t)\|^3_{C^{0,\alpha}(\tilde\Omega)}dt\le M(U)<\infty\,. \qquad (4b)$$

Then (u,p) satisfies in $(t_1,t_2)\times\tilde\Omega$ the local energy conservation:

$$\partial_t\frac{|u|^2}{2}+\nabla_x\cdot\left(u\left(\frac{|u|^2}{2}+p\right)\right)=0\quad\text{in }\mathcal{D}'((t_1,t_2)\times\tilde\Omega)$$
$$\Leftrightarrow\forall\phi\in\mathcal{D}(\tilde\Omega)\,\frac{d}{dt}\langle\phi,\frac{|u|^2}{2}\rangle-\langle\nabla_x\phi,u\left(\frac{|u|^2}{2}+p\right)\rangle=0\,\text{in}\,\mathcal{D}'(t_1,t_2). \qquad (5)$$

(5) is proven with any given $\phi \in \mathcal{D}(\tilde{\Omega})$ hence with a support in an open set S_ϕ such that $\overline{S_\phi} \subset\subset \tilde{\Omega}$. Introduce $\eta > 0$ small enough, $\Omega_{3\eta}\,, \Omega_{2\eta}\,, \Omega_\eta$ such that

$$S_\phi \subset\subset \Omega_{3\eta} \subset\subset \Omega_{2\eta} \subset\subset \Omega_\eta \subset\subset \tilde{\Omega} \subset\subset \Omega$$
$$d(\Omega_{3\eta}, \partial\Omega_{2\eta}) = d(\Omega_{2\eta}, \partial\Omega_\eta) = d(\Omega_\eta, \partial\tilde{\Omega}) = \eta$$

and $\theta \in \mathcal{D}(\tilde{\Omega})$ equal to 1 in Ω_η

$$-\Delta\theta p = -\theta(\sum_{i,j} \partial_{x_i}\partial_{x_j} u_i u_j) - (\nabla p \cdot \nabla\theta + p\Delta\theta) \tag{6}$$

Then with elliptic theory and

$$\sum_{i,j} \partial_{x_i}\partial_{x_j} u_i u_j \in L^{\frac{2}{3}}((t_1, t_2); C^{0,\alpha}(\tilde{\Omega}))$$

one has:

$$\int_{t_1}^{t_2} \|p\|^{\frac{2}{3}}_{C^{0,\alpha}(\Omega_{3\eta})}(.,t)dt \le C(U)$$
$$\Rightarrow \partial_t u_{|\Omega_{3\eta}} = -\nabla\cdot(u_{|\Omega_{3\eta}} \otimes u_{|\Omega_{3\eta}}) - \nabla p_{|\Omega_{3\eta}} \in L^{\frac{3}{2}}((t_1,t_2);H^{-1}(\Omega_{3\eta}))\,. \tag{7}$$

The formula

$$\partial_t \frac{|u|^2}{2} + \nabla_x \cdot \left(u\left(\frac{|u|^2}{2}+p\right)\right) = 0$$

is well defined on $(t_1,t_2)\times\Omega_{3\eta}$ Moreover:

$$\partial_t u_{|\Omega_{3\eta}} = -\nabla\cdot(u_{|\Omega_{3\eta}} \otimes u_{|\Omega_{3\eta}}) - \nabla p_{|\Omega_{3\eta}} \in L^{\frac{3}{2}}((t_1,t_2);H^{-1}(\Omega_{3\eta}))$$

$\forall v \in L^1(\tilde\Omega)\,, \overline{v}$ denotes its extension by 0 outside $\tilde\Omega\,.$

$x \mapsto \rho(x) \in C^\infty(\mathbb{R}^n)$ a mollifier, $\rho(x) \geq 0$, with support in $|x| \leq 1$,

$$\int_{\mathbb{R}^n} \rho(x)dx = 1 \ \text{ with}$$

$$\text{For} \quad \epsilon < \eta \quad \text{fixed} \quad \rho_\epsilon = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon}) \quad \forall v \in \mathcal{D}'(\mathbb{R}^n) \quad v^\epsilon = v \star \phi_\epsilon$$

With the properties of support of distributions under convolution one has the following:

Lemma

For any $\psi \in \mathcal{D}(\Omega_{3\eta})$ and any $\epsilon < \eta$ one has:

$$\langle \partial_t u + \nabla\cdot(u\otimes u) - \nabla p, \psi\rangle = \langle (\partial_t(\overline{u})^\epsilon + \nabla\cdot(\overline{(u\otimes u)})^\epsilon + \nabla(\overline{p})^\epsilon)\cdot\psi\rangle = 0 \quad (8)$$

Insert in (8) $\psi_\epsilon = \phi(\overline{u})^\epsilon \in L^{\frac{3}{2}}((0,T);H^1(\tilde{\Omega}))$ while $\partial_t(\overline{u})^\epsilon \in L^{\frac{3}{2}}((0,T);H^{-1}(\tilde{\Omega}))$, Therefore:

$$0 = \langle \partial_t(\overline{u})^\epsilon(\overline{u})^\epsilon\phi\rangle + \langle(\overline{(u\otimes u)})^\epsilon : \nabla(\overline{u})^\epsilon\phi\rangle + \langle(\overline{p})^\epsilon\nabla\cdot(\overline{u})^\epsilon\phi\rangle = 0 \qquad (9)$$

Obviously with the local regularity on $(t_1,t_2)\times\Omega_{3\eta}$ for $\epsilon\to 0$ one has:

$$\begin{aligned}
&\lim_{\epsilon\to 0}\partial_t\langle(\overline{u})^\epsilon(\overline{u})^\epsilon\phi\rangle = \lim_{\epsilon\to 0}\langle\partial_t\frac{|(\overline{u})^\epsilon|^2}{2}\phi\rangle = \partial_t\langle\frac{|u|^2}{2}\phi\rangle\\
&\text{On}\quad \Omega_{3\eta}\quad \nabla\cdot(\overline{u})^\epsilon = (\overline{\nabla\cdot u}^\epsilon) = 0\\
&\Rightarrow \langle(\overline{p})^\epsilon\nabla\cdot(\overline{u})^\epsilon\phi\rangle = \langle(\overline{p})^\epsilon(\overline{u})^\epsilon\nabla\cdot\phi\rangle \to \langle pu\cdot\nabla\phi\rangle\,.
\end{aligned} \qquad (10)$$

Now the result follows from the limit of :

$$\begin{aligned}&\langle(\overline{u\otimes u})^\epsilon, \nabla_x((\overline{u})^\epsilon\phi)\rangle = \\ &\langle((\overline{u\otimes u})^\epsilon - (\overline{u})^\epsilon\otimes(\overline{u})^\epsilon), \nabla_x((\overline{u})^\epsilon\phi)\rangle + \langle((\overline{u})^\epsilon\otimes(\overline{u})^\epsilon, \nabla_x((\overline{u})^\epsilon\phi)\rangle\end{aligned} \tag{11}$$

On the support of ϕ , $\nabla\cdot(\overline{u})^\epsilon = 0$, therefore, with the Lebesque theorem one has:

$$\langle((\overline{u})^\epsilon\otimes(\overline{u})^\epsilon, \nabla_x((\overline{u})^\epsilon\phi)\rangle = \int_\Omega \frac{|(\overline{u})^\epsilon|^2}{2}(\overline{u})^\epsilon\cdot\nabla\phi dx \to \int_\Omega \frac{|u|^2}{2} u\cdot\nabla\phi dx$$

Then the only thing to show is that, in $\mathcal{D}'((t_1,t_2))\,,\forall\phi\in\mathcal{D}(\tilde\Omega)$:

$$\lim_{\epsilon\to 0}\langle((\overline{u\otimes u})^\epsilon - (\overline{u})^\epsilon\otimes(\overline{u})^\epsilon), \nabla_x((\overline{u})^\epsilon\phi)\rangle = 0\,.$$

First observe that:

$$\nabla\cdot(\phi(x)(\overline{u}^\epsilon)) = \nabla_x\Big(\int_{\mathbb{R}^n_x} \rho_\epsilon(x-y)\phi(x)I_2(y)u(y)dy\Big)$$
$$= \nabla_x\Big(\int_{\mathbb{R}^n_x} (\rho_\epsilon(x-y)\phi(x) - \rho_\epsilon(y)\phi(y))I_2(y)u(y)dy\Big) \quad \Rightarrow \tag{12}$$
$$|\nabla\cdot(\phi(x)(\overline{u}^\epsilon))| \le C(\phi)\epsilon^{\alpha(\Omega)-1}\|u\|_{C^{0,\alpha(\tilde\Omega)}}.$$

Regularization versus Weak Convergence .

Second For weak convergence or for statistical average one has:

$$\overline{(u_\epsilon \otimes u_\epsilon)} - \overline{u}_\epsilon \otimes \overline{u}_\epsilon = \overline{(\overline{u_\epsilon} - u_\epsilon) \otimes (\overline{u_\epsilon} - u_\epsilon))}$$

which comes from the identity in double averaging or weak limit:

$$\overline{u_\epsilon \otimes \overline{u_\epsilon}} = \overline{u_\epsilon} \otimes \overline{u_\epsilon}$$

However here it is not an average but a regularization (as in it classical in some models of turbulence) and (cf Constantin, E and Titi)) on the support of ϕ one has:

$$(\overline{u \otimes u})^\epsilon - ((\overline{u})^\epsilon \otimes (\overline{u})^\epsilon) = (\overline{u} - (\overline{u})^\epsilon) \otimes (\overline{u} - (\overline{u})^\epsilon) + \int (\delta_y \overline{u} \otimes \delta_y \overline{u}) \rho_\epsilon(y) dy$$

$$\delta_y \overline{u} = \overline{u}(x-y) - \overline{u}(x) .$$

Keeping in mind that $\epsilon < \eta$ one has for $x \in \Omega_{3\eta}$

$$|\overline{u} - (\overline{u})^\epsilon|_{L^\infty(\Omega_{3\eta})} = |\int_{\mathbb{R}^d} (\overline{u}(x) - \overline{u}(x-y))\rho^\epsilon(y)dy| \leq \epsilon^\alpha \|u\|_{C^{0,\alpha}(\tilde{\Omega})}$$
$$|\nabla_x(\overline{u})^\epsilon|_{L^\infty(\Omega_{3\eta})} \leq C\epsilon^{\alpha-1}\|u\|_{C^{0,\alpha}(\tilde{\Omega})}$$
$$\| \int (\delta_y \overline{u} \otimes \delta_y \overline{u})\rho_\epsilon(y)dy| \leq \int (|u(x-y) - u((x)|^2 \rho_\epsilon(y)dy \leq \epsilon^{2\alpha}\|u\|^2_{C^{0,\alpha}(\tilde{\Omega})}$$

which gives:

$$\langle ((\overline{u \otimes u})^\epsilon - (\overline{u})^\epsilon \otimes (\overline{u})^\epsilon), \nabla_x((\overline{u})^\epsilon \phi)\rangle \leq \epsilon^{3\alpha-1} C(\tilde{\Omega}, \phi) \left(\|u\|_{C^{0,\alpha}(\tilde{\Omega})}\right)^3$$

And this concludes the proof.
Here no use of the impermeability condition. Only the estimate on the pressure around $\tilde{\Omega}$

Consider the Euler equation in $(0,T)\times\Omega$ with $\partial\Omega\in C^1$, $\vec{n}$ denoting the outward normal the and impermeability boundary condition:

$$x\in\partial\Omega\Rightarrow u(x,t)\cdot\vec{n}(x)=0 \tag{13}$$

The incompressibility $\nabla\cdot u=0$ implies that $u(x,t)\cdot\vec{n}(x)$ is well defined in $L^2((0,T);H^{-\frac{1}{2}}(\partial\Omega)$ therefore (13) is well defined.
Introduce the function and the set

$$d(x)=d(x,\partial\Omega)=\inf_{y\in\partial\Omega}|x-y|\geq 0\,,U_{\eta_0}=\{x\in\Omega\,,d(x)<\eta_0\} \tag{14}$$

which have the following properties. For $0<\eta_0$ small enough $d(x)_{|U_{\eta_0}}\in C^\infty(U_{\eta_0})$, for any $x\in U_{\eta_0}$ there exists a unique $\sigma(x)\in\partial\Omega$ such that $d(x))=|x-\sigma(x)|$ and moreover one has:

$$\forall x\in U_{\eta_0}\qquad \nabla_x d(x)=-n(\sigma(x))\,. \tag{15}$$

Theorem II ; Global and Local hypothesis

Let (u,p) be in $(0,T)\times\Omega\subset\mathbb{R}_t\times\mathbb{R}^d$ a weak solution of the Euler equation in $(t_1,t_2)\times\Omega\subset\subset(0,T)\times\Omega$. with the hypothesis:
1 For some η_0

$$p\in L^\infty((0,T):H^{-\beta}(U_{\eta_0})):\quad \text{with}\quad \beta<\infty\,, \tag{16a}$$

$$\lim_{\eta\to 0}\sup_{t\in(0,T)\,d(x)<\eta<\eta_0}|((\frac{|u|^2}{2}+p)u(x,t))\cdot n(\sigma(x))|=0\,. \tag{16b}$$

2 For any open set $V=(t_1,t_2)\times\Omega\subset\subset(0,T)\times\Omega$ the function u satisfies the hypothesis (4b) of the theorem I

$$\text{with}\quad \alpha>\frac{1}{3}\quad \int_{t_1}^{t_2}\|u(.,t)\|^3_{C^{0,\alpha}(V)}dt\le M(V)<\infty\,.$$

Then, (u,p) globally conserves the energy, i.e. it satisfies for any $0\le t_1<t_2\le T$ the relation:

$$\|u(t_2)\|_{L^2(\Omega)}=\|u(t_1)\|_{L^2(\Omega)}. \tag{17}$$

Start with any open subset $\tilde\Omega \subset\subset \Omega$ and η_0 small enough, in particular such that

$$\tilde\Omega \subset\subset \Omega\backslash\overline{U_{\eta_0}} \tag{18}$$

then introduce a smooth function $x \mapsto \theta(x) \in \mathcal{D}(\Omega)$ equal to 1 for $d(x) \geq \frac{\eta_0}{2}$ and from the formula

$$\sum_{i,j} \partial_{x_i}\partial_{x_j} u_i u_j \in L^{\frac{2}{3}}((t_1, t_2); C^{0,\alpha}(\Omega))$$

with (16b) and as in the proof of the theorem I deduce that

$$p \in L^{\frac{2}{3}}((0, T); C^{0,\alpha}(\tilde\Omega))$$

Then with the hypothesis (17) one concludes that the relation

$$\frac{d}{dt}\langle\frac{|u|^2}{2}, \psi\rangle + \langle\nabla\cdot((\frac{|u|^2}{2} + p)u), \psi\rangle = 0 \tag{19}$$

holds for any $\psi \in \mathcal{D}(\tilde\Omega)$ in the sense of $\mathcal{D}'(0, T)$.

Eventually introduce a function $s \mapsto \phi(s)$ equal to 1 for $s > \frac{1}{2}$ and equal to 0 for $s < \frac{1}{4}$. With $0 < \eta < \eta_0$ one has:

$$\psi_\eta(x) = \phi(\frac{d(x)}{\eta}) \in \mathcal{D}(\Omega)$$

$$\nabla\psi_\eta(x) = -\frac{1}{\eta}\phi'(\frac{d(x)}{\eta})\vec{n}(\sigma(x)) \text{ for } \frac{\eta}{4} < d(x) < \frac{\eta}{2}\,; \text{ otherwise} = 0\,.$$

With the theorem I and $\psi = \psi_\eta$ in (19) one has

$$\begin{aligned}
&\int_\Omega \frac{|u(t_2,x)|^2}{2}\phi(\frac{d(x)}{\eta}) - \int_\Omega \frac{|u(t_1,x)|^2}{2}\phi(\frac{d(x)}{\eta}) \\
&\quad = \int_{t_1}^{t_2}\int_\Omega ((u\otimes u)+p)u)(x,t)\cdot \vec{n}(\sigma(x))\frac{1}{\eta}\phi'(\frac{d(x)}{\eta})dxdt
\end{aligned} \tag{20}$$

and with hypothesis (16b):

$$\lim_{\eta\to 0} \sup_{t\in(0,T)} \sup_{d(x)<\eta<\eta_0} |((\frac{|u|^2}{2}+p)u(x,t))\cdot n(\sigma(x))| = 0$$

the results follows from the Lebesgue Theorem by letting $\eta \to 0$.

1. The hypothesis (4a) in the theorem I or (16b) in the theorem II is used to control the effect of what comes from the boundary to the open set $(0,T)\times\Omega$ through the action of the pressure. They are not needed in domains without boundary.
2. Since on $\partial\Omega$ one has $u(x)\cdot\vec{n}(\sigma(x))=0$ the hypothesis (16b) are satisfied if for some $\eta<\eta_0$ one has

$$\begin{aligned}&(u,p)\in L^\infty((0,T)\times U_\eta))\quad\text{and}\\ &u(x,t)\cdot\vec{n}(\sigma(x))\in C^0(\overline{(0,T)\times U_\eta}).\end{aligned}\tag{21}$$

3. Examples of "wild" admissible solutions which which do not conserve the energy as constructed in B., Szekelyhidi and Weidemann, seem to indicate that the hypothesis (16b) may be compulsory to always enforce the conservation of energy.

The above improvement is of interest as it gives a sufficient condition for non-anomalous energy dissipation in the zero viscosity limit which is not in contradiction with the presence of a Prandtl type boundary layer. This is the object of the very easy but essential theorem below:

Theorem III

$u_\nu(x,t)$ a family of Leray solutions of Navier-Stokes in $\mathbb{R}_t^+ \times \Omega$:

$$\begin{aligned} &\partial_t u_\nu + (u_\nu \cdot \nabla_x) u_\nu - \nu \Delta u_\nu + \nabla p_\nu = 0, \quad \nabla \cdot u_\nu = 0, \\ &u_\nu(t,x) = 0 \text{ on } \quad \mathbb{R}_t^+ \times \partial\Omega \quad \text{and} \quad u_\nu(\cdot, x) = u_0 \in L^2(\Omega)\,. \end{aligned} \tag{22}$$

Assume that on $(0,T)\times\Omega$ the family u_ν satisfies the hypotheses of the Theorems I and II uniformly in ν; more precisely:

• With $\alpha>\frac{1}{3}$ and $\tilde\Omega\subset\subset\Omega$ one has: $\int_0^T\|u_\nu(t,.)\|^3_{C^{0,\alpha}(\tilde\Omega)}ds\le M(\tilde\Omega)$;

•$(t,x)\in(0,T)\times\times\{x/d(x,\partial\Omega)<\eta_0\}\Rightarrow\quad |u_\nu(t,x)|+|p_\nu(t,x)|\le M_2$;

• There exists a ν-independent modulus of continuity $s\mapsto\omega(s)$ with $\lim_{s\to0}\omega(s)=0$ such that for $d(x,\partial\Omega)<\eta_0$ one has:

$$|u_\nu(t,x)\cdot\vec n(\sigma(x))|\le\omega(d(x)). \tag{23}$$

Then (if necessary extracting subsequences) $\overline{u_\nu}$ converges in

$$L^\infty((0,T);L^2(\Omega))\,\mathrm{weak}$$

to a function $\overline{u_\nu}$ which is a weak solution of the Euler equation, and which satisfies the hypothesis of the Theorems I and II , so that there is no anomalous energy dissipation in the zero viscosity limit:

$$\lim_{\nu\to0}\nu\int_0^T\int_\Omega|\nabla_x u_\nu(t,x)|^2dxdt=0. \tag{24}$$

Proof.

Introduces $m = \sup_\nu \nu \int_0^T \int_\Omega |\nabla_x u_\nu(t,x)|^2 dxdt$ and a sequence ν_i such that $m = \lim_{i\to\infty} \nu_i \int_0^T \int_\Omega |\nabla_x u_{\nu_i}(t,x)|^2 dxdt$. Extract yet another one, ν_j, such that $u_j = u_{\nu_j}$ converges to a limit $\overline{u_j}$ in weak star $L^\infty(0,T); L^2(\Omega))$. From the Leray energy inequality:

$$\|u_j(T)\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)} + 2\nu_j \int_0^T \int_\Omega |\nabla_x u_J(t,x)|^2 dxdt \leq 0 \tag{25}$$

weak convergence and energy conservation gives

$$\|\overline{u_j}(T)\|_{L^2(\Omega)} - \|u_0\|_{L^2(\Omega)} + 2m \leq 0\,, \text{and} \|\overline{u_j}(T)\|^2_{L^2(\Omega)} = \|u_0\|^2_{L^2(\Omega} \tag{26}$$

Hence $m = 0$.

□

1. As observed above some $C^{0,\alpha}$ with $\alpha > 1/3$ regularity implies the energy conservation and the absence of anomalous energy dissipation. However simple examples like the shear flow in a canal show that such regularity may not be necessary for the absence of anomalous energy dissipation.
2. It is only in the presence of a smooth (Lispchitz) solution of the Euler equation that weak convergence to such solution turns out to be equivalent to the absence of anomalous energy dissipation. This is an avatar (in the sense of weak convergence) of the Kolmogorov $1/3$ law and the object of the Kato theorem.

Relative estimate with $\nabla u \in L^\infty((0,T)\times\Omega)$

$$
\begin{aligned}
&\partial_t(u_\nu - u) + u_\nu\cdot\nabla u_\nu - u\cdot\nabla u - \nu\Delta u_\nu + \nabla p_\nu - \nabla p = 0\\
&(u_\nu\cdot\nabla u_\nu - u\cdot\nabla u, u_\nu - u) = (u_\nu - u, S(u)(u_\nu - u))\,;\\
&\qquad S(u) = \frac{\nabla u + (\nabla u)^t}{2} \in L^\infty((0,T)\times\Omega)\,;\\
&\frac{d}{dt}\frac{1}{2}|u_\nu - u|^2_{L^2(\Omega)} + \nu\int_\Omega |\nabla u_\nu|^2 dx \le |(u_\nu - u, S(u)(u_\nu - u))|\\
&\qquad + \nu\int_\Omega (\nabla u_\nu\cdot\nabla u)dx - \nu\int_{\partial\Omega}(\partial_{\vec n}u_\nu)_\tau u d\sigma\,. \text{ The bad term!}\,.
\end{aligned}
$$

Without physical boundary u_ν converges to u in $C((0,T);L^2(\Omega))$ and $\lim_{\nu\to 0}\int_0^T \|\nabla u_\nu\|^2_{L^2(\Omega)}dt = 0$. Otherwise the situation is much more subtle!!!

Relative estimate with $\nabla u \in L^\infty((0,T)\times\Omega)$

$$\partial_t(u_\nu - u) + u_\nu\cdot\nabla u_\nu - u\cdot\nabla u - \nu\Delta u_\nu + \nabla p_\nu - \nabla p = 0$$

$$(u_\nu\cdot\nabla u_\nu - u\cdot\nabla u, u_\nu - u) = (u_\nu - u, S(u)(u_\nu - u))\,;$$

$$S(u) = \frac{\nabla u + (\nabla u)^t}{2} \in L^\infty((0,T)\times\Omega)\,;$$

$$\frac{d}{dt}\frac{1}{2}|u_\nu - u|^2_{L^2(\Omega)} + \nu\int_\Omega |\nabla u_\nu|^2 dx \le |(u_\nu - u, S(u)(u_\nu - u))|$$

$$+\nu\int_\Omega(\nabla u_\nu\cdot\nabla u)dx - \nu\int_{\partial\Omega}(\partial_{\vec n}u_\nu)_\tau u d\sigma\,.\ \text{The bad term!}\,.$$

Without physical boundary u_ν converges to u in $C((0,T);L^2(\Omega))$ and $\lim_{\nu\to0}\int_0^T \|\nabla u_\nu\|^2_{L^2(\Omega)}dt = 0$. Otherwise the situation is much more subtle!!!

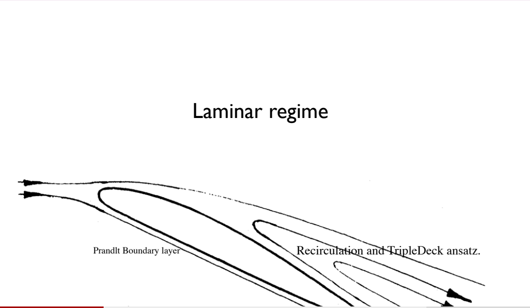
Recirculation and Vorticity Control

It is enough to have a moderate recirculation:

$$\lim_{\nu\to 0}\nu\int_0^T\int_{\partial\Omega}\left((\frac{\partial u_\nu}{\partial \vec n}(\sigma,t))_\tau u_\tau(\sigma,t)\right)_- d\sigma dt = 0$$

or a moderate backward vorticity using:

$$(u_\nu = 0, u\cdot\vec n = 0) \Rightarrow \nu(\frac{\partial u_\nu}{\partial \vec n}(\sigma,t))_\tau u_\tau = \nu((\nabla\wedge u_\nu)\wedge\vec n)\cdot u$$



The Bardos-Titi version of the Kato theorem

In 1984. T.Kato gave, in an internal Berkley proceeding Math. Sci. Res. Inst. Publ. a criteria for the convergence to the smooth solution. Now what happens is that this criteria can be generalized to include the above considerations and much more about the energy:

Theorem

Are equivalent:

$$\forall w(x,t) \in L^\infty((0,T) \times \partial\Omega, w \cdot \vec{n} = 0\,,$$

$$\lim_{\nu \to 0} \nu \int_0^T \int_{\partial\Omega} (\frac{\partial u_\nu}{\partial \vec{n}}(\sigma,t))_\tau w(\sigma,t) d\sigma dt = 0 \qquad (27)$$

$$\lim_{\nu \to 0} \nu \int_0^T \int_{\partial\Omega} ((\frac{\partial u_\nu}{\partial \vec{n}}(\sigma,t))_\tau u_\tau(\sigma,t))_- d\sigma dt = 0 \qquad (28)$$

$$u_\nu(t) \to u(t) \ \textit{in } L^2(\Omega) \textit{ uniformly in } t \in [0,T]\,, \qquad (29)$$

$$u_\nu(t) \to u(t) \ \textit{weakly in } L^2(\Omega) \textit{ for each } t \in [0,T]\,, \qquad (30)$$

$$\lim_{\nu \to 0} \nu \int_0^T \int_\Omega |\nabla u_\nu(x,t)|^2 dx dt = 0\,, \qquad (31)$$

$$\lim_{\nu \to 0} \nu \int_0^T \int_{\Omega \cap \{d(x,\partial\Omega) < \nu\}} |\nabla u_\nu(x,t)|^2 dx dt = 0\,. \qquad (32)$$

$$\lim_{\nu \to 0} \frac{1}{\nu} \int_0^T \int_{\Omega \cap \{d(x,\partial\Omega) < \nu\}} |u_\nu(x,t)|^2 dx dt = 0\,. \qquad (33)$$

The proof: An updated version of the basic result of Kato

The fact (27) implies (28) is trivial then that it implies (29) has already been observed. It implies (30) which gives (31) with the energy inequality

$$\|u_\nu(t)\|^2_{L^2(\Omega)} + 2\nu \int_0^t \int_\Omega |\nabla u_\nu(s)|^2 ds \leq \|u_0\|^2_{L^2(\Omega)} \tag{34}$$

and of course (32).
With Poincaré inequality (32) implies (33).

With "dual" Kato corrector (33) implies (27)

Construct a family of divergence free vector fields $\hat{w}_\nu \in \mathcal{C}^\infty(\overline{\Omega} \times (0, T))$ with support in $\{(x, t) \in (d(x, \partial\Omega) < \nu \times (0, T))\}$ which coincides with w on the boundary and with gradient bounded in L^∞ by $C\nu^{-1}$.

$$w \in L^{lip}(\partial\Omega \times (0, T)); \quad w \cdot \vec{n} = 0\,;$$

$$\Theta \in \mathcal{D}(\overline{0, 1}), \quad \Theta(0) = 0\,, \quad \Theta'(0) = 1\,;$$

$$\hat{w}_\nu(x, t) = \nabla \wedge ((\vec{n}(\sigma(x)) \wedge w(\sigma(x), t)\nu\Theta(\frac{d(x, \partial\Omega)}{\nu}))$$

$$\Rightarrow \text{on} \quad \partial\Omega \quad \hat{w}_\nu(x, t) = w(x, t)$$

$$\Rightarrow \text{In} \quad \Omega\,, \qquad \nabla \cdot \hat{w}_\nu = 0\,, \quad \text{and} \quad \text{support} \quad \hat{w}_\nu \subset \{d(x, \partial\Omega) < \nu\}\,;$$

$$\Rightarrow \|\nabla_x \hat{w}_\nu\|_\infty \le C\nu^{-1}\,, \sup_t \|\nabla_x \hat{w}_\nu(x, t)\|_{L^2(\Omega)} \le C\nu^{-\frac{1}{2}}\,.$$

With multiplication of the Navier-Stokes equation by $\hat{w}_\nu$ and integration (33) $\Rightarrow$ (27)

$$\nu \int_{\partial\Omega} (\frac{\partial u_\nu}{\partial \vec{n}}(\sigma, t))_\tau w(\sigma, t) d\sigma =$$
$$\nu(\nabla u_\nu, \nabla \hat{w}_\nu)_{L^2(\Omega)} - (u_\nu \otimes u_\nu, \nabla \hat{w}_\nu)_{L^2(\Omega)} + (\partial_t u_\nu, \hat{w}_\nu)_{L^2(\Omega)} .$$

Then

$$\sup_t \|\nabla_x \hat{w}_\nu(x,t)\|_{L^2(\Omega)} \le C\nu^{-\frac{1}{2}} \Rightarrow \int_0^T |\nu(\nabla u_\nu, \nabla \hat{w}_\nu)_{L^2(\Omega)}| dt \to 0$$

$$|\nabla_x \hat{w}_\nu|_\infty \le C\nu^{-1} \quad \text{and} \lim_{\nu \to 0} \frac{1}{\nu} \int_0^T \int_{\Omega \cap \{d(x,\partial\Omega) < \nu\}} |u_\nu(x,t)|^2 dx dt = 0$$

$$\Rightarrow \int_0^T (u_\nu \otimes u_\nu, \nabla \hat{w}_\nu)_{L^2(\Omega)} dt = 0 .$$

The D'Alembert Paradox and the Kato Theorem

Even in the absence of a theorem the general feeling is that the case where the Kato criteria does not apply is not the exception but the general situation. This can be confirmed by comparison with the d'Alembert Paradox.

Potential flows and D'Alembert paradox 1

$$\Omega \subset \mathbb{R}^3\,, \mathbb{R}^3 \backslash \Omega \quad \text{compact}\,.$$

$$\text{In } \Omega \qquad \nabla \cdot u = 0 \quad \text{and} \quad \nabla \wedge u = 0\,. \quad \text{On} \quad \partial\Omega \quad u \cdot \vec{n} = 0\,,$$

$$\lim_{|x|\to\infty} u(x) = \vec{u}_\infty = (u_\infty, 0, 0)$$

then $u = \nabla \phi$ is a potential flow and a stationary solution:

$$u \cdot \nabla u = \sum_{1 \le j \le d} u_j \partial_{x_j} u_i = u \cdot \nabla u = \sum_{1 \le j \le d} u_j \partial_{x_j} u_j = \frac{1}{2} \nabla_x |u|^2$$

$$u \cdot \nabla u - \nabla(\frac{|u|^2}{2}) = 0\,.$$

$$\nabla \cdot u = \Delta \phi = 0 \quad \text{on} \quad \partial\Omega \quad u \cdot \vec{n} = \partial_{\vec{n}} \phi = 0$$

D'Alembert Paradox 2

With Green formula :

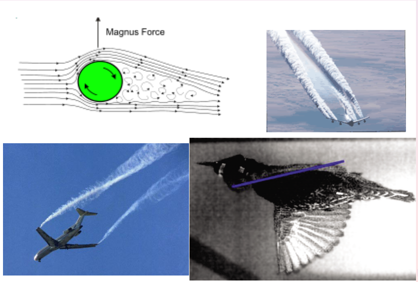
$$
\begin{aligned}
F &= \int_{\partial\Omega} p\vec{n}d\sigma = \int_{\partial\Omega} (p\vec{n} + (\vec{n}\cdot u)u)d\sigma \\
&= \int_{\Omega\cap\{|x|<R\}} (\nabla p + u\cdot\nabla u)dx - \int_{|x|=R} (\frac{\vec{x}}{|x|}p + (\frac{\vec{x}}{|x|}\cdot u)u)d\sigma \\
&= -\lim_{R\to\infty} \int_{|x|=R} (\frac{\vec{x}}{|x|}p + (\frac{\vec{x}}{|x|}\cdot u)u)d\sigma = 0
\end{aligned}
$$

If stationary fluids are described by the Euler equations birds and plane cannot glide!

Then to make the plane fly some energy has to be used-dissipated and this is related to last form of the Kato criteria

$$
\lim_{\nu\to 0} \frac{1}{\nu}\int_0^T \int_{\Omega\cap\{d(x,\partial\Omega)<\nu\}} |u_\nu(x,t)|^2 dxdt = 0
$$

Conjecture The catch : The Kato Criteria The viscosity and the no slip boundary condition remain in the solution of the Navier-Stokes equation (even for $\nu \to 0$) and in the behavior of the fluid. Even for initial data stationary or close to the stationary solution the viscosity limit is not the potential flow. And the anomalous dissipation of energy contributes to the motion.



MANY THANKS FOR THE ATTENTION.