Measure-valued - strong uniqueness for hyperbolic systems

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joint work with Tomasz Debiec, Eduard Feireisl, Ondřej Kreml, Agnieszka Świerczewska-Gwiazda and Emil Wiedemann

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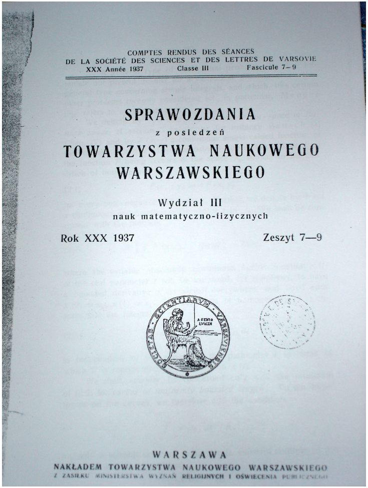

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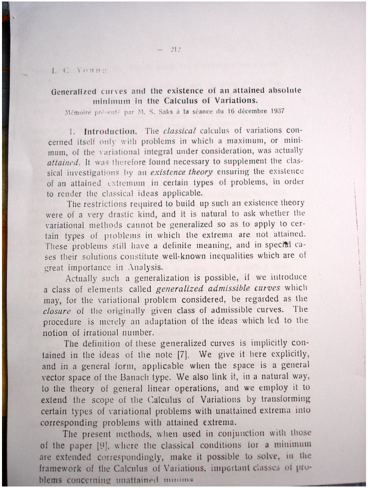

L. C. Young

Generalized curves and the existence of an attained absolute minimum in the Calculus of Variations.

Mémoire présenté par M. S. Saks à la séance du 16 décembre 1937

1. **Introduction.** The *classical* calculus of variations concerned itself only with problems in which a maximum, or minimum, of the variational integral under consideration, was actually *attained*. It was therefore found necessary to supplement the classical investigations by an *existence theory* ensuring the existence of an attained extremum in certain types of problems, in order to render the classical ideas applicable.

The restrictions required to build up such an existence theory were of a very drastic kind, and it is natural to ask whether the variational methods cannot be generalized so as to apply to certain types of problems in which the extrema are not attained. These problems still have a definite meaning, and in special cases their solutions constitute well-known inequalities which are of great importance in Analysis.

Actually such a generalization is possible, if we introduce a class of elements called *generalized admissible curves* which may, for the variational problem considered, be regarded as the *closure* of the originally given class of admissible curves. The procedure is merely an adaptation of the ideas which led to the notion of irrational number.

The definition of these generalized curves is implicitly contained in the ideas of the note [7]. We give it here explicitly, and in a general form, applicable when the space is a general vector space of the Banach type. We also link it, in a natural way, to the theory of general linear operations, and we employ it to extend the scope of the Calculus of Variations by transforming certain types of variational problems with unattained extrema into corresponding problems with attained extrema.

The present methods, when used in conjunction with those of the paper [9], where the classical conditions for a minimum are extended correspondingly, make it possible to solve, in the framework of the Calculus of Variations, important classes of problems concerning unattained minima

Notation

$C_0(\mathbb{R}^d)$ – closure of the space of continuous functions on $\mathbb{R}^d$ with compact support w.r.t. the $\|\cdot\|_\infty$-norm.
$(C_0(\mathbb{R}^d))^* \cong \mathcal{M}(\mathbb{R}^d)$ – the space of signed Radon measures with finite mass. The duality pairing is given by

$$\langle \mu, f \rangle = \int_{\mathbb{R}^d} f(\lambda)\, d\mu(\lambda).$$

Definition

A map $\mu : \Omega \to \mathcal{M}(\mathbb{R}^d)$ is called weakly* measurable if the functions $x \mapsto \langle \mu(x), f \rangle$ are measurable for all $f \in C_0(\mathbb{R}^d)$.

Young measures

- The main feature of Young measure theory is that it allows us to pass to a limit in the expression $f(v^j)$ with nonlinear f and only weakly-star convergent v^j.
- What's the strategy?
  Instead of considering $f(v^j)$ we embed the problem in a larger space, but gain linearity, i.e. $\langle f, \delta_{v^j(x)} \rangle$.
- If $f \in C_0(\mathbb{R}))$ using the duality

$$(L^1(\Omega; C_0(\mathbb{R}^d)))^* \cong L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^d)),$$

  Banach-Alaoglu theorem and weak-star continuity of linear operators allows for limit passage to get $\langle f, \nu_x \rangle$.
- What's the cost?
  We end up with a weaker notion of solution: *measure-valued solution*

Can the Young measures describe a concentration effect?

Definition

A bounded sequence $\{z^j\}$ in $L^1(\Omega)$ converges in biting sense to a function $z \in L^1(\Omega)$, written $z^j \stackrel{b}{\to} z$ in Ω, provided there exists a sequence $\{E_k\}$ of measurable subsets of Ω, satisfying $\lim_{k\to\infty} |E_k| = 0$, such that for each k

$$z^j \rightharpoonup z \quad \text{in} \quad L^1(\Omega \setminus E_k).$$

Remarks

Biting limit can be also express as $\lim_{n\to\infty} \lim_{j\to\infty} T^n(z^j)$, where by $T^n(\cdot)$ we denote standard truncation operator.

Lemma

Let u^j be a sequence of measurable functions and ν_x a Young measure associated to a subsequence u_{j_k}. Then $f(\cdot, u^{j_k}) \stackrel{b}{\rightarrow} \langle \nu_x, f \rangle$ for every Carathéodory function $f(\cdot,\cdot)$ s.t. $f(\cdot, u^{j_k})$ is a bounded sequence in $L^1(\Omega)$. Here $\langle \nu_x, f \rangle = \int_{\mathbb{R}^d} f \, d\nu_x$.

Remarks

In view of the above facts the classical Young measures prescribe only the oscillation effect, not the concentration one. The attempt to prescribe also concentration effect by some generalizations of the Young measures was initiated by DiPerna and Majda

Lemma

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Incompressible Euler equations

This system models the flow of an inviscid, incompressible fluid with constant density in the absence of external forces, where $v(t,x)$ is the velocity of the fluid and the $p(t,x)$ the pressure

$$\begin{aligned} v_t + \operatorname{div}(v \otimes v) + \nabla p &= 0, \\ \operatorname{div} v &= 0. \end{aligned}$$

Generalized Young measures

A (generalized) Young measure on $\mathbb{R}^d$ with parameters in $\mathbb{R}^d \times \mathbb{R}^+$ is a triple $(\nu_{x,t}, m, \nu^\infty_{x,t})$, where

- $\nu_{x,t} \in \mathcal{P}(\mathbb{R}^d)$ for a.e. $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$ (oscillation measure)
- $m \in \mathcal{M}^+(\mathbb{R}^d \times \mathbb{R}^+)$ (concentration measure)
- $\nu^\infty_{x,t} \in \mathcal{P}(\mathcal{S}^{d-1})$ for m-a.e. $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$ (concentration-angle measure)

R. J. DiPerna and A. J. Majda, Oscillations and concentrations in weak solutions of the incompressible fluid equations, Comm. Math. Phys. 1987.

J. J. Alibert and G. Bouchitté, Non-uniform integrability and generalized Young measures, J. Convex Anal. 1997.

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J. J. Alibert and G. Bouchitté, Non-uniform integrability and generalized Young measures, J. Convex Anal. 1997.

Measure-valued solutions to incompressible Euler system

We say that (ν, m, ν^∞) is a measure-valued solution of IE with initial data u_0 if for every $\phi \in C^1_{c,\mathrm{div}}([0,T) \times \mathbb{T}^n; \mathbb{R}^n)$ it holds that

$$\int_0^T \int_{\mathbb{T}^n} \partial_t \phi \cdot \overline{u} + \nabla \phi : \overline{u \otimes u} dxdt + \int_{\mathbb{T}^n} \phi(\cdot, 0) \cdot u_0 dx \quad = 0.$$

Where

$$\overline{u} = \langle \lambda, \nu \rangle$$
$$\overline{u \otimes u} = \langle \lambda \otimes \lambda, \nu \rangle + \langle \beta \otimes \beta, \nu^\infty \rangle m$$

If the solution is generated by some approximation sequences, then the black terms on right-hand side correspond to the biting limit of sequences whereas the blue ones corespond to concentration measure

Admissibility

Let us set

$$E_{mvs}(t) := \int_{\mathbb{T}^n} \frac{1}{2}\overline{|u|^2}(t,x)dx$$

for almost every t, where

$$\overline{|u|^2} = \langle |\lambda|^2, \nu \rangle + \langle |\beta|^2, \nu^\infty \rangle m$$

and

$$E_0 := \int_{\mathbb{T}^n} \frac{1}{2}|u_0|^2(x)dx.$$

We then say that a measure-valued solution is admissible if

$$E_{mvs}(t) \leq E_0$$

in the sense of distributions.

Weak-strong uniqueness for mvs to concrete systems

- Y. Brenier, C. De Lellis, L. Székelyhidi, Jr., Weak-strong uniqueness for measure-valued solutions. Comm. Math. Phys. 2011,

  Incompressible Euler -oscillation and concentration measure,
- S. Demoulini, D. M. A. Stuart, A. E. Tzavaras, Weak-strong uniqueness of dissipative measure-valued solutions for polyconvex elastodynamics. Arch. Ration. Mech. Anal. 2012

  In weak formulation only oscillation measure, in entropy inequality there appears non-negative concentration measure

Weak-strong uniqueness for mvs to concrete systems

- P. G., A. Świerczewska-Gwiazda, E. Wiedemann, Weak-Strong Uniqueness for Measure-Valued Solutions of Some Compressible Fluid Models, Nonlinearity, 2015
  Oscillatory and vector-valued concentration measure both in weak formulation and entropy inequality
- E. Feireisl, P. G., A. Świerczewska-Gwiazda, E. Wiedemann, Dissipative measure-valued solutions to the compressible Navier–Stokes system, Calculus of Variations and Partial Differential Equations, 2016
  Instead of vector-valued concentration measure the dissipation defect is introduced
- J. Březina, E. Feireisl, Measure-valued solutions to the complete Euler system, arXiv:1702.04870

Weak-Strong Uniqueness

Theorem (Y. Brenier, C. De Lellis, L. Székelyhidi, Jr., 2011)

Let $U \in C^1([0,T] \times \mathbb{T}^n)$ *be a solution of IE . If* (ν, m, ν^∞) *is an admissible measure-valued solution with the same initial data, then*

$$\nu_{t,x} = \delta_{U(t,x))} \text{ for a.e. } t, x, \text{ and } m = 0.$$

Remark:

Some generalization of this result: Emil Wiedemann, *Weak-strong uniqueness in fluid dynamics*, arXiv:1705.04220

Skeach of the proof:

Let's define relative energy (entropy):

$$E_{rel}(t) := \int_{\mathbb{T}^n} \frac{1}{2}\overline{|u-U|^2}(t,x)dx$$

where

$$\overline{|u-U|^2} = \langle |\lambda - U|^2, \nu\rangle + \langle |\beta|^2, \nu^\infty\rangle m$$

then

$$\frac{d}{dt}E_{rel}(t) \leq \|U\|_{C^1} \cdot E_{rel}(t).$$

Compressible Euler system

We consider now the isentropic Euler equations,

$$\begin{aligned}\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) &= 0,\\ \partial_t \rho + \operatorname{div}(\rho u) &= 0.\end{aligned}$$

We will use the notation for the so-called pressure potential defined as

$$P(\rho) = \rho \int_1^{\rho} \frac{p(r)}{r^2} dr.$$

Measure-valued solutions to compressible Euler

- We need a slight refinement which allows us to treat sequences whose components have different growth.
- Let $(u_k, w_k)_k$ be a sequence such that (u_k) is bounded in $L^p(\Omega;\mathbb{R}^l)$ and (w_k) is bounded in $L^q(\Omega;\mathbb{R}^m)$ $(1 \le p, q < \infty)$. Define the *nonhomogeneous unit sphere*

$$\mathbb{S}^{l+m-1}_{p,q} := \{(\beta_1,\beta_2) \in \mathbb{R}^{l+m} : |\beta_1|^{2p} + |\beta_2|^{2q} = 1\}.$$

Then, there exists a a subsequence and measures

$$\nu \in L^\infty_w(\Omega;\mathcal{P}(\mathbb{R}^{l+m})), m \in \mathcal{M}^+(\bar\Omega),\ \ \nu^\infty \in L^\infty_w(\Omega, m;\mathcal{P}(\mathbb{S}^{l+m-1}_{p,q}))$$

such that in the sense of measures

$$f(x,u_n(x),w_n(x))dx \overset{*}{\rightharpoonup} \int_{\mathbb{R}^{l+m}} f(x,\lambda_1,\lambda_2)d\nu_x(\lambda_1,\lambda_2)dx + \int_{\mathbb{S}^{l+m-1}_{p,q}} f^\infty(x,\beta_1,\beta_2)d\nu^\infty_x(\beta_1,\beta_2)m.$$

General hyperbolic conservation law

- Y. Brenier, C. De Lellis, L. Székelyhidi, Jr., Weak-strong uniqueness for measure-valued solutions. Comm. Math. Phys. 2011,
  General hyperbolic systems - only oscillation measure, both in weak formulation and entropy inequality
- S. Demoulini, D. M. A. Stuart, A. E. Tzavaras, Weak-strong uniqueness of dissipative measure-valued solutions for polyconvex elastodynamics. Arch. Ration. Mech. Anal. 2012
  In weak formulation only oscillation measure, in entropy inequality there appears non-negative concentration measure

General hyperbolic conservation law

- C. Christoforou, A. Tzavaras, Relative entropy for hyperbolic-parabolic systems and application to the constitutive theory of thermoviscoelasticity, Arch. Ration. Mech. Anal. 2017
  An analogue result for more general form of a system, hyperbolic-parabolic case, also only with a non-negative concentration measure in entropy inequality
- P. G., O. Kreml, A. Świerczewska-Gwiazda. Dissipative measure valued solutions for general hyperbolic conservation laws, arXiv:1801.01030
  Concentration measure both in the weak formulation and the entropy inequality

General hyperbolic conservation law

We consider the hyperbolic system of conservation laws in the form

$$\partial_t A(u) + \partial_\alpha F_\alpha(u) = 0 \tag{1}$$

with the initial condition $u(0) = u_0$. Here $u : [0, T] \times \mathbb{T}^d \to \mathbb{R}^n$. There exists an open convex set $X \subset \mathbb{R}^n$ such that the mappings $A : \overline{X} \to \mathbb{R}^n$, $\mathrm{F}_\alpha : X \to \mathbb{R}^n$ are C^2 maps on X, A is continuous on $\bar{X}$ and $\nabla A(u)$ is nonsingular for all $u \in X$.

Definition

We say that $(\nu, m_A, m_{F_\alpha}, m_\eta)$, is a dissipative measure-valued solution of system (1) if $\nu \in L^\infty_{\text{weak}}\left((0,T)\times\mathbb{T}^d; \mathcal{P}\left(\overline{X}\right)\right)$ is a parameterized measure and together with concentration measures $m_A \in (\mathcal{M}([0,T]\times\mathbb{T}^d))^n$, $m_{F_\alpha} \in (\mathcal{M}([0,T]\times\mathbb{T}^d))^{n\times n}$ satisfy

$$\int_Q \langle \nu_{t,x}, A(\lambda)\rangle \cdot \partial_t\varphi dxdt + \int_Q \partial_t\varphi \cdot m_A(dxdt)$$
$$+ \int_Q \langle \nu_{t,x}, F_\alpha(\lambda)\rangle \cdot \partial_\alpha\varphi dxdt + \int_Q \partial_\alpha\varphi \cdot m_{F_\alpha}(dxdt)$$
$$+ \int_{\mathbb{T}^d} \langle \nu_{0,x}, A(\lambda)\rangle \cdot \varphi(0) dx + \int_{\mathbb{T}^d} \varphi(0)\cdot m_A^0(dx) = 0$$

for all $\varphi \in C_c^\infty(Q)^n$.

Definition

Moreover, the total energy balance holds for all $\zeta \in C_c^\infty([0, T))$

$$\int_Q \langle v_{t,x}, \eta(\lambda)\rangle \zeta'(t)dxdt + \int_Q \zeta'(t)m_\eta(dxdt)$$
$$+ \int_{\mathbb{T}^d} \langle v_{0,x}, \eta(\lambda)\rangle \zeta(0)dx + \int_{\mathbb{T}^d} \zeta(0)m_\eta^0(dx) \geq 0$$

with a dissipation measure $m_\eta \in \mathcal{M}^+([0, T] \times \mathbb{T}^d)$. In particular we assume that measures m_A^0 and m_η^0 are well defined.

Hypotheses

(H1) There exists an entropy-entropy flux pair (η, q_α), $\eta(u) \geq 0$ and $\lim_{|u|\to\infty} \eta(u) = \infty$
This yields the existence of a C^1 function $G : \overline{X} \to \mathbb{R}^n$ such that

$$\nabla\eta = G \cdot \nabla A, \quad \nabla q_\alpha = G \cdot \nabla F_\alpha, \qquad \alpha = 1, ..., d.$$

(H2) The symmetric matrix

$$\nabla^2\eta(u) - G(u) \cdot \nabla^2 A(u)$$

is positive definite for all $u \in X$.

(H3) The vector $A(u)$ and the fluxes $F_\alpha(u)$ are bounded by the entropy, i.e.

$$|A(u)| \leq C(\eta(u) + 1), \quad |F_\alpha(u)| \leq C(\eta(u) + 1), \qquad \alpha = 1, ..., d.$$

Define for a strong solution U taking values in a compact set $D \subset X$ the relative entropy

$$\begin{aligned}\eta(u|U) &:= \eta(u) - \eta(U) - \nabla\eta(U) \cdot \nabla A(U)^{-1}(A(u) - A(U)) \\ &= \eta(u) - \eta(U) - G(U) \cdot (A(u) - A(U))\end{aligned}$$

and the relative flux as

$$F_\alpha(u|U) := F_\alpha(u) - F_\alpha(U) - \nabla F_\alpha(U)\nabla A(U)^{-1}(A(u) - A(U))$$

for $\alpha = 1, ..., d$.

If we assume (H1) – (H3) hold and $\lim_{|u|\to\infty} \frac{A(u)}{\eta(u)} = 0$ then

$$|F_\alpha(u|U)| \leq C\eta(u|U).$$

An analogue fact under more restrictive assumptions

$$\lim_{|u|\to\infty} \frac{A(u)}{\eta(u)} = \lim_{|u|\to\infty} \frac{F_\alpha(u)}{\eta(u)} = 0,$$

was proved in Christoforou & Tzavaras 2017. Note however that this condition is satisfied for polyconvex elastodynamics but is not satisfied e.g. for compressible Euler equations.

Relations between concentration measures

Let $f(y,u)$ be a nonnegative continuous function on $Y \times \overline{X}$ and let $g(y,u)$ be a vector-valued function, also continuous on $Y \times \overline{X}$ such that

$$\lim_{|u|\to\infty} |g(y,u)| \leq C \lim_{|u|\to\infty} f(y,u).$$

Let m_f and m_g denote the concentration measures related to $f(\cdot,u_n)$ and $g(\cdot,u_n)$ respectively, where $f(\cdot,u_n)$ is a sequence bounded in L^1. Then

$$|m_g| \leq C m_f,$$

i.e. $|m_g|(A) \leq C m_f(A)$ for any Borel set $A \subset Y$.

G., Kreml, Świerczewska-Gwiazda 2017

Theorem

Let $(\nu, m_A, m_{F_\alpha}, m_\eta)$, $\alpha = 1, ..., d$, *be a dissipative measure-valued solution to* (1) *generated by a sequence of approximate solutions. Let* $U \in W^{1,\infty}(Q)$ *be a strong solution to* (1) *with the same initial data* $\eta(u_0) \in L^1(\mathbb{T}^d)$, *thus* $\nu_{0,x} = \delta_{u_0(x)}$, $m_A^0 = m_{F_\alpha}^0 = m_\eta^0 = 0$. *Then* $\nu_{t,x} = \delta_{U(x)}$, $m_A = m_{F_\alpha} = m_\eta = 0$ *and* $u = U$ *a.e. in* Q.

Examples

Compressible Euler system

$$\partial_t \rho + \mathrm{div}(\rho v) = 0,$$
$$\partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) + \nabla p(\rho) = 0.$$

The associated entropy is given by

$$\eta(\rho, v) = \frac{1}{2}\rho |v|^2 + P(\rho),$$

here the pressure potential $P(\rho)$ is related to the original pressure $p(\rho)$ through

$$P(\rho) = \rho \int_1^\rho \frac{p(r)}{r^2} dr.$$

Examples

$$A(u) = \begin{pmatrix} \rho \\ \rho v \end{pmatrix}, \quad F(u) = \begin{pmatrix} \rho v \\ v \otimes v + p(\rho) \end{pmatrix}.$$

$$\eta(\rho, v) = \frac{1}{2}\rho|v|^2 + P(\rho),$$

We show that

$$\frac{|A(u)|}{\eta(u)} \to 0$$

as $|u| \to \infty$ and

$$\frac{|F(u)|}{\eta(u)} \leq C.$$

Examples

Shallow water magnetohydrodynamics

$$\begin{aligned} \partial_t h + \operatorname{div}_x(hv) &= 0, \\ \partial_t(hv) + \operatorname{div}_x\left(hv \otimes v - hb \otimes b\right) + \nabla_x(gh^2/2) &= 0, \\ \partial_t(hb) + \operatorname{div}_x(hb \otimes v - hv \otimes b) + v \operatorname{div}_x(hb) &= 0, \end{aligned}$$

where $g > 0$ is the gravity constant, $h\colon Q \to \mathbb{R}_+$ is the thickness of the fluid, $v\colon Q \to \mathbb{R}^2$ is the velocity, $b\colon Q \to \mathbb{R}^2$ is the magnetic field.

Polyconvex elasticity

Consider the evolution equations of nonlinear elasticity

$$\left.\begin{aligned} \partial_t F &= \nabla_x \mathbf{v} \\ \partial_t \mathbf{v} &= \operatorname{div}_x \left(D_F W(F) \right) \end{aligned}\right. \quad \text{in } \mathcal{X},$$

for an unknown matrix field $F \colon \mathcal{X} \to \mathbb{M}^{k \times k}$, and an unknown vector field $\mathbf{v} \colon \mathcal{X} \to \mathbb{R}^k$. Function $W \colon \mathcal{U} \to \mathbb{R}$ is given. For many applications, $\mathcal{U} = \mathbb{M}_+^{k \times k}$ where $\mathbb{M}_+^{k \times k}$ denotes the subset of $\mathbb{M}^{k \times k}$ containing only matrices having positive determinant.

Extension

- This general framework will not cover systems of conservation laws, which may fail to be hyperbolic, typically incompressible inviscid systems.
- We propose an extension of this framework to cover the case of incompressible fluids, in case of which the assumption that ∇A is a nonsingular matrix is not satisfied.
- We distinguish from the flux the part L (Lagrange multiplier) which is perpendicular to the vector $G(U)$ (which coincides with the gradient of the entropy of the strong solution in the case $A = \mathrm{Id}$).
- Thus we assume that there exists a subspace Y, such that $G(U) \in Y$ and $L \in Y^{\perp}$, where U is a strong solution to the considered system.

Let us then consider a system in the following form

$$\partial_t A(u) + \partial_\alpha F_\alpha(u) + L = 0.$$

Examples covered by our theory:

- incompressible Euler
- incompressible magnetohydrodynamics
- inhomogeneous incompressible Euler
- incompressible inhomogeneous magnetohydrodynamics

Thank you for your attention