Well-posedness for a toy nonlinear model in kinetic theory

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The motivation in kinetic theory

Boltzmann equation (Maxwell 1867, Boltzmann 1872):

$$\partial_t f + v \cdot \nabla_x f = Q(f, f) \qquad \text{on } f = f(x, v, t) \geq 0$$

Nonlinear PDE with Q bilinear integral operator acting only along v
For long-distance interactions Q has fractional ellipticity in v
Limit case (electrons in plasmas) *Landau-Coulomb operator*

$$Q(f, f) = \nabla_v \cdot \big(A[f] \nabla_v f + B[f] f \big)$$

$$\text{with } \begin{cases} A[f](v) = a \displaystyle\int_{\mathbb{R}^3} \left(I - \frac{w}{|w|} \otimes \frac{w}{|w|} \right) |w|^{-1} f(t, x, v - w) \, \mathrm{d}w, \\ B[f](v) = b \displaystyle\int_{\mathbb{R}^3} |w|^{-3} \, w \, f(t, x, v - w) \, \mathrm{d}w \end{cases}$$

Global well-posedness major open mathematical problem

Questions

- Question 1: regularity of solutions to the Landau-Coulomb equation with bounded hydrodynamical fields?
- Toy model: Quadratic nonlinearity and preserves mass $\int_{x,v} f$

$$\partial_t f + v \cdot \nabla_x f = \rho[f] \nabla_v \cdot (\nabla_v f + vf) \quad \text{with} \quad \rho[f] := \int_v f$$

- Question 2: Global well-posedness and regularity for this toy model
- Follow De Giorgi-Nash's strategy on Hilbert's 19th problem
- Question 3: L^∞ bound and Hölder regularity for solutions to

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f \quad (+\text{source})$$

  with A and B rough and $\lambda < A < \Lambda$ (partial answer to question 1)
- Question 4: Harnack inequality for such equation?

A regularity result on the Landau-Coulomb equation

Theorem (Golse-Imbert-CM-Vasseur)

Let f be an essentially bounded weak solution in $B_1 \times B_1 \times (-1, 0]$ so that

$$\begin{cases} M_1 \le M(x,t) = \displaystyle\int_{\mathbb{R}^d} f(x,v,t)\,\mathrm{d}v \le M_0 & \text{(local mass)} \\ E(x,t) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^d} f(x,v,t)|v|^2\,\mathrm{d}v \le E_0 & \text{(local energy)} \\ H(x,t) = \displaystyle\int_{\mathbb{R}^d} f(x,v,t)\ln f(x,v,t)\,\mathrm{d}v \le H_0 & \text{(local entropy)} \end{cases}$$

holds in this domain, then f is α-Hölder continuous with respect to $(x, v, t) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times (-\frac{1}{2}, 0]$ and

$$\|f\|_{C^\alpha(B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times (-\frac{1}{2},0])} \le C\left(\|f\|_{L^2(B_1 \times B_1 \times (-1,0])} + \|f\|^2_{L^\infty(B_1 \times B_1 \times (-1,0])}\right)$$

The De Giorgi-Nash-Moser theory (1)

- Hilbert's 19th problem: analytic regularity of minimizers u of an energy functional $\int_U L(\nabla u)\,\mathrm{d}x$, where $L : \mathbb{R}^d \to \mathbb{R}$ Lagrangian satisfies growth, smoothness and convexity conditions
- Euler-Lagrange equations for the minimizers take the form

$$\nabla \cdot \Big[\nabla L(\nabla u)\Big] = 0 \quad \text{i.e.} \quad \sum_{ij} \underbrace{\big[(\partial_{ij} L)(\nabla u)\big]}_{b_{ij}} \partial_{ij} u = 0$$

- Dirichlet energy $L(p) = |p|^2$, minimal surfaces $L(p) = \sqrt{1 + |p|^2}$
- With suitable assumptions on L and the domain, control of ∇u
- However existence-uniqueness-regularity requires more: if $u \in C^{1,\alpha}$ with $\alpha > 0$ then $b_{ij} \in C^\alpha$ and Schaüder estimates imply $u \in C^{2,\alpha}$ (then bootstrap yields higher regularity. . .)

The De Giorgi-Nash-Moser theory (2)

- Equation on derivative $f := \partial_k u$ (divergence form):

$$\sum_{ij} \partial_i \left[\underbrace{(\partial_{ij} L)(\nabla u)}_{a_{ij}} \partial_j f \right] = 0$$

- De Giorgi 1956 – Nash 1958: with controls (but no regularity) on a_{ij} then $f = \nabla u$ is Hölder (Nash considered the parabolic case)
- Proof of De Giorgi: (1) iterative gain of integrability (2) "isoperimetric argument" to control oscillations
- Proof of Moser 1964: (1) iterative gain of integrability (2) relating positive and negative Lebesgue norms by studying $g := \ln f$
- We use the De Giorgi-Moser iteration and the control of oscillations *à la* De Giorgi, but hypoelliptic-like structure creates new difficulties
- (Non-divergence theory by Krylov-Safonov not considered here)

Hörmander's theory of hypoellipticity (1)

- Starting point: 3 pages note of Kolmogorov Annals of Math. 1934 *"Zufällige Bewegungen (Zur Theorie der Brownschen Bewegung)"*
- This paper considered dimension $d = 1$ transport with constant drift and diffusion (thus sometimes called "Kolmogorov equation")

$\partial_t f + v \cdot \partial_x f \; (+b \partial_v f) = a \partial_v^2 f$ and fundamental solutions from δ_{x_0, v_0}

$$\frac{1}{3a\pi^2 t^2} \exp\left\{ -\frac{1}{\pi^2 a} \left(\frac{|v - v_0|^2}{t} + \frac{3|x - x_0 - t v_0|^2}{t^3} - \frac{3(x - x_0 - t v_0) \cdot (v - v_0)}{t^2} \right) \right\}$$

Hörmander's theory of hypoellipticity (2)

- Hörmander 1967's seminal paper starts from observing the regularisation of this fundamental solution and builds a general theory based on commutator estimates
- Regularisation Gevrey instead of analytic for parabolic equations
- Simpler case when no first order part and missing directions of diffusion ("Hörmander type I"): DGNM theory already extended
- Hörmander original theory is local but recently global estimates derived under the impulsion of hypocoercivity
- Example of commutator estimates in a (very) simple case:

$$\partial_t f + Bf + A^* A f = 0, \quad B = v \cdot \partial_x, \quad A = \partial_v$$

$$[A, B] = C = \partial_x, \quad \frac{\mathrm{d}}{\mathrm{d}t} \langle Af, Cf \rangle = -\| Cf \|^2 + \dots$$

The toy model (question 2)

Toy nonlinear model for the Landau-Coulomb equation in $x \in \mathbb{T}^d$:

$$\partial_t f + v \cdot \nabla_x f = \rho[f] \nabla_v \cdot (\nabla_v f + v f), \qquad \rho[f](t,x) := \int f(t,x,v) \mathrm{d}v$$

Theorem (Imbert-CM)

This equation is globally well-posed for $f_{in} \in H^k$, $k \geq d/2$, with $C_1 \mu \leq f_{in} \leq C_2 \mu$, and the unique solutions are C^∞ for positive times.

Goal: developping a methodology for future study

(1) Blow-up criterion by energy estimate with interpolation

** blow-up controlled by pointwise v-derivative **

(2) Integral-to-pointwise bounds (iteration or barrier)

(3) Hölder regularity (oscillation)

(4) Schauder estimate (hypoelliptic trajectorial estimates)

Pointwise control

- Freeze ρ then solutions f to $\partial_t f + v \cdot \nabla_x f = \rho \nabla_v \cdot (\nabla_v f + vf)$ preserve sign (e.g. positive/negative parts are sub-solutions)
- Linearity of the equation and μ steady state implies that if $C_1 \mu \leq f_{in}(\cdot, \cdot) \leq C_2 \mu$ then $C_1 \mu \leq f(t, \cdot, \cdot) \leq C_2 \mu$
- Hence L^∞ bound free for this toy model without De Giorgi-Nash
- If solution satisfies $C_1 \mu \leq f(t, \cdot, \cdot) \leq C_2 \mu$ then bounds of ellipticity on the coefficient: $C_1 \leq \rho(t, \cdot) \leq C_2$
- The latter bound opens the way for the study of Hölder regularity along the line of our previous theorem

Energy estimates

Denote $\mu(v) := (2\pi)^{-d/2} e^{-|v|^2/2}$ and change unknown $g := f\mu^{-1/2}$:

$$\partial g + v \cdot \nabla_x g = r[g] L[g] \quad \text{with} \quad r[g] := \int_v g \mu^{1/2} \, \mathrm{d}v$$

$$\text{and} \quad L[g] := \mu^{-1/2} \nabla_v (\mu \nabla_v (\mu^{-1/2} g)) = \left(\Delta_v g + \frac{d}{2} g - \frac{|v|^2}{4} g \right)$$

Natural space of symmetry: $L^2(\mathrm{d}x\,\mathrm{d}v)$.
Denote $h := \mu^{1/2} \nabla_v (\mu^{-1/2} g)$, and write energy estimates:
At the zero-th derivative level:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{x,v} |g|^2 \leq -C_1 \int_{x,v} |h|^2$$

where we used $\int L[g] g = \int \mu^{-1/2} \nabla_v (\mu \nabla_v (\mu^{-1/2} g)) g = -\int |h|^2$

Blow-up criterion I

Study high-order v derivative for $\ell \geq 1$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{x,v}|\partial_{v_i}^{\ell} g|^2 = -\ell\int_{x,v}(\partial_{v_i}^{\ell-1}\partial_{x_i} g)\partial_{v_i}^{\ell} g - \int_{x,v} r[g]\left|\nabla_v\left(\frac{\partial_{v_i}^{\ell} g}{\sqrt{\mu}}\right)\right|^2\mu$$
$$+\frac{1}{4}\binom{\ell}{1}\int_{x,v} r[g]\left|\partial_{v_i}^{\ell-1} g\right|^2 + \frac{1}{2}\binom{\ell}{2}\int_{x,v} r[g]\left|\partial_{v_i}^{\ell-1} g\right|^2$$

Using $\int_{x,v}(\partial_{v_i}^{\ell-1}\partial_{x_i} g)\partial_{v_i}^{\ell} g \lesssim \int_{x,v}|\partial_{v_i}^{\ell} g|^2 + \int_{x,v}|\partial_{x_i}^{\ell} g|^2$ and the control $r[g] \lesssim 1$ this yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{x,v}|\partial_{v_i}^{\ell} g|^2 \lesssim_k \|g\|^2_{H^{\ell}_{x,v}}$$

Blow-up criterion II

Study high-order x derivatives for $k > d/2$: since x-derivatives commute with the operators $v \cdot \nabla_x$ and Fokker-Planck

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{x,v} |\partial^k_{x_i} g|^2 = \sum_{0 \le \beta \le k} \binom{k}{\beta} \int_{x,v} \partial^{k-\beta}_{x_i} r[g] \partial^{\beta}_{x_i} L[g] \partial^k_{x_i} g$$

$$\lesssim_k -C_1 \int_{x,v} |\partial^k_{x_i} h|^2 + \sum_{0 \le \beta_i < k} \int_{x,v} |\partial^{k-\beta_i}_{x_i} r[g]|.|\partial^{\beta_i}_{x_i} h||\partial^k_{x_i} h|$$

where we used

$$\int L[\partial^*_x g] \partial^{**}_x g = \int \mu^{-1/2} \nabla_v (\mu \nabla_v (\mu^{-1/2} \partial^*_x g)) \partial^{**}_x g = -\int \partial^*_x h \partial^{**}_x h$$

Blow-up criterion III

Standard interpolation: given $\beta < k$, for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ s.t.

$$\left\| \partial_x^{k-\beta} r \partial_x^\beta h \right\|_{L^2(\mathbb{T}^d)} \leq \varepsilon \|r\|_{L^\infty_x(\mathbb{T}^d)} \|h\|_{H^k_x(\mathbb{T}^d)} + C_\varepsilon \|r\|_{H^k_x(\mathbb{T}^d)} \|h\|_{L^\infty_x(\mathbb{T}^d)}.$$

Using lower and upper bounds on $r[g]$ and the negative term in the previous estimate we get with ε small enough:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{x,v} |\partial^k_{x_i} g|^2 \leq -C_1 \int_{x,v} |\partial^k_{x_i} h|^2 + C_2 \varepsilon \|h\|_{H^k_x L^2_v} \left(\int_{x,v} |\partial^k_{x_i} h|^2 \right)^{1/2} + C_\varepsilon \|r\|_{H^k_x} \|h\|_{L^\infty_x L^2_v} \left(\int_{x,v} |\partial^k_{x_i} h|^2 \right)^{1/2}$$

which yields summing up to k and taking ε small:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|g\|^2_{H^k_x L^2_v} \lesssim -\frac{C_1}{2} \|h\|^2_{H^k_x L^2_v} + \|h\|^2_{L^\infty_x L^2_v} \|g\|^2_{H^k_x L^2_v}$$

Local-in-time and continuation

Local well-posedness:
Standard with $\ell = k > d/2$ and Sobolev embedding on $\|h\|^2_{L^\infty_x L^2_v}$ (either decompose then in terms of g or use time-integrability from previous negative terms)

Continuation requires a pointwise bound in x and L^2 in v on $h = \nabla_v g + (v/2)g$, that is independent of the energy being estimated. This is where we shall use the extension of the De Giorgi-Nash regularisation theory:
Hölder regularisation first but since we need the pointwise control of a full derivative we shall develop hypoelliptic *Schauder estimates*.

Recall on the regularity theory

Theorem (Golse-Imbert-CM-Vasseur)

$$\textit{(Simplified) Equation} \quad \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A(t,x,v)\nabla_v f)$$

where the $d \times d$ symmetric matrix A satisfies the ellipticity condition $0 < \lambda Id \leq A \leq \Lambda Id$ but is, besides that, merely measurable.
We define for $z = (t,x,v)$ the cube $Q_r(z) = B_{r^3}(x) \times B_r(v) \times (t - r^2, t]$.
Then for $0 < r_1 < r_0$, if f is a solution in $Q_{r_0(z_0)}$ then

$$\|f\|_{L^\infty(Q_{r_1}(z_0))} + \|f\|_{C^\alpha(Q_{r_1}(z_0))} \leq C\|f\|_{L^2(Q_{r_0}(z_0))}$$

where C depends on $z_0, r_0, r_1, \lambda, \Lambda, d$ and $\alpha \in (0,1)$ depends on λ, Λ, d.

Gain of L^∞ in [Pascucci-Polidoro 2004]
Related Hölder regularity results in [Wang-Zhang 2011]

Gain of integrability - The elliptic case (following Moser)

- We consider, with $f = f(v)$ and g source term nicely behaved:

$$\nabla_v \left(A(v) \nabla_v f \right) = g$$

- Core energy estimate (valid for subsolutions):

$$\|f\|_{H^1(Q_{r_1})} \lesssim \frac{1}{(r_0 - r_1)^2} \|f\|_{L^2(Q_{r_0})} + \|g\|_{L^2(Q_{r_1})}$$

- Sobolev embedding translates the gain H^1 into L^p, $p > 2$
- Iteration by applying the argument to any subsolution $f^{p/2}$, $p \geq 2$, for a sequence of radii $r_n \to r_\infty > 0$, to get finally L^∞ in Q_{r_∞}
- Uses the ellipticity of the operator in all directions $v \in \mathbb{R}^d$

The parabolic case (following Moser)

- Parabolic case (one step closer to our setting) with $f = f(v,t)$:

$$\partial_t f = \nabla_v \left(A(v,t) \nabla_v f \right)$$

- Core energy estimate:

$$\left(\int_{v \in B_{r_1}} f^2 \, \mathrm{d}v \right)_{t=T} + \int_{T-r_1^2}^{T} \int_{v \in B_{r_1}} |\nabla_v f|^2 \, \mathrm{d}v \, \mathrm{d}t \lesssim \frac{1}{(r_0 - r_1)^2} \int_{T-r_0^2}^{T} \int_{v \in B_{r_0}} f^2 \, \mathrm{d}v \, \mathrm{d}t$$

- Similar iteration argument in both variables v, t
- Again uses ellipticity of the operator in all directions $v \in \mathbb{R}^d$

Difficulties in the non elliptic case

- Coming back to our equation $\partial_t f + v \cdot \nabla_x f \le \nabla_v (A \nabla_v f)$ we derive the corresponding energy estimate:

$$\left(\int_{x \in B_{r_1^3}} \int_{v \in B_{r_1}} f^2 \, \mathrm{d}x \, \mathrm{d}v \right)_{t=T} + \int_{T-r_1^2}^{T} \int_{x \in B_{r_1^3}} \int_{v \in B_{r_1}} |\nabla_v f|^2 \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \lesssim \frac{1}{(r_0 - r_1)^2} \int_{T-r_0^2}^{T} \int_{x \in B_{r_0^3}} \int_{v \in B_{r_0}} f^2 \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t$$

- Problem 1: control only on v-gradients, not x-gradients
- Key tool in kinetic theory to remedy this: averaging lemma [Golse-Perthame-Sentis 1985]
- Problem 2: the iteration requires to work on subsolutions ($f^{p/2}$, $p > 2$) for which averaging lemma do not hold in general

Strategy

Theorem (Averaging lemma)

$$\partial_t f + v \cdot \nabla_x f = (1 - \Delta_{t,x})^{\beta} \nabla_v^k g, \quad f, g \in L^p_{t,x,v}, \quad k \geq 0, \ \beta \in (0, 1/2)$$

implies regularity (for $p > 1$) on $\int_v f \, \mathrm{d}v \in W^{s,p}_{t,x}$ *($s > 0$ small)*

- Averages "transversal" to cancellations of symbol of the hyperbolic transport operator (gain of regularity limited by order 1 of operator)
- It degenerates if RHS g not controlled $\Rightarrow$ problem for subsolutions

$$\partial_t f + v \cdot \nabla_x f \leq \nabla_v \cdot H_0 + H_1 \text{ with } H_0, H_1 \in L^2$$

- Comparison principle: $0 \leq f \leq F$ with true solution F on which energy estimate $L^2_{t,x} H^1_v$ and averaging lemma $H^s_{t,x} L^1_v$ imply $H^{s'}_{t,x,v}$ $(0 < s' < s)$ and thus some gain of integrability $L^{p>2}$ by Sobolev embedding, inherited by f

Control of oscillation: the classical theory (2)

- De Giorgi's strategy: always consider oscillation as a whole without separating controls on suprema and infima, and control decrease of oscillation when reducing the size of the cube considered
- Main Lemma of decrease of oscillations: for f solution in Q_2 with $|f| \leq 1$ then $\mathrm{osc}_{Q_{1/2}} f \leq 2 - \delta$ for some $\delta > 0$
- It implies Hölder regularity at the point at which cubes shrink
- It is implied by the following Lemma of decrease of supremum bound: for f solution in Q_2 with $|f| \leq 1$ and $|\{f \leq 0\} \cap Q_1| \geq (1/2)|Q_1|$ then $\sup_{Q_{1/2}} f \leq 1 - \delta$
- This decrease of the supremum bound follows from the *isoperimetric argument of De Giorgi* ("intermediate-value lemma")

De Giorgi's isoperimetric argument (1)

- Original statement is proved by constructive direct calculation:

Lemma (Intermediate-value)

Consider $f \in H^1$ on Q_2 with $f \leq 1$ and

$$\left| \left\{ f \geq \frac{1}{2} \right\} \cap Q_1 \right| \geq \delta_1 > 0 \quad \text{and} \quad |\{f \leq 0\} \cap Q_1| \geq \delta_2 > 0$$

then there is $\nu > 0$ depending on δ_1, δ_2 and the H^1 norm so that

$$\left| \left\{ 0 < f < \frac{1}{2} \right\} \cap Q_1 \right| \geq \nu$$

- In our setting we do not have an H^1 bound in all variables, and H^s with small $0 < s < 1/2$ seems insufficient
- We argue by contradiction for solutions to the equation

De Giorgi's isoperimetric argument (2)

Lemma (Hypoelliptic version of the intermediate-value lemma)

For all $\delta_1, \delta_2 > 0$ and $f \leq 1$ solution of our equation on Q_2 and

$$\left| \left\{ f \geq \frac{1}{2} \right\} \cap Q_1 \right| \geq \delta_1 \quad \text{and} \quad |\{f \leq 0\} \cap Q_1| \geq \delta_2$$

there is $\nu > 0$ depending on δ_1, δ_2 and the bounds on A so that

$$\left| \left\{ 0 < f < \frac{1}{2} \right\} \cap Q_1 \right| \geq \nu$$

We consider a contradiction sequence f_k, A_k (the diffusion matrix must be let depending on k, in order to prove something universal and independent of scaling-zooming)

De Giorgi's isoperimetric argument (3)

- The sequence satisfies $\lambda \mathrm{Id} \le A_k \le \Lambda \mathrm{Id}$ and $f_k \le 1$ and

$$\left| \left\{ f_k \ge \frac{1}{2} \right\} \cap Q_1 \right| \ge \delta_1 \quad \text{and} \quad |\{f_k \le 0\} \cap Q_1| \ge \delta_2$$

$$\left| \left\{ 0 < f_k < \frac{1}{2} \right\} \cap Q_1 \right| \xrightarrow[k \to \infty]{} 0$$

- The positive part f_k^+ is a subsolution bounded in L^2 and thus bounded in $L^2_{t,x} H^1_v \cap L^\infty_{t,x,v} \cap H^s_{t,x,v}$ with $\nabla_v f_k^+ \in L^2_{t,x} L^{2+\varepsilon}_v$ (previous results)
- Consider ζ smooth, $\zeta(z) = 0$ in $z \le 0$, $\zeta(z) = 1/2$ in $z \in [1/2, 1]$, then $g_k = \zeta(f_k)$ satisfies $0 \le g_k \le 1/2$ and $\left| \left\{ 0 < g_k < \frac{1}{2} \right\} \cap Q_1 \right| \to 0$
- g_k converges strongly $L^2_{t,x,v}$ in Q_1
- The ζ- error term $\displaystyle\int \zeta''(f_k) A_k \nabla_v f_k \cdot \nabla_v f_k \phi \xrightarrow[k \to \infty]{} 0$ by the gain of integrability on $\nabla_v f_k^+ = 1_{f_k \ge 0} \nabla_v f_k$

De Giorgi isoperimetric argument (4)

- We have built a solution g valued in $\{0, 1/2\}$ with

$$\left| \left\{ g \geq \frac{1}{2} \right\} \cap Q_1 \right| \geq \delta_1 \quad \text{and} \quad |\{g \leq 0\} \cap Q_1| \geq \delta_2$$

- For almost every x, by the classical isoperimetric lemma in v (using the H^1_v control) it is constant in v, and $\nabla_v g = 0$
- The equation on g_k is

$$\partial_t g_k + v \cdot \nabla_x g_k = \nabla_v h_k \quad \text{with} \quad h_k := A_k \nabla_v g_k$$

- The equation on g is

$$\partial_t g + v \cdot \nabla_x g = \nabla_v h \quad \text{where} \quad h_k = A_k \nabla_v g_k \rightharpoonup_{L^2(Q_1)} h$$

- It remains to identify the product of weak limits $h_k = A_k \nabla_v g_k \rightharpoonup h$

De Giorgi isoperimetric argument (5)

- Integrating the equation on g_k against $g_k \phi$ we have

$$\lim_{k \to \infty} \int h_k \cdot \nabla_v (g_k \phi) = \int \frac{g^2}{2} (\partial_t \phi + v \cdot \nabla_x \phi)$$

- Integrating the limit equation on g against $g\phi$ we have

$$\int h \nabla_v (g\phi) = \int \frac{g^2}{2} (\partial_t \phi + v \cdot \nabla_x \phi)$$

- Moreover since $g_k \to g$ strongly in L^2 we have

$$\int g_k \left(h_k \cdot \nabla_v \phi \right) \to \int g \left(h \nabla_v \phi \right)$$

- Since $\nabla_v g = 0$ we deduce that $\int h_k \cdot (\nabla_v g_k) \phi \to 0$

De Giorgi isoperimetric argument (6)

- We have at the same time the coercivity

$$\int h_k(\nabla_v g_k)\phi = \int A_k \nabla_v g_k \cdot \nabla_v g_k \phi \geq \frac{1}{\Lambda} \int |h_k|^2 \phi$$

  and therefore $h = 0$
- Finally we end up with $\partial_t g + v \cdot \nabla_x g = 0$ with some mass at 0 and 1/2
- The free transport equation and $\nabla_v g = 0$ implies $\nabla_x g = 0$ and $\partial_t g = 0$, which reaches a contradiction
- Note that this proof is not quantitative and a quantitative proof would be interesting *per se*: tentative statement

$$\left| \left\{ 0 < f < \frac{1}{2} \right\} \cap Q_1 \right| \geq C \left| \{ f \leq 0 \} \cap Q_1 \right|^2 \left| \{ f \geq 1/2 \} \cap Q_1 \right|^{2 - \frac{2}{1+2d}}$$

  where C depends on $\|\nabla_v f\|_{L^2_{t,x,v}}$ and $\|\partial_t + v \cdot \nabla_x f\|_{L^2_{t,x} H^{-1}_v}$

From isoperimetric estimate to supremum bound (1)

- The $L^2 \to L^\infty$ gain implies: if the L^2 mass locally is below a certain threshold related to the constant in the gain $L^2 - L^\infty$, then the supremum has to be lower than $1/2$ in a smaller cube
- We "zoom" in the upper values as long as the mass there is not below the threshold: each time it is so, the intermediate region must contain some mass by the isoperimetric argument and the number of iteration is limited by the total volume of the cube
- In a finite number of iteration we must reach a zoom scale where the threshold condition is satisfied, and therefore where we diminish the upper bound

From isoperimetric estimate to supremum bound (2)

- Iteration: $f_1 = f$ and $f_{k+1} = 2f_k - 1$ which preserves $f_k \leq 1$
- We have $|\{f_1 \leq 0\} \cap Q_1| \geq \delta_1$ with $\delta_1 = |Q_1|/2$ and it propagates: $\{f_k \leq 0\} \subset \{f_{k+1} \leq 0\}$
- We prove that for some k_0 large enough (finite) then $|\{f_{k_0} \geq 0\} \cap Q_1| \leq \delta_2$ is small enough to get $\|f_{k_0}^+\|_{L^\infty(Q_{1/2})} \leq 1/2$ by applying the gain of integrability to $f_{k_0}^+$
- This means back on f: $f \leq 1 - 2^{-1-k_0}$ on $Q_{1/2}$
- k_0 exists since an amount ν of mass necessary each time:
- As long as $|\{f_{k+1} \geq 0\} \cap Q_1| \geq \delta_2$ we have $|\{f_k \geq 1/2\} \cap Q_1| \geq \delta_2$
- With $|\{f_k \leq 0\} \cap Q_1| \geq \delta_1$ this implies $|\{0 \leq f_k \leq 1/2\} \cap Q_1| \geq \nu$
- $|Q_1| \geq |\{f_{k+1} \leq 0\} \cap Q_1| \geq |\{f_k \leq 0\} \cap Q_1| + \underbrace{|\{0 \leq f_k \leq 1/2\} \cap Q_1|}_{\geq \nu}$

Hölder regularity for the toy model

- Assume $C_1\mu \le f_{in}(\cdot,\cdot) \le C_2\mu$ from now on
- (Gain $L^2 \to L^\infty$ since g is sub-solution to $\partial_t g + v\cdot\nabla_x g = r\Delta_v g$)
- For the decrease of oscillation write

$$\partial g + v\cdot\nabla_x g = r[g]\Delta_v g + r[g]\left(\frac{d}{2}g - \frac{|v|^2}{4}g\right)$$

treat $s = r[g](gd/2 - g|v|^2/4)$ as a source term: it is L^∞ from the bound $C_1\mu^{1/2} \le g(t,\cdot,\cdot) \le C_2\mu^{1/2}$

- Deduce $g \in C^{0,\alpha}_{loc}$ (local argument in cylinders cf. $v\cdot\nabla_x$)

$$|r(x)-r(y)| = \left|\int_v g(x,v) - g(y,v)\right| \le \int_{B_R} + \int_{B_R^c} \lesssim R^a|x-y|^\alpha + e^{-R^2/2}$$

Optimise to get $r \in C^{0,\alpha'}_x$. In fact more precise localisation of the estimates based on scaling.

Pointwise norms respecting hypoelliptic scaling I

Consider the following scaling rule

$$z := (t, x, v), \qquad rz := (r^2 t, r^3 x, rv). \tag{1}$$

(changing the operator $\partial_t - v \cdot \nabla_x - \Delta_v$ by a factor r^2) and translation rule

$$z_1 \circ z_2 = (t_1, x_1, v_1) \circ (t_2, x_2, v_2) := (t_1 + t_2, x_1 + x_2 + t_2 v_1, v_1 + v_2)$$

and finally the general cylinder

$$\begin{aligned} Q_r(z_0) &:= \left\{ z : \frac{1}{r}(z_0^{-1} \circ z) \in Q_1 \right\} \\ &= \left\{ (t, x, v) : |t - t_0| \leq r^2, \ |x - x_0 - (t - t_0) v_0| \leq r^3, \ |v - v_0| \leq r \right\} \end{aligned}$$

Pointwise norms respecting hypoelliptic scaling II

Domain (open connected set) $\mathcal{Q} \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and $\alpha \in (0,1]$, a function $g : \mathcal{Q} \to \mathbb{R}$ lies in $\mathcal{C}^\alpha(\mathcal{Q})$ (hypoelliptic Hölder space) if it is bounded and there is $C > 0$ s.t.

$$\forall\, z_0 \in \mathcal{Q},\ r > 0 \ \text{ s.t. } \ Q_r(z_0) \subset \mathcal{Q}, \quad \|g - g(z_0)\|_{L^\infty(Q_r(z_0))} \le C r^\alpha.$$

The smallest such constant C is denoted by $[g]_{\mathcal{C}^\alpha(\mathcal{Q})}$. The $\mathcal{C}^\alpha$-norm of g is then $\|g\|_{\mathcal{C}^\alpha(\mathcal{Q})} := \|g\|_{L^\infty(\mathcal{Q})} + [g]_{\mathcal{C}^\alpha(\mathcal{Q})}$.

A function g lies in $\mathcal{H}^\alpha(\mathcal{Q})$ (hypoelliptic first-order space) if $h(t,x,v) := g(t, x+tv, v)$ is differentiable in t and $g(t,x,v)$ is twice differentiable in v, and $\partial_t g + v \cdot \nabla_x g$, $D^2_v g \in C^\alpha(\mathcal{Q})$:

$$\text{semi-norm: } [g]_{\mathcal{H}^\alpha(\mathcal{Q})} := [\partial_t g + v \cdot \nabla_x g]_{\mathcal{C}^\alpha(\mathcal{Q})} + [D^2_v g]_{\mathcal{C}^\alpha(\mathcal{Q})}$$

$$\text{norm: } \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})} := \|g\|_{L^\infty(\mathcal{Q})} + \|(\partial_t + v\cdot\nabla_x) g\|_{L^\infty(\mathcal{Q})} + \|D^2_v g\|_{L^\infty(\mathcal{Q})} + [g]_{\mathcal{H}^\alpha(\mathcal{Q})}$$

Hypoelliptic estimates on trajectories I

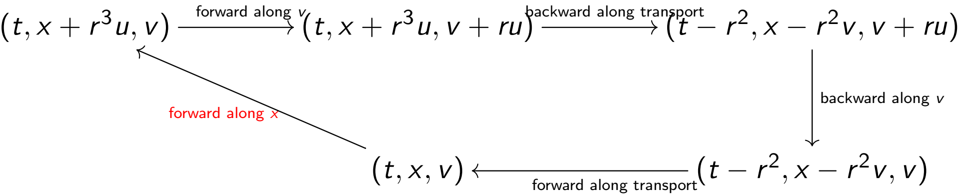
Lemma

The hypoelliptic Hölder regularity along free transport and v-diffusion allow to recover the following full (i.e. in all directions) pointwise controls:

$$[g]_{\mathcal{C}^1(\mathcal{Q})} \leq \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})},$$
$$[\nabla_v g]_{\mathcal{C}^1(\mathcal{Q})} \leq \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}.$$

The main difficulty is to obtain the Hölder regularity on the x and t directions from the higher regularity along the directions $\partial_t + v \cdot \nabla_x$ and ∇_v. This is an hypoelliptic commutator estimate in disguise. Take two points $z_1 \in Q_r(z_0) \subset \mathcal{Q}$ with $z_1 = z_0 + (0, r^3 u, 0)$ and $z_0 = (t, x, v)$ with $u \in \mathbb{S}^{d-1}$ and $r > 0$, and follow

Hypoelliptic estimates on trajectories II



(observe that all four points $(t, x + r^3 u, v)$, $(t, x + r^3 u, v + ru)$, $(t - r^2, x - r^2 v, v + ru)$, $(t - r^2, x - r^2 v, v)$ belong to $Q_r(z_0)$ with $z_0 = (t, x, v)$))

Write then the four Taylor expansions for g, and bootstrap a similar reasoning for the variations of $\nabla_v g$

Controlling regularity by oscillations

$[g]_{\mathcal{P}^\alpha}$ on domain $\mathcal{Q}$ smallest constant C s.t.

$$\forall z \in \mathbb{R}^{2d+1},\ Q_r(z_0) \subset \mathcal{Q}, \qquad \inf_{P \in \mathbb{P}} \|g - P\|_{L^\infty(Q_r(z_0))} \leq C r^{2+\alpha}$$

$$\mathbb{P} = \left\{ P(t,x,v) = a + bt + q \cdot v + \frac{1}{2} A v \cdot v \mid a, b \in \mathbb{R}, q \in \mathbb{R}^d, A \in \mathbb{R}^{d \times d} \right\}$$

For $[f]_{\mathcal{H}^\alpha} < \infty$, Taylor polynomial of f denoted $\mathcal{T}_z f$, lies in $\mathbb{P}$:

$$\mathcal{T}_z f(\bar{z}) = f(z) + \partial_t f(z)(s-t) + \nabla_v f(z) \cdot (w-v) + \frac{1}{2} D_v^2 f(z)(w-v) \cdot (w-v)$$

Similar definition $\mathcal{P}_0^\alpha$ but with the Taylor polynomial

$$\forall z \in \mathbb{R}^{2d+1},\ Q_r(z_0) \subset \mathcal{Q}, \qquad \|g - \mathcal{T}_{z_0}\|_{L^\infty(Q_r(z_0))} \leq C r^{2+\alpha}$$

Inequalities and interpolation

Lemma (Two simple inequalities comparing semi-norms)

$\mathcal{H}^\alpha \lesssim \mathcal{P}^\alpha$ *and* $\mathcal{P}^\alpha_0 \lesssim \mathcal{P}^\alpha \leq \mathcal{P}^\alpha_0$ *(proof close to the parabolic case)*

Lemma (Crucial interpolation inequalities)

Let $g \in \mathcal{H}^\alpha(\mathcal{Q})$ *with* $\alpha \in (0,1]$, *then for any* $\varepsilon > 0$ *there are* $C > 0$ *s.t.*

$$\begin{aligned}
\|(\partial_t + v \cdot \nabla_x) g\|_{L^\infty(\mathcal{Q})} &\leq \varepsilon^\alpha [g]_{\mathcal{H}^\alpha(\mathcal{Q})} + C\varepsilon^{-1} \|g\|_{L^\infty(\mathcal{Q})}, \\
\|D^2_v g\|_{L^\infty(\mathcal{Q})} &\leq \varepsilon^\alpha [g]_{\mathcal{H}^\alpha(\mathcal{Q})} + C\varepsilon^{-2} \|g\|_{L^\infty(\mathcal{Q})}, \\
[g]_{\mathcal{C}^\alpha(\mathcal{Q})} &\leq \varepsilon^{\frac{1-\alpha}{\alpha}} \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})} + C\varepsilon^{-1} \|g\|_{L^\infty(\mathcal{Q})}, \\
\|\nabla_v g\|_{\mathcal{C}^\alpha(\mathcal{Q})} &\leq \varepsilon^{\frac{1-\alpha}{\alpha}} \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})} + C\varepsilon^{-1} \|g\|_{L^\infty(\mathcal{Q})}.
\end{aligned}$$

Proof follows from the main hypoelliptic trajectorial estimate. Crucial for extending the Schauder regularity estimate from constant coefficients to variable Hölder coefficients.

Hypoelliptic Schauder estimates

The goal is to control the $\mathcal{H}^\alpha$ norm of g in terms of the $\mathcal{C}^\alpha$ norm of $\mathcal{L}g$ where $\mathcal{L}$ is the operator defining the equation.

Inspired from structure of the argument of Safonov in the parabolic case:

(1) Gradient bounds for $\mathcal{L} = \partial_t + v \cdot \nabla_x - \Delta_v$: gain of one derivative pointwise in terms of control of solution and source on a neighborhood

(2) Iteration of this gain by differentiating the equation

(3) Prove $\|g\|_{\mathcal{P}^\alpha} \lesssim \|\mathcal{L}g\|_{\mathcal{C}^\alpha}$ for constant coefficients as in the parabolic case (uses the fundamental Kolmogorov solution and semi-norm $\mathcal{P}^\alpha_0$)

(4) General constant coefficients by scaling
** here constant coefficients does not concern the transport $v \cdot \nabla_x$ **

(5) Finally extend to variable Hölder coefficients by local approximation and controlling the errors thanks to the interpolation inequalities (includes variations in x and t)

Gradient bounds

Gradient bounds

For h solution to $\partial_t h + v \cdot \nabla_x h = \Delta_v h + s$ there is C s.t.

$$|\nabla_{x,v} h(0,0,0)| \leq C(\|h\|_{L^\infty(Q_1)} + \|s\|_{L^\infty(Q_1)} + \|\nabla_{x.v} s\|_{L^\infty(Q_1)}).$$

Hypoelliptic extension of Bernstein's method: sub-solution $\mathcal{L}w \leq 0$ built as

$$w = \nu_0 h^2 - \nu_1 t + \left\{\zeta^4 (\partial_{x_i} h)^2 + \zeta^3 (\partial_{x_i} h \partial_{v_i} h) + \zeta^2 (\partial_{v_i} h)^2\right\}$$

with cutoff function ζ and maximum principle

Uses hypocoercivity-type estimates on the weight, and "default of distributivity" of $\mathcal{L}(g_1 g_2)$ that is reminiscent to "carré du champ" and Γ-calculus of Bakry-Émery

Then with no source term: $|\partial_t^n D_x^\alpha D_v^\beta g(0,0,0)| \leq \frac{C\|g\|_{L^\infty(Q_r)}}{r^{2n+3|\alpha|+|\beta|}}$

The hypoelliptic Schauder estimates

Schauder estimates for constant coefficients

Let $\alpha \in (0,1)$ and a C^∞ function $h : \mathbb{R}^{2d+1} \to \mathbb{R}$ with compact support. There exists a constant C only depending on dimension and α such that

$$[h]_{\mathcal{P}^\alpha} \lesssim [\partial_t h + v \cdot \nabla_x h - \Delta_v h]_{\mathcal{C}^\alpha}$$

Proof based on fundamental Kolmogorov solution.
Then treat general constant coefficients by scaling and finally variable coefficients by local approximation (Hölder regularity of the coefficients) controlling the errors of the form

$$[g]_{\mathcal{H}^\alpha} \lesssim [\mathcal{L}g]_{\mathcal{C}^\alpha} + \|\nabla_v g\|_{L^\infty} + \|D^2_v g\|_{L^\infty} + \|g\|_{L^\infty} + \|\partial_t g + v \cdot \nabla_x g\|_{L^\infty}$$

from the interpolation inequalities

Finally inequality $\|g\|_{\mathcal{H}^\alpha} \lesssim \|\mathcal{L}g + g\|_{\mathcal{C}^\alpha}$ and localisation by scaling...

Regularity of non-negative subsolutions revisited (1)

- With the $L^2 \to L^\infty$ gain at hand we return to the regularity of subsolutions to $\partial_t f + v \cdot \nabla_x f \leq \nabla_v \cdot H_0 + H_1$:

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot H_0 + H_1 - \mu$$

with a measure $\mu \geq 0$ and $H_0, H_1 \in L^2$ and $0 \leq f \in L^\infty_{t,x,v} \cap L^2_{t,x} H^1_v$

- We perform an estimate on the mass rather than quadratic: integrating $\int_{t,x,v} f \phi$ (for a cutoff function ϕ) yields

$$\|\mu\|_{M^1(Q_{r_1})} \lesssim \|f\|_{L^2(Q_{r_0})} + \|H_0\|_{L^2(Q_{r_0})} + \|H_1\|_{L^2(Q_{r_0})}$$

which gives a control on the size of the unknown error μ

Regularity of non-negative subsolutions revisited (2)

- We then write $-\mu = (1 - \Delta_{t,x,v})^{1/4} g$ with $g \in L^p$, $p \in (1, 2)$ by ellipticity of the fractional Laplacian

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot H_0 + H_1 + (1 - \Delta_{t,x,v})^{1/4} g$$

  Refined averaging lemma in $L^p \implies W^{s,p}_{t,x} L^1_v$ regularity for a small s
- Interpolate with L^∞ to deduce $H^{s'}_{t,x} L^1_v$ regularity $(0 < s' < s)$
- Finally we combine it with the energy estimate $L^2_{t,x} H^1_v$ to get $f \in H^{s''}_{t,x,v}$ for $0 < s'' < s' < s$
- Note that because of the interpolation with L^∞ to "bring back" the regularity obtained in L^2 this argument does not supersede the previous comparison principle in the $L^2 \to L^\infty$ iteration, which is still needed

Gain of integrability on $\nabla_v f$ (1)

- Another interesting property on subsolutions of $\partial_t f + v \cdot \nabla_x f \leq \nabla_v (A \nabla_v f)$ resembling the parabolic case

Theorem

There is $\varepsilon > 0$ universal so that

$$\int_{Q_{r_1}} |\nabla_v f|^{2+\varepsilon} \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t \lesssim_{r_0, r_1, \lambda, \Lambda, d} \left(\int_{Q_{r_0}} |\nabla_v f|^2 \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t \right)^{\frac{2+\varepsilon}{2}}$$

- It follows (iteration) from (Gehring lemma): given $q > 1$ there is θ small enough s.t. if for all $z \in \Omega$ ("almost reversed Hölder inequality")

$$⨍_{Q_r(z)} g^q \leq C_\theta \left(⨍_{Q_{8r}(z)} g \,\mathrm{d}z \right)^q + \theta ⨍_{Q_{8r}(z)} g^q \,\mathrm{d}z$$

then $\left(⨍_{Q_r} g^{q+\varepsilon} \,\mathrm{d}z \right)^{1/(q+\varepsilon)} \lesssim \left(⨍_{Q_{4r}} g^q \,\mathrm{d}z \right)^{1/q}$ for some $\varepsilon > 0$

Gain of integrability on $\nabla_v f$ (2)

- Proof of Gehring lemma is based on the following inequalities

$$(1) \qquad \int_{\mathcal{Q}_r} |\nabla_v f|^2 \,\mathrm{d}z \le \frac{C}{r^2} \int |f - \tilde{f}_{2r}|^2 \,\mathrm{d}z$$

$$(2) \qquad \sup_{t \in (T - r^2, T]} \int_{\mathcal{Q}_r^t} |f - \tilde{f}_r|^2 \,\mathrm{d}z \le C r^2 \int_{Q_{3r}} |\nabla_v f|^2 \,\mathrm{d}z$$

$$(3) \qquad \left(\int |f - \tilde{f}|^{2+\eta} \,\mathrm{d}z \right)^{1/(2+\eta)} \lesssim \left(\int |\nabla_v f|^2 \,\mathrm{d}z \right)^{1/2}$$

  proved by the energy estimate (written removing the x, v-average $\tilde{f}_{\dots}$), fractional Poincaré in x, v, Sobolev embedding and the $H^s_{x,v}$ regularity for subsolutions (averages $\tilde{f}_{\dots}$ and cubes $\mathcal{Q}_{\dots}$ along free flow)
- Boostrap the estimate on $\nabla_v f$ using $\int |f - \tilde{f}|^2$ as pivot: gain of integrability on $f - \tilde{f}$ reason for smallness of θ

Control of oscillation: the classical theory (1)

- We cannot differentiate PDE as coefficients non regular: relate local suprema and infima (oscillation), and control this difference
- Gain of integrability suggest the "L^∞" setting: Hölder regularity
- In the parabolic case, it takes time for the diffusive effect to manifest $\to$ time delays when comparing suprema and infima (cf. cubes)
- Moser's strategy: gains $L^\epsilon \to L^\infty$ and $L^{-\infty} \to L^{-\epsilon}$ and then compare L^ϵ and $L^{-\epsilon}$ by studying the equation for $g := \ln f$ and proving/using a suitable Poincaré inequality
- Moser manages to compare suprema and infima which is an independent property called Harnack inequality
- Harnack inequality implies Hölder regularity but reverse not true

The iteration

- The previous argument proves: there is $\kappa > 1$ such that for all $q > 1$:

$$\|(f^q)^\kappa\|^2_{L^2(Q_{r_1})} \le C \left(\frac{1}{(r_0 - r_1)^2} + \frac{1}{r_0(r_1 - r_0)} \right)^\kappa \|f^q\|^{2\kappa}_{L^2(Q_{r_0})}$$

- Choose $q = q_n = 2\kappa^n$ and $r_{n+1} = r_n - \frac{1}{a(n+1)^2}$ (a large enough)
- We obtain

$$\|f\|_{L^{q_{n+1}}(Q_{n+1})} \le C_{n+1}^{\frac{1}{q_{n+1}}} \|f\|_{L^{q_n}(Q_n)} \quad \text{with} \quad C_n \sim c(a^2 n^4 + bn^2)^\kappa$$

and

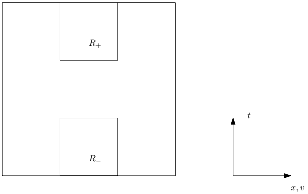
$$\prod_{n=0}^{+\infty} C_n^{\frac{1}{2\kappa^n}} < +\infty, \qquad r_n \to r_\infty > 0$$

which proves the convergence of the iteration

Harnack inequality (I)

- Harnack inequalities were first inspired from observing fundamental solutions in simple cases (non-negative solutions)
- In the elliptic case: $\sup_B f \leq \gamma \min_B f$ for some universal constant γ
- In the parabolic case a time delay must be taken into account:

$$\sup_{R_+} f \leq \gamma \min_{R_-} f \qquad \text{for some universal constant } \gamma$$

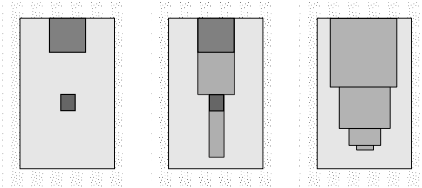


Harnack inequality (II)

- Strategies:
  (1) Moser's approach: equation on $\ln f$ and "Poincaré inequality" to bridge the gap between negative and positive Lebesgue norms
  (2) DiBenedetto's approach: Hölder regularity + reduction of oscillations + propagation-spreading of the lower bound
- Argument (2) independent of the equation
- Reduction of oscillation already proved above
- We adapt this strategy (technical variants due to scalings)
- Adaptation of strategy (1) open

Harnack inequality (III)

(a) Propagation-spreading of the minima obtained by repeatedly applying the "supremum bound" argument (on $-f$) ("doubling" property)



(b) Assume by contradiction the Harnack inequality wrong for small enough constant and construct a sequence violating the L^∞ bound lying in $Q^-[1]$, using the decrease of oscillation "backward" and (a):