Journal of Hyperbolic Differential Equations
Vol. 3, No. 1 (2006) 1–26

COUPLING EULER AND VLASOV EQUATIONS IN THE CONTEXT OF SPRAYS: THE LOCAL-IN-TIME, CLASSICAL SOLUTIONS

CÉLINE BARANGER

CEA/DIF, BP 12, 91680 Bruyères le Châtel, France
celine.baranger@cea.fr

LAURENT DESVILLETTES

Centre de Mathématiques et Leurs Applications
CNRS UMR 8536, Ecole Normale Supérieure de Cachan
61, Avenue du Président Wilson, 94235 Cachan Cedex, France
desville@cmla.ens-cachan.fr

Received 12 Apr. 2005
Accepted 7 Oct. 2005

Communicated by E. Tadmor

Abstract. Sprays are complex flows made of liquid droplets surrounded by a gas. They can be modeled by the introducing a system coupling a kinetic equation (for the droplets) of Vlasov type and a (Euler-like) fluid equation for the gas. In this paper, we prove that, for the so-called thin sprays, this coupled model is well-posed, in the sense that existence and uniqueness of classical solutions holds for small time, provided the initial data are sufficiently smooth and their support have suitable properties.

Keywords:

1. Introduction

In the framework of sprays (that is, gases in which droplets form a dispersed phase), couplings between an equation of fluid mechanics and a kinetic equation were introduced by Williams [18], cf. also [2].

In this modeling, the gas is described by macroscopic quantities depending on the time t and the position x: its density $\rho(t,x)$ and its velocity $u(t,x)$. The evolution of those quantities is ruled by a system of partial differential equations such as the Navier–Stokes or Euler (compressible or incompressible). We shall investigate here the case of the compressible Euler equation.

In order to describe the dispersed phase (the droplets), we use their distribution function in the phase space ("pdf"): it is defined as $f \equiv f(t,x,v) \geq 0$, density of droplets which at time t and point x have velocity v. This function is the solution of a kinetic equation.

We concentrate here on the so-called "thin sprays" [13], in which the coupling between the gas and the droplets is made only through a drag term (whereas in so-called thick sprays, it is also made through the volume fraction).

The system reads

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u + P(\rho) Id_N) = -\int m_p F f dv, \tag{1.2}$$

$$\partial_t f + \nabla_x \cdot (vf) + \nabla_v \cdot (Ff) = 0, \tag{1.3}$$

where m_p is the mass of one single droplet (supposed to be a constant), $m_p F$ is the drag force, and P is the pressure. We consider here a gas which is isentropic, so that P depends on ρ only (and no equation of energy appears). We restrict ourselves to the case of perfect gases, that is

$$P(\rho) = A\rho^\gamma, \quad A > 0, \quad \text{and} \quad \gamma > 1. \tag{1.4}$$

The drag force $m_p F$ is due to the resistance of the fluid to the motion of the droplets. It is possible to find in [13, 16] some analysis on the modeling of this term. One of the most standard formula is the following:

$$m_p F = \frac{1}{2}\pi r^2 \rho(t,x) C_d |u(t,x) - v|(u(t,x) - v),$$

where r is the radius of the droplets (a constant in this work) and C_d is the drag coefficient. This coefficient is sometimes taken as

$$C_d = \frac{24}{\text{Re}}\left(1 + \frac{1}{6}\text{Re}^{2/3}\right),$$

where $\text{Re} = \frac{2\rho|u-v|r}{\mu}$ is the Reynolds number and μ the dynamic viscosity of the fluid. This formula is used, for example, in [1]. We shall assume here (as in [3–6, 8]) that $C_d = \frac{24}{\text{Re}}$, so that

$$F = \frac{C\mu}{r^2 \rho_l}(u(t,x) - v),$$

where ρ_l is the (constant) density of the droplets (and where C is a generic constant). This assumption is reasonable as long as the Reynolds number is not too large. In many works (cf. [3–6, 8]), the viscosity is supposed to be constant, and the drag force becomes

$$m_p F = C(u(t,x) - v). \tag{1.5}$$

It is, however, possible to consider also that the viscosity is proportional to the density of the fluid $\mu = \rho\nu$, with ν kinematic viscosity of the fluid (cf. [7]). This leads to a drag force

$$m_p F = C\rho(t,x)(u(t,x) - v). \tag{1.6}$$

We shall make this assumption in this work. For the sake of simplicity, we shall also suppose that all the constants of the model are equal to 1 (taking other values does not lead to any difficulty).

Let us comment a little bit on the modeling: depending on the physical context, the fluid equation can be the Euler or Navier–Stokes equation (with or without turbulent viscosity), compressible or incompressible. Note that though the drag force is proportional to the viscosity of the gas, the Euler equation (and not Navier–Stokes equation) is used in some realistic simulations (cf. [11]).

Whenever the exchange of temperature is important in the study, one has to replace the isentropic Euler equation by the full Euler system (with 5 equations in dimension 3) and to add one extra variable (of temperature or internal energy) in the "pdf" of the droplets.

When the volume occupied by the droplets is not neglegted in front of the volume occupied by the gas, one has to add a new unknown ($\alpha \equiv \alpha(t,x)$) representing the volumic fraction of gas: this leads to the theory of thick sprays (cf. [13]). One takes then into account various complex phenomena for the droplets (collisions, breakups, etc.).

Here are the typical values of a computation on a thin-air mixture: the Reynolds number relative to the gas is $\mathrm{Re}_{\mathrm{gas}} = \frac{\rho}{\mu}\frac{L^2}{T} \approx 10^5$ (L and T are the typical length and time scale). Therefore, no (molecular) diffusion is taken into account in the Euler equation. The Reynolds number relative to the droplets is $\mathrm{Re} = \frac{2r\rho_l|u-v|}{\mu} \approx 1$.

The already existing mathematical studies on the fluid-kinetic coupling (in the context of sprays) concern models in which the fluid is described by its velocity u (but not its density ρ), and in which some diffusion is present. In [5], Domelevo and Roquejoffre show the existence and uniqueness of global smooth solutions of 1D Burger's viscous equation when it is coupled with a kinetic equation. For the same system, but in the polydispersed case (that is, different radiuses of droplets are present), Domelevo proves the existence of solutions in [4]. Hamdache [8] shows the global existence and the large time behavior of the solution of a coupled system of Vlasov and Stokes equations (in all dimensions). Finally, in [7], Goudon studies the existence and uniqueness of smooth solutions to the coupling between the viscous Burger's equation and a kinetic equation.

In this work, we combine two ingredients in order to obtain the existence (and uniqueness) of solutions (locally in time) to our system (that is, (1.1)–(1.4) and (1.6)). On one hand, we use the classical theory of local (in time) solutions for symmetrisable hyperbolic systems of conservation laws (cf. [10, 17], for example), and on the other hand, the theory of characteristics for the control of H^s norms of f and of its support (like in the works on the Vlasov–Poisson system, such as in [14]).

It is easy to verify that our theorem also holds when (1.6) is replaced by (1.5), or when P is not a power function of ρ (but some well-behaved function). We think that the extension to systems in which an energy equation appears should also be not too difficult. Finally, polydispersion (when droplets have different radii) could certainly be taken into account.

Note however that the well-posedness (even for small times) of systems for sprays (when no diffusion is present) is not obvious: (non diffusive) equations for diphasic flows are known to be linearly ill-posed (non hyperbolic) in certain regimes (cf. [9, 12]), and they have some similarity with equations for sprays. This similarity is most apparent for thick sprays: our method does not work in this case and the corresponding equations might be ill-posed.

When some diffusion is present at the level of the fluid equations (that is, Euler is replaced by Navier–Stokes), the conjecture is that the corresponding system for thick sprays is well-posed (diphasic equations of Navier–Stokes type are known to be linearly well-posed (cf. [15])). It looks however quite difficult, even in this case, to prove rigorously the existence of local smooth solutions.

Finally, we think that there is no hope to obtain (for general initial data) global smooth (H^s) solutions for the system we consider because the shocks created by the "Euler part" of the system have no reason to be smoothed by the "Vlasov part", the coupling being made through source terms only. For having an idea on how the smoothness disappears at the level of the Euler equations, we refer, for example, to [10].

In all the sequel, we shall use the following notations (when $h \equiv h(t,x)$, $s \in \mathbb{N}$, $T > 0$, $p \in [1, +\infty]$ and α is a multi-index):

$$\|h\|_s(t) = \sum_{|\alpha| \le s} \sqrt{\int_{\mathbb{R}^N} |D_x^\alpha(h)|^2(t,x)dx},$$
$$\|h\|_{s,T} = \max_{0 \le t \le T} \|h\|_s(t),$$
$$\|h\|_{L^p,T} = \max_{0 \le t \le T} \|h(t,\cdot)\|_{L^p}.$$

Sometimes, the same notations are used for $f \equiv f(t,x,v)$ (with x replaced by (x,v)).

In Sec. 2, we present our main result. Then, Sec. 3 is devoted to some (classical) preliminary results for the Euler and Vlasov equations taken separately. The rest of the paper presents the proof of our main theorem. In Sec. 4, we define an approximation scheme and show a priori estimates for its solution. The convergence of this scheme towards our equation is proven in Sec. 5. Finally, a few complementary results are presented in Sec. 6.

2. Main Theorem of Existence and Uniqueness

Once all the constants have been eliminated from the equations, we end up with the following system (in dimension $N \ge 1$):

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \tag{2.1}$$
$$\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u + P(\rho) Id_N) = \int_{\mathbb{R}^N} f(\rho v - \rho u) dv, \tag{2.2}$$
$$\partial_t f + \nabla_x \cdot (vf) + \nabla_v \cdot (f(\rho u - \rho v)) = 0, \tag{2.3}$$

where $P(\rho) = \rho^\gamma$, and $\gamma > 1$.

We take as initial data:

$$\forall x \in \mathbb{R}^N, \quad \rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x), \tag{2.4}$$

$$\forall (x,v) \in \mathbb{R}^N \times \mathbb{R}^N, \quad f(0,x,v) = f_0(x,v). \tag{2.5}$$

Finally, we define $G = \,]0,+\infty[\, \times \mathbb{R}^N$ as the space in which $(\rho,\ \rho u)$ will take its values. We prove the following theorem.

Theorem 2.1. *We consider* $N \in \mathbb{N}^*$, $G = \,]0,+\infty[\, \times \mathbb{R}^N$, $s \in \mathbb{N}$ *such that* $s > N/2+1$ *and* $s \geq N$, *and* G_1, G_2 *open sets of* G *such that* $\overline{G_1} \subset G_2$, *and such that* $\overline{G_1}$, $\overline{G_2}$ *are compact. Let* $(\rho_0, \rho_0\, u_0) : \mathbb{R}^N \to G_1$ *be functions satisfying* $\tilde{\rho}_0 = \rho_0 - 1 \in H^s(\mathbb{R}^N)$ *and* $u_0 \in H^s(\mathbb{R}^N)$. *Let also* $f_0 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ *be a function of* $C_c^1(\mathbb{R}^N \times \mathbb{R}^N) \cap H^s(\mathbb{R}^N \times \mathbb{R}^N)$.

Then, one can find $T > 0$ *such that there exists a solution* $(\rho, \rho u; f)$ *to system* (2.1)–(2.5) *belonging to* $C^1([0,T] \times \mathbb{R}^N, G_2) \times C_c^1([0,T] \times \mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}_+)$. *Moreover,* $\tilde{\rho}(= \rho - 1)$, $u \in L^\infty([0,T], H^s(\mathbb{R}^N))$ *and* $f \in L^\infty([0,T], H^s(\mathbb{R}^N \times \mathbb{R}^N))$.

Moreover, if $(\rho_1, \rho_1 u_1; f_1)$ *and* $(\rho_2, \rho_2 u_2; f_2)$ *belong to* $C^1([0,T] \times \mathbb{R}^N, G_2) \times C_c^1([0,T] \times \mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}_+)$, *if they satisfy* (2.1)–(2.5), *and if they are such that* $\tilde{\rho_1}$, $\tilde{\rho_2}$, u_1, $u_2 \in L^\infty([0,T], H^s(\mathbb{R}^N))$, f_1, $f_2 \in L^\infty([0,T], H^s(\mathbb{R}^N \times \mathbb{R}^N))$, *then* $\rho_1 = \rho_2$, $u_1 = u_2$ *and* $f_1 = f_2$.

Remark 2.2. This theorem shows the existence (and uniqueness) of solutions corresponding to a gas which is at rest at infinity (and of density 1), and which contains particles which are localized in a certain bounded domain.

Proof of Theorem 2.1. In a first step, we shall restrict ourselves to initial data such that $\tilde{\rho_0}$ and $\rho_0 u_0$ lie in $C_c^\infty(\mathbb{R}^N)$, while f_0 belongs to $C_c^\infty(\mathbb{R}^N \times \mathbb{R}^N)$. In Sec. 6.2, we shall explain how to regularize the initial data in order to obtain the result for all initial data described in Theorem 2.1.

Sections 3–6 are devoted to the sequel of the proof of Theorem 2.1. □

3. Preliminary Results

3.1. *Symmetrisation*

We prove in this section the following proposition, which enables to obtain a symmetrized form for the Euler equation.

Proposition 3.1. *The system* (2.1)–(2.2) *can be written under the symmetrized form*

$$S(U)\partial_t U + \sum_i (SA_i)(U)\partial_{x_i} U = S(U)b(U,f), \tag{3.1}$$

where $U = \binom{\rho}{\rho u}$,

$$S = \begin{pmatrix} P'(\rho) + \dfrac{|\rho u|^2}{\rho^2} & -\dfrac{{}^t(\rho u)}{\rho} \\ -\dfrac{\rho u}{\rho} & Id_N \end{pmatrix},$$

(*for* $i = 1, \dots, N$),

$$A_i = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\dfrac{\rho u_1 \rho u_i}{\rho^2} & \dfrac{\rho u_i}{\rho} & 0 & 0 & \dfrac{\rho u_1}{\rho} & 0 & 0 \\ -\dfrac{\rho u_2 \rho u_i}{\rho^2} & 0 & \dfrac{\rho u_i}{\rho} & 0 & \dfrac{\rho u_2}{\rho} & 0 & 0 \\ \vdots & 0 & 0 & \ddots & \vdots & 0 & \vdots \\ P'(\rho) - \dfrac{(\rho u_i)^2}{\rho^2} & 0 & 0 & 0 & \dfrac{2\rho u_i}{\rho} & 0 & 0 \\ \vdots & 0 & 0 & 0 & \vdots & \ddots & \vdots \\ -\dfrac{\rho u_N \rho u_i}{\rho^2} & 0 & 0 & 0 & \dfrac{\rho u_N}{\rho} & 0 & \dfrac{\rho u_i}{\rho} \end{pmatrix},$$

and

$$b = \begin{pmatrix} 0 \\ \displaystyle\int f(\rho v_1 - \rho u_1)dv \\ \displaystyle\int f(\rho v_2 - \rho u_2)dv \\ \vdots \\ \displaystyle\int f(\rho v_N - \rho u_N)dv \end{pmatrix}.$$

Moreover, the symmetric definite positive matrix $S(U)$ *is a smooth function of* U *satisfying*

$$cId_N \le S(U) \le c^{-1} Id_N \tag{3.2}$$

when $U \in G_1$ (*or* G_2), *for some constant* $c > 0$ (*depending on* G_1 (*or* G_2)).

Finally, all the matrices $SA_i(U)$ *are symmetric.*

Proof. The eigenvalues of S are

$$\lambda_1 = \frac{1}{2}\left(P'(\rho) + \frac{(\rho u)^2}{\rho^2} + 1\right) + \frac{1}{2}\sqrt{\left(P'(\rho) + \frac{(\rho u)^2}{\rho^2} - 1\right)^2 + 4\frac{(\rho u)^2}{\rho^2}},$$

$$\lambda_2 = \frac{1}{2}\left(P'(\rho) + \frac{(\rho u)^2}{\rho^2} + 1\right) - \frac{1}{2}\sqrt{\left(P'(\rho) + \frac{(\rho u)^2}{\rho^2} - 1\right)^2 + 4\frac{(\rho u)^2}{\rho^2}},$$

$$\lambda_i = 1 \quad \text{for } 3 \le i \le N+1.$$

A simple computation shows that the matrices SA_i are symmetric. It remains to prove that S satisfies (3.2). This is due to the fact that $\lambda_1 \leq P'(\rho)+\frac{(\rho u)^2}{\rho^2}+1$, so that $\lambda_1 \leq C$. Moreover, $\lambda_1\lambda_2 = P'(\rho)$, so $\lambda_2 \geq \frac{P'(\rho)}{P'(\rho)+\frac{(\rho u)^2}{\rho^2}+1}$ and therefore $\lambda_2 \geq C > 0$. □

3.2. *The transport equation*

Let ρ, u be any smooth functions. Then, the transport equations (2.3) and (2.5), which can be rewritten as

$$\partial_t f + v \cdot \nabla_x f + (\rho u - \rho v) \cdot \nabla_v f = N\rho f, \quad f(0,x,v) = f_0(x,v),$$

has a unique solution

$$f(t,x,v) = f_0(X(0;x,v,t), V(0;x,v,t))e^{\int_0^t N\rho(X(\tau;x,v,t),\tau)d\tau}, \tag{3.3}$$

where the characteristic curves $X(t;x,v,s)$, $V(t;x,v,s)$ are defined by

$$\begin{aligned} &\frac{dX}{dt}(t;x,v,s) = V(t;x,v,s), \\ &X(s;x,v,s) = x, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\frac{dV}{dt}(t;x,v,s) = (\rho u)(t, X(t;x,v,s)) - \rho(t, X(t;x,v,s))V(t;x,v,s), \\ &V(s;x,v,s) = v. \end{aligned} \tag{3.5}$$

It is clear thanks to (3.3) that if f_0 has a compact support, then $f(t,\cdot,\cdot)$ will also have a compact support for all t. We denote

$$X_M(t) = \sup_{\substack{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N \\ f(t,x,v)>0}} |x|, \tag{3.6}$$

and

$$V_M(t) = \sup_{\substack{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N \\ f(t,x,v)>0}} |v|. \tag{3.7}$$

In other words, Supp $f(t,\cdot,\cdot) \subset B(0, X_M(t)) \times B(0, V_M(t))$.

4. The Construction Scheme

We recall that we denote $U = {}^t(\rho, \rho u)$. According to Sec. 3.1, the system (2.1)–(2.3) can be written as

$$S(U)\partial_t U + \sum_{i=1}^N (SA_i)(U)\partial_{x_i} U = S(U)b(U,f), \tag{4.1}$$

$$\partial_t f + \nabla_x \cdot (vf) + \nabla_v \cdot (f(\rho u - \rho v)) = 0, \tag{4.2}$$

where S satisfies (3.2).

We also recall that up to Sec. 6.2, we suppose that the initial data are such that

$$U_0 - \overline{U_0} \in C_c^\infty(\mathbb{R}^N), \quad f_0 \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}^N), \tag{4.3}$$

where $\overline{U_0} = \binom{1}{0}$.

In this section, we write an inductive approximation of the system. Then, we show *a priori* estimates on a time interval $[0, T_*]$, where $T_* > 0$ is the same for all the steps of the approximation.

We note that since $s > N/2 + 1 > N/2$, $H^s(\mathbb{R}^N)$ is embedded in $L^\infty(\mathbb{R}^N)$ (the inclusion being continuous). Using the fact that U_0 takes its values in G_1 (such that $\overline{G_1} \subset G_2$), we can find $R \equiv R(G_1, G_2, s, U_0) > 0$ (defined once and for all) such that for any function U, if $\|U - U_0\|_s \le R$, then U takes its values in G_2. Finally, Supp $f_0 \subset B(0, X_M(0)) \times B(0, V_M(0))$, with the notations (3.6) and (3.7).

We define by induction the quantities $\theta_k > 0$ and $(U^k,\, f^k)$ in this way:

- $\theta_0 = +\infty$, and for $t \in [0, \theta_0[$, $(U^0(t), f^0(t)) = (U_0, f_0)$.
- We now suppose that $\theta_k > 0$ is defined, together with the smooth functions U^k, f^k on the time interval $[0, \theta_k[$. We suppose moreover that $\forall t \in [0, \theta_k[, x \in \mathbb{R}^N$, one has $U^k(t, x) \in G_2$. Then, we define (U^{k+1}, f^{k+1}) on $[0, \theta_k[$ as the unique smooth solution of the linear system

$$S(U^k)\partial_t U^{k+1} + \sum_{i=1}^N (SA_i)(U^k)\partial_{x_i} U^{k+1} = S(U^k) b(U^k, f^k), \tag{4.4}$$

$$U^{k+1}(x, 0) = U_0(x), \tag{4.5}$$

$$\partial_t f^{k+1} + \nabla_x \cdot (v f^{k+1}) + \nabla_v \cdot (f^{k+1}(\rho^k u^k - \rho^k v)) = 0, \tag{4.6}$$

$$f^{k+1}(0, x, v) = f_0(x, v). \tag{4.7}$$

Finally, we introduce $\theta_{k+1} > 0$ as the supremum of the times $\theta < \theta_k$ satisfying $U^{k+1}(t, x) \in G_2$ for all $t \in [0, \theta[, x \in \mathbb{R}^N$.

Note that since the system (4.4) is linear, symmetric and has smooth coefficients (on $[0, \theta_k[$), it admits a unique smooth solution U^{k+1} (on $[0, \theta_k[$). Moreover f^{k+1} is the unique smooth solution of a linear Vlasov equation with smooth coefficients (it can be explicitly computed by the method of characteristics as in Sec. 3.2). Finally, $\theta_{k+1} > 0$ since U^{k+1} is smooth and $U_0 \in G_1$. All of this ensures that the induction is well-defined.

Then, one can observe that the support (in the x and v variables) of $U^k - \overline{U_0}$ and f^k are compact (and depend on k and $t \in [0, \theta_k[$). This property is true for $k = 0$ (thanks to (4.3)) and it can then be proven by induction. We denote by X_M^k and V_M^k the quantities X_M and V_M related to f^k (defined by (3.6) and (3.7)).

We finally define T_k as the supremum of the times $T \in [0, \theta_k[$ such that

$$\|U^k - U_0\|_{s,T} \le R, \tag{4.8}$$

$$\|f^k\|_{s,T} \le 2\|f_0\|_s, \tag{4.9}$$

$$\|f^k\|_{L^\infty,T} \le 2\|f_0\|_{L^\infty}, \tag{4.10}$$

$$\forall t \in [0,T], \quad X_M^k(t) \leq 2X_M(0), \tag{4.11}$$

$$\forall t \in [0,T], \quad V_M^k(t) \leq 2V_M(0), \tag{4.12}$$

and such that $T_{k+1} \leq T_k$.

It is clear (by induction) that for all $k \in \mathbb{N}$, $T_k > 0$. We now prove the decisive a priori estimate, namely the existence of $T_* > 0$ such that $\forall k \in \mathbb{N}$, $T_k \geq T_*$.

Proposition 4.1. *We consider initial data such that* (4.3) *holds and define the sequences* θ_k, U^k, f^k *by* (4.4)–(4.7), *and* T_k *by* (4.8)–(4.12).

Then one can find $T_* > 0$ *which depend only upon* G_1, G_2, s, U_0 *and* f_0 *such that* $\forall k \in \mathbb{N}$, $T_k \geq T_*$.

Proof. We recall that (on the time interval $[0, T_k[$) f^k and $U^k - \overline{U_0}$ are smooth and have a compact support, so that we can manipulate them (and in particular perform integrations by parts) without taking care of their behaviour at infinity.

In the sequel, we use C for any constant ($C(f_0)$ for any constant depending only on f_0, etc.). Though R depends only on G_1, G_2, s and U_0, we keep its dependence in the various constants, for the sake of readability of the proof.

The proof of Proposition 4.1 is divided in three steps. □

Step 1. For all $k \geq 0$,

$$\|\partial_t U^{k+1}\|_{s-1,T_{k+1}} \leq C(s, G_2, R, U_0, f_0). \tag{4.13}$$

Proof of Step 1.1. Using Eq. (4.4), we get for $t \in [0, T_{k+1}[$,

$$\|\partial_t U^{k+1}\|_{s-1}(t) \leq \sum_i \|A_i(U^k)\partial_{x_i} U^{k+1}\|_{s-1,T_{k+1}} + \|b(U^k, f^k)\|_{s-1,T_{k+1}}, \tag{4.14}$$

where

$$\|b(U^k, f^k)\|_{s-1,T_{k+1}} = \sum_{|\alpha| \leq s-1} \left\| \begin{pmatrix} 0 \\ D^\alpha(\rho^k \int f^k v dv) - D^\alpha(\rho^k u^k \int f dv) \end{pmatrix} \right\|_{0,T_{k+1}}.$$

We use the following (classical) result (cf. [10]): If h, $g \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $|\alpha| \leq s$, then

$$\|D^\alpha(hg)\|_{L^2} \leq C(s)(\|h\|_{L^\infty}\|g\|_s + \|g\|_{L^\infty}\|h\|_s). \tag{4.15}$$

We get (for $|\alpha| \leq s$)

$$\begin{aligned} \|D^\alpha(b(U^k, f^k))\|_0 \leq C(s)\Bigg(& \|\rho^k\|_{L^\infty} \left\| \int f^k v dv \right\|_s + \left\| \int f^k v dv \right\|_{L^\infty} \|\rho^k\|_s \\ & + \|\rho^k u^k\|_{L^\infty} \left\| \int f^k dv \right\|_s + \left\| \int f^k dv \right\|_{L^\infty} \|\rho^k u^k\|_s \Bigg). \end{aligned}$$

For $t \in [0, T_k[$, we can bound by R the H^s norm of $U^k - U_0$, so

$$\begin{aligned}\|D^\alpha(b(U^k, f^k))\|_0 \le C(s,R)\bigg(&\left\|\int f^k v dv\right\|_s + \left\|\int f^k v dv\right\|_{L^\infty} \\ &+ \left\|\int f^k dv\right\|_s + \left\|\int f^k dv\right\|_{L^\infty}\bigg).\end{aligned}$$

For $t \in [0, T_k[$, we can also bound the support of f^k, and its H^s and L^∞ norms:

$$\begin{aligned}\left\|\int f^k dv\right\|_{L^\infty} &\le \|f^k\|_{L^\infty}\left\|\int 1_{\mathrm{Supp}\, f^k} dv\right\|_{L^\infty} \\ &\le 2^N \|f^k\|_{L^\infty}(V_M^k)^N \le C(f_0), \qquad (4.16)\\ \left\|\int f^k v dv\right\|_{L^\infty} &\le 2^N \|f^k\|_{L^\infty}(V_M^k)^{N+1} \le C(f_0), \qquad (4.17)\\ \left\|\int f^k v dv\right\|_s &= \sum_{|\alpha|\le s}\sqrt{\int\left(\int v D_x^\alpha(f^k) dv\right)^2 dx} \\ &\le C(V_M^k)^{N/2+1}\|f^k\|_s \le C(f_0),\\ \left\|\int f^k dv\right\|_s &\le (V_M^k)^{N/2}\|f^k\|_s \le C(f_0).\end{aligned}$$

Finally, we obtain

$$\|b(U^k, f^k)\|_{s,T_k} \le C(s, R, f_0), \qquad (4.18)$$

and (in particular)

$$\|b(U^k, f^k)\|_{s-1,T_{k+1}} \le C(s, R, f_0). \qquad (4.19)$$

Then,

$$\|A_i(U^k)\partial_{x_i} U^{k+1}\|_{s-1,T_{k+1}} = \sum_{|\alpha|\le s-1}\|D^\alpha(A_i(U^k)\partial_{x_i}U^{k+1})\|_{0,T_{k+1}}$$

depends only on derivatives of U^{k+1} of order $\le s$. We recall that $\|U^{k+1} - U_0\|_{s,T_{k+1}} \le R$, and we use the following (classical) result (cf. [10]): if $u \mapsto g(u)$ is a smooth function on G_2 and if $x \mapsto u(x)$ is a continuous function with values in G_2 such that $u \in L^\infty(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$, then for $|\alpha| \le s$ (and $s \ge 1$),

$$\|D^\alpha g(u)\|_{L^2} \le C(s) \sup_{u\in G_2}\sup_{|\beta|\le s-1}|D^\beta g(u)|\,\|u\|_{L^\infty}^{s-1}\|u\|_s. \qquad (4.20)$$

We obtain (for $|\alpha| \le s-1$) thanks to the Sobolev embedding $H^{s-1}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$,

$$\begin{aligned}\|D^\alpha(A_i(U^k)\partial_{x_i}U^{k+1})\|_{0,T_{k+1}} &\le \|D^\alpha((A_i(U^k) - A_i(\overline{U_0}))\partial_{x_i}U^{k+1})\|_{0,T_{k+1}} \\ &\quad + \|A_i(\overline{U_0})D^\alpha(\partial_{x_i}U^{k+1})\|_{0,T_{k+1}} \\ &\le C(s)(\|A_i(U^k) - A_i(\overline{U_0})\|_{s-1,T_{k+1}}\|\partial_{x_i}U^{k+1}\|_{L^\infty,T_{k+1}} \\ &\quad + \|A_i(\overline{U_0})\|_{L^\infty,T_{k+1}}\|\partial_{x_i}U^{k+1}\|_{s-1,T_{k+1}}) \\ &\le C(s, G_2, R, U_0). \qquad (4.21)\end{aligned}$$

Putting together (4.14), (4.19) and (4.21), we get (4.13). □

Then, we turn to the

Step 2. For all $k \geq 0$, $T \in [0, \inf(1, T_{k+1})[$,

$$\sup_{t\in[0,T]} \|U^{k+1} - U_0\|_s(t) \leq Tc(G_2)^{-1}C(s, R, G_2, U_0, f_0). \tag{4.22}$$

Proof of Step 2. We begin with an abstract (classical) lemma which enables to obtain energy estimates for solutions of symmetrized hyperbolic systems.

Lemma 4.2. *Let $S \equiv S(t, x)$, $A_i \equiv A_i(t, x)$ be smooth matrices (on $[0, T]$) such that S and SA_i are symmetric. We suppose moreover that $c\, Id_N \leq S(t, x) \leq c^{-1} Id_N$ for some $c > 0$. Then all (smooth and compactly supported) vectors $W \equiv W(t, x)$ and $F \equiv F(t, x)$ satisfying the system*

$$S\partial_t W + \sum_i SA_i\partial_{x_i} W = F, \tag{4.23}$$

$$W(x, 0) = W_0(x), \tag{4.24}$$

can be estimated in the following way: for all $t \in [0, T]$,

$$\|W\|_0(t) \leq c^{-1}\left(\|W_0\|_0 + \frac{1}{2}\left\|\partial_t S + \sum_i \partial_{x_i}(SA_i)\right\|_{L^\infty,T} \int_0^t \|W\|_0(\tau)d\tau + \int_0^t \|F\|_0(\tau)d\tau\right). \tag{4.25}$$

Proof of Lemma 4.2. By multiplying (4.23) by ${}^tW$ and by integrating over $x \in \mathbb{R}^N$, we obtain

$$\frac{1}{2}\partial_t \int {}^tWSW dx = \frac{1}{2}\int {}^tW\left(\partial_t S + \sum_i \partial_{x_i}(SA_i)\right)W dx + \int {}^tWF dx. \tag{4.26}$$

Then, (4.25) is obtained by differentiating $\int {}^tWSW dx$ and by using the estimate ${}^tWSW \geq c\,{}^tWW$. □

We now turn back to the proof of Step 2. We study $W^{k+1} = U^{k+1} - U_0$. Then, W^{k+1} is solution of

$$S(U^k)\partial_t W^{k+1} + \sum_i (SA_i)(U^k)\partial_{x_i} W^{k+1} = S(U^k)b(U^k, f^k) + H^k,$$

$$W^{k+1}(x, 0) = 0,$$

with $H^k = -\sum_i (SA_i)(U^k)\partial_{x_i} U_0$. We recall that W^{k+1} is smooth (C^∞) and has a compact support in $[0, T_{k+1}] \times \mathbb{R}^N$.

We look for an estimate on the norm H^s of W^{k+1}. We denote $W_\alpha = D^\alpha W^{k+1}$ (for $|\alpha| \leq s$). The function W_α satisfies

$$S(U^k)\partial_t W_\alpha + \sum_i (SA_i)(U^k)\partial_{x_i} W_\alpha = S(U^k)D^\alpha(S^{-1}(U^k)H^k + b(U^k, f^k)) + F_\alpha,$$

with $F_\alpha = S(U^k)\sum_i (A_i(U^k)\partial_{x_i} W_\alpha - D^\alpha(A_i(U^k)\partial_{x_i} W^{k+1}))$.

We use formula (4.26) in order to obtain

$$\begin{aligned}\frac{1}{2}\partial_t \int {}^tW_\alpha S(U^k)W_\alpha dx = \frac{1}{2}\int {}^tW_\alpha \left(\partial_t S(U^k) + \sum_i \partial_{x_i}(SA_i)(U^k)\right) W_\alpha dx \\ + \int {}^tW_\alpha S(U^k) D^\alpha (S^{-1}(U^k)H^k + b(U^k, f^k))dx \\ + \int {}^tW_\alpha F_\alpha dx.\end{aligned}$$

Up to time T_k, U^k takes its values in G_2, on which S and SA_i are smooth (more precisely, one can bound the derivatives (of any order) of those matrices by a constant depending on G_2 only). We also recall that thanks to the Sobolev inequalities, $H^{s-1}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$. We estimate

$$\begin{aligned}&\|\partial_t U^k\|_{L^\infty,T_k} \le C(s)\|\partial_t U^k\|_{s-1,T_k} \le C(s,G_2,R,U_0,f_0) \quad \text{from (4.13)},\\ &\|\partial_{x_i} U^k\|_{L^\infty,T_k} \le C(s)\|\partial_{x_i} U^k\|_{s-1,T_k}\\ &\qquad \le C(s)(\|U^k - U_0\|_{s,T_k} + \|U_0 - \overline{U_0}\|_{s,T_k}) \le C(s,R,U_0),\end{aligned}$$

so that (since $T_{k+1} \le T_k$)

$$\|\partial_t S(U^k) + \sum_i \partial_{x_i}(SA_i)(U^k)\|_{L^\infty,T_{k+1}} \le C(s,G_2,R,U_0,f_0). \tag{4.27}$$

We now use the following (classical) inequality (cf. [10]): if $h \in H^s(\mathbb{R}^N)$, $\nabla h \in L^\infty(\mathbb{R}^N)$, $g \in H^{s-1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $|\alpha| \le s$,

$$\|D^\alpha(hg) - hD^\alpha(g)\|_0 \le C(s)(\|\nabla h\|_{L^\infty}\|g\|_{s-1} + \|g\|_{L^\infty}\|h\|_s). \tag{4.28}$$

Then, for $|\alpha| \le s$,

$$\begin{aligned}A_i(U^k)\partial_{x_i}W_\alpha - D^\alpha(A_i(U^k)\partial_{x_i}W^{k+1}) = (A_i(U^k) - A_i(\overline{U_0}))D^\alpha(\partial_{x_i}W^{k+1})\\ - D^\alpha((A_i(U^k) - A_i(\overline{U_0}))\partial_{x_i}W^{k+1}),\end{aligned}$$

and (according to (4.28))

$$\begin{aligned}\|F_\alpha\|_{0,T_{k+1}} &\le \|S(U^k)\|_{L^\infty,T_{k+1}} C(s) \sum_i (\|D(A_i(U^k) - A_i(\overline{U_0}))\|_{L^\infty,T_{k+1}}\\ &\qquad \times \|\partial_{x_i}W^{k+1}\|_{s-1,T_{k+1}} + \|\partial_{x_i}W^{k+1}\|_{L^\infty,T_{k+1}}\|A_i(U^k) - A_i(\overline{U_0})\|_{s,T_{k+1}})\\ &\le C(s,G_2,R,U_0).\end{aligned} \tag{4.29}$$

According to (4.15), we get (for $|\alpha| \le s$)

$$\begin{aligned}\|S(U^k)D^\alpha(S^{-1}(U^k)H^k)\|_{0,T_{k+1}} &\le \|S(U^k)\|_{L^\infty,T_k} \sum_i \|D^\alpha(A_i(U^k)\partial_{x_i}U_0)\|_{0,T_k}\\ &\le C(s,G_2)\sum_i (\|A_i(U^k) - A_i(\overline{U_0})\|_{L^\infty,T_k}\|\partial_{x_i}U_0\|_{s,T_k}\\ &\quad + \|\partial_{x_i}U_0\|_{L^\infty,T_k}\|A_i(U^k) - A_i(\overline{U_0})\|_{s,T_k}\\ &\quad + C\|\partial_{x_i}U_0\|_{s,T_k}) \le C(s,G_2,R,U_0).\end{aligned} \tag{4.30}$$

Finally, thanks to (4.18), (and for $|\alpha| \le s$)

$$\|S(U^k)D^\alpha(b(U^k, f^k))\|_{0,T_{k+1}} \le C(G_2)C(s, R, f_0). \tag{4.31}$$

We now use estimate (4.21) for $t \in [0, T_{k+1}[$. We get

$$\begin{aligned}\|W_\alpha\|_0(t) \le c(G_2)^{-1}\Bigg[&\|W_\alpha\|_0(0) + \frac{1}{2}\left\|\partial_t S + \sum_i \partial_{x_i}(SA_i)\right\|_{L^\infty, T_{k+1}} \int_0^t \|W_\alpha\|_0(\tau)d\tau \\ &+ \int_0^t (\|F_\alpha\|_0 + \|S(U^k)D^\alpha(S^{-1}(U^k)H^k)\|_0 \\ &+ \|S(U^k)D^\alpha(b(U^k, f^k))\|_0)d\tau\Bigg].\end{aligned}$$

By using (4.27)–(4.31), we obtain

$$\begin{aligned}\|W_\alpha\|_0(t) \le c(G_2)^{-1}\Bigg[&\|W_\alpha\|_0(0) + C(s, G_2, R, U_0, f_0)\int_0^t \|W_\alpha\|_0(\tau)d\tau \\ &+ \int_0^t (C(s, G_2, R, U_0) + C(s, G_2, R, U_0) + C(G_2)C(s, R, f_0))d\tau\Bigg].\end{aligned}$$

Summing for all $|\alpha| \le s$ these estimates, we end up (for $t \in [0, T_{k+1}[$) with

$$\begin{aligned}\|W^{k+1}\|_s(t) \le c(G_2)^{-1}\Bigg(&\|W^{k+1}\|_s(0) + C(s, G_2, R, U_0, f_0) \\ &\times \int_0^t \|W^{k+1}\|_s(\tau)d\tau\Bigg) + c(G_2)^{-1}tC(s, R, G_2, U_0, f_0).\end{aligned}$$

Thanks to Gronwall's lemma, for all $t \in [0, T]$ with $T \le T_{k+1}$,

$$\begin{aligned}\|W^{k+1}\|_s(t) \le c(G_2)^{-1}(&\|W^{k+1}\|_s(0) + TC(s, R, G_2, U_0, f_0)) \\ &\times e^{c(G_2)^{-1}C(s,R,f_0,U_0,G_2)T}.\end{aligned}$$

When $T \le 1$ and since $W^{k+1}(0) = 0$, we end up with (4.22). □

We now turn to the

Step 3. The following inequalities hold for all $T \in [0, \inf(1, T_{k+1})[$,

$$\|f^{k+1}\|_{L^\infty, T} \le \|f_0\|_{L^\infty} e^{C(G_2)T}, \tag{4.32}$$

$$\|f^{k+1}\|_{s,T} \le \sqrt{2}e^{C(G_2)T}\|f_0\|_s + TC(s, R, f_0, G_2, U_0), \tag{4.33}$$

$$\sup_{t\in[0,T]} V_M^{k+1}(t) \le V_M(0)e^{C(G_2)T} + C(G_2)T, \tag{4.34}$$

$$\sup_{t\in[0,T]} X_M^{k+1}(t) \le X_M(0) + C(G_2, f_0)T. \tag{4.35}$$

Proof Step 3. Using notations similar to those of Sec. 3.2, we define the characteristics

$$\begin{aligned}\frac{dX^{k+1}}{dt}(t;x,v,s) &= V^{k+1}(t;x,v,s),\\ X^{k+1}(s;x,v,s) &= x, \end{aligned}\tag{4.36}$$

$$\begin{aligned}\frac{dV^{k+1}}{dt}(t;x,v,s) &= (\rho^k u^k)(X^{k+1}(t;x,v,s),t)\\ &\quad - \rho^k(X^{k+1}(t;x,v,s),t)V^{k+1}(t;x,v,s),\\ V^{k+1}(s;x,v,s) &= v.\end{aligned}\tag{4.37}$$

Estimate (4.32) is a direct consequence of the formula:

$$f^{k+1}(t,x,v) = f_0(X^{k+1}(0;x,v,t), V^{k+1}(0;x,v,t))e^{\int_0^t N\rho^k(X^{k+1}(\tau;x,v,t),\tau)d\tau}.$$

Then, writing in an implicit way (4.36)–(4.37), we obtain

$$\begin{aligned}X^{k+1}(t;x,v,s) &= x + \int_s^t V^{k+1}(\tau;x,v,s)d\tau,\\ V^{k+1}(t;x,v,s) &= e^{-\int_s^t \rho^k(X^{k+1}(\tau;x,v,s),\tau)d\tau} v\\ &\quad + \int_s^t e^{-\int_\tau^t \rho(X^{k+1}(\tilde\tau;x,v,s),\tilde\tau)d\tilde\tau}(\rho^k u^k)(X^{k+1}(\tau;x,v,s),\tau)d\tau\end{aligned}$$

so that

$$\begin{aligned}V^{k+1}(0;x,v,t) &= v e^{-\int_t^0 \rho^k(X^{k+1}(\tau;x,v,t),\tau)d\tau}\\ &\quad + \int_t^0 e^{-\int_\tau^0 \rho^k(X^{k+1}(\tilde\tau;x,v,t),\tilde\tau)d\tilde\tau}(\rho^k u^k)(X^{k+1}(\tau;x,v,t),\tau)d\tau.\end{aligned}$$

Then,

$$|v| \le |V^{k+1}(0;x,v,t)|e^{\int_0^t \|\rho^k\|_{L^\infty}(\tau)d\tau} + \int_0^t \|\rho^k u^k\|_{L^\infty}(\tau)e^{\int_\tau^t \|\rho^k\|_{L^\infty}(\tilde\tau)d\tilde\tau}d\tau. \tag{4.38}$$

Since

$$\begin{aligned}V_M^{k+1}(t) &= \sup_{f^{k+1}(t,x,v)>0} |v|\\ &= \sup_{f_0^{k+1}(X^{k+1}(0;x,v,t),V^k(0;x,v,t))>0} |v|\\ &\le \sup_{\substack{|X^{k+1}(0;x,v,t)|\le X_M(0)\\ |V^{k+1}(0;x,v,t)|\le V_M(0)}} |v|,\end{aligned}$$

we get (for $t \in [0,T[$, $T \in [0,T_{k+1}[$)

$$V_M^{k+1}(t) \le V_M(0)e^{\int_0^t \|\rho^k\|_{L^\infty}(\tau)d\tau} + \int_0^t \|\rho^k u^k\|_{L^\infty}(\tau)e^{\int_\tau^t \|\rho^k\|_{L^\infty}(\tilde\tau)d\tilde\tau}d\tau. \tag{4.39}$$

Then, we obtain (4.34) by noticing that $T \le 1$.

We proceed similarly for X_M^k. Using the formula for V_M^{k+1}, we get

$$x = X^{k+1}(0;x,v,t) + \int_0^t v e^{-\int_t^\tau \rho^k(X^{k+1}(s;x,v,t),s)ds} d\tau$$
$$+ \int_0^t \int_t^\tau e^{-\int_{\tilde\tau}^\tau \rho(X^{k+1}(s;x,v,t),s)ds}(\rho^k u^k)(X^{k+1}(\tilde\tau;x,v,t),\tilde\tau) d\tilde\tau d\tau,$$

so that

$$|x| \le |X^{k+1}(0;x,v,t)| + |V^{k+1}(0;x,v,t)| \int_0^t e^{\int_0^\tau \|\rho^k\|_{L^\infty}(s)ds} d\tau$$
$$+ \int_0^t \|\rho^k u^k\|_{L^\infty}(\tau) e^{\int_\tau^t \|\rho^k\|_{L^\infty}(s)ds} d\tau \int_0^t e^{\int_\tau^t \|\rho^k\|_{L^\infty}(s)ds} d\tau$$
$$+ \int_0^t \|\rho^k u^k\|_{L^\infty}(\tau) \int_0^\tau e^{\int_{\tilde\tau}^\tau \|\rho^k\|_{L^\infty}(s)ds} d\tilde\tau d\tau,$$

and (for $t \in [0,T[$, $T \in [0,T_{k+1}[$)

$$X_M^{k+1}(t) \le X_M(0) + V_M(0) \int_0^t e^{\int_0^\tau \|\rho^k\|_{L^\infty}(s)ds} d\tau$$
$$+ \int_0^t \|\rho^k u^k\|_{L^\infty}(\tau) e^{\int_\tau^t \|\rho^k\|_{L^\infty}(s)ds} d\tau \int_0^t e^{\int_\tau^t \|\rho^k\|_{L^\infty}(s)ds} d\tau$$
$$+ \int_0^t \|\rho^k u^k\|_{L^\infty}(\tau) \int_0^\tau e^{\int_{\tilde\tau}^\tau \|\rho^k\|_{L^\infty}(s)ds} d\tilde\tau d\tau. \quad (4.40)$$

Using the fact that $T \le 1$, we get (4.35).

It remains to prove estimate (4.33) on the H^s norm of f^{k+1}. For this, we take α derivatives with respect to x and β derivatives with respect to v (with $|\alpha|+|\beta| \le s$) of f^{k+1}. As a result, we get a coupled system of Vlasov equations whose characteristic fields are the same as those of the equation satisfied by f^{k+1}, and whose right-hand side contains derivatives of order $\le |\alpha| + |\beta|$ of f^{k+1}, and derivatives of order $\le |\alpha|$ of U^k.

The equation for a derivative of arbitrary order of f^{k+1} writes:

$$D_x^\alpha D_v^\beta f^{k+1}(t,x,v) = D_x^\alpha D_v^\beta f_0(X^k(0;x,v,t), V^k(0;x,v,t))$$
$$\times e^{\int_0^t C_{\alpha,\beta}\rho^k(X(\tau;x,v,t),\tau)d\tau} + \int_0^t e^{\int_\tau^t C_{\alpha,\beta}\rho^k(X(\tilde\tau;x,v,t),\tilde\tau)d\tilde\tau}$$
$$\times B_{\alpha,\beta}(X^k(\tau;x,v,t), V^k(\tau;x,v,t))d\tau,$$

where $C_{\alpha,\beta} \in \mathbb{R}$ (in fact it is always some positive multiple of N) and where $B_{\alpha,\beta}$ is a linear combination of the $D_x^{\alpha'} D_v^{\beta'} f^{k+1}$, with $|\alpha'|+|\beta'| \le |\alpha|+|\beta|$, whose coefficients are themselves linear combinations of v and $D_x^\gamma U^k$, with $|\gamma| \le |\alpha|$.

We write down explicitly the term $B_{3,0}$, which contains the terms of highest order (of derivatives) of U^k (among those for which $|\alpha| + |\beta| \leq 3$):

$$\begin{aligned} B_{3,0} &= 3\nabla_x \rho^k \nabla_{xx} f^{k+1} - 3(\nabla_x(\rho^k u^k) - v\nabla_x \rho^k)\nabla_{xxv} f^{k+1} \\ &\quad + 3\nabla_{xx}\rho^k \nabla_x f^{k+1} - 3(\nabla_{xx}(\rho^k u^k) - v\nabla_{xx}\rho^k)\nabla_{xv} f^{k+1} \\ &\quad + \nabla_{xxx}(\rho^k) f^{k+1} - (\nabla_{xxx}(\rho^k u^k) - v\nabla_{xxx}\rho^k)\nabla_v f^{k+1}. \end{aligned} \tag{4.41}$$

We look for the L^2 norm of $D_x^\alpha D_v^\beta f^{k+1}$. We have (for $T \in]0, T_{k+1}[$ and $t \in [0, T[$)

$$\begin{aligned} &\iint (D_x^\alpha D_v^\beta f^{k+1}(t,x,v))^2 dxdv \\ &\quad \leq 2\iint (D_x^\alpha D_v^\beta f_0(X^k(0;x,v,t), V^k(0;x,v,t)))^2 \\ &\qquad \times e^{2\int_0^t C_{\alpha,\beta}\rho^k(X^k(\tau;x,v,t),\tau)d\tau} dxdv \\ &\qquad + 2\iint \left(\int_0^t e^{\int_\tau^t C_{\alpha,\beta}\rho^k(X^k(\tilde\tau;x,v,t),\tilde\tau)d\tilde\tau} \right. \\ &\qquad \left. \times B_{\alpha,\beta}(X^k(\tau;x,v,t), V^k(\tau;x,v,t))d\tau \right)^2 dxdv. \end{aligned} \tag{4.42}$$

1 In the first integral of (4.42), we use the change of variables $(x,v) \mapsto (X^k(0;x,v,t), V^k(0;x,v,t))$, whose Jacobian is $e^{\int_0^t N\rho^k(X^k(\tau;x,v,t),\tau)d\tau}$.

We obtain

$$\begin{aligned} &\iint 2(D_x^\alpha D_v^\beta f_0(X^k(0;x,v,t), V^k(0;x,v,t)))^2 e^{2\int_0^t C_{\alpha,\beta}\rho^k(X(\tau;x,v,t),\tau)d\tau} dxdv \\ &\quad \leq \iint 2(D_x^\alpha D_v^\beta f_0(X,V))^2 e^{\int_0^t C'_{\alpha,\beta}\rho^k(X(\tau;x,v,t),\tau)d\tau} dXdV \\ &\quad \leq 2e^{tC(G_2)} \|D_x^\alpha D_v^\beta f_0\|_0^2. \end{aligned} \tag{4.43}$$

We now estimate the second integral. Using Cauchy–Schwarz' inequality on the square of the time integral:

$$\begin{aligned} &2\iint \left(\int_0^t e^{\int_\tau^t C_{\alpha,\beta}\rho^k(X(\tilde\tau;x,v,t),\tilde\tau)d\tilde\tau} B_{\alpha,\beta}(X^k(\tau;x,v,t), V^k(\tau;x,v,t))d\tau \right)^2 dxdv \\ &\quad \leq 2\iint \left(\int_0^t e^{\int_\tau^t C_{\alpha,\beta}\rho^k(X(\tilde\tau;x,v,t),\tilde\tau)d\tilde\tau} d\tau \right) \\ &\qquad \times \left(\int_0^t e^{\int_\tau^t C_{\alpha,\beta}\rho^k(X(\tilde\tau;x,v,t),\tilde\tau)d\tilde\tau} B_{\alpha,\beta}^2(X^k(\tau;x,v,t), V^k(\tau;x,v,t))d\tau \right) dxdv \\ &\quad \leq 2te^{tC(G_2)} \int_0^t \iint e^{\int_\tau^t C_{\alpha,\beta}\rho^k(X(\tilde\tau;x,v,t),\tilde\tau)d\tilde\tau} \\ &\qquad \times B_{\alpha,\beta}^2(X^k(\tau;x,v,t), V^k(\tau;x,v,t)) dxdvd\tau. \end{aligned}$$

Then we use the change of variables $(x, v) \mapsto (X^k(\tau; x, v, t), V^k(\tau; x, v, t))$, whose Jacobian is $e^{\int_\tau^t N\rho^k(X(\tilde{\tau}; x,v,t),\tilde{\tau})d\tau}$. We obtain in this way

$$\begin{aligned} 2 \iint \left(\int_0^t e^{\int_\tau^t C_{\alpha,\beta}\rho^k(X(\tilde{\tau};x,v,t),\tilde{\tau})d\tilde{\tau}} B_{\alpha,\beta}(X^k(\tau;x,v,t), V^k(\tau;x,v,t))d\tau \right)^2 dxdv \\ \leq 2te^{tC(G_2)} \int_0^t \iint B_{\alpha,\beta}^2(X,V)dXdVd\tau. \end{aligned} \tag{4.44}$$

We now show that the L^∞ norms of $\nabla_x f^{k+1}$ and $\nabla_v f^{k+1}$ are bounded. We notice that

$$\begin{aligned} &\partial_t(\nabla_v f^{k+1}) + v \cdot \nabla_x(\nabla_v f^{k+1}) + (\rho^k u^k - \rho^k v) \cdot \nabla_v(\nabla_v f^{k+1}) \\ &\quad = 2N\rho^k(\nabla_v f^{k+1}) - \nabla_x f^{k+1}, \end{aligned}$$

so that

$$\begin{aligned} \nabla_v f^{k+1}(t,x,v) = {} & e^{\int_0^t 2N\rho^k(X^k(\tau;x,v,t),\tau)d\tau} \nabla_v f_0(X^k(0;x,v,t), V^k(0;x,v,t)) \\ & - \int_0^t e^{\int_\tau^t 2N\rho^k(X^k(s;x,v,t),s)ds} \\ & \times \nabla_x f^{k+1}(X^k(\tau;x,v,t), V^k(\tau;x,v,t),\tau)d\tau \end{aligned}$$

and therefore (for $t \in [0, T[$)

$$\|\nabla_v f^{k+1}\|_{L^\infty}(t) \leq \|\nabla_v f_0\|_{L^\infty} e^{C(G_2)t} + e^{C(G_2)t} \int_0^t \|\nabla_x f^{k+1}\|_{L^\infty}(\tau)d\tau.$$

With the same kind of arguments (and noticing that U^k has its derivatives in x of first order in L^∞ since it is bounded in H^s), we get (for $t \in [0, T[$)

$$\begin{aligned} \|\nabla_x f^{k+1}\|_\infty(t) \leq {} & \|\nabla_x f_0\|_\infty e^{C(G_2)t} + C(s,R)\|f_0\|_\infty te^{C(G_2)t} \\ & + C(f_0,s,R)e^{C(G_2)t} \int_0^t \|\nabla_v f^{k+1}\|_\infty(\tau)d\tau. \end{aligned}$$

Thanks to Gronwall's lemma, we obtain (for $t \in [0, T[$, $T \leq 1$):

$$\|\nabla_x f^{k+1}\|_{L^\infty}(t) + \|\nabla_v f^{k+1}\|_{L^\infty}(t) \leq C(s, R, f_0, G_2). \tag{4.45}$$

We now prove that for $\tau \in [0, T[$:

$$\iint B_{\alpha,\beta}^2(X,V)dXdV \leq C(s, R, f_0, G_2, U_0). \tag{4.46}$$

We write down in a detailed way the proof in the case $N = 3$, $s = 3$ (the most important for applications). A summary of the proof in the general case is presented in Sec. 6.3.

In this case ($N = 3$, $s = 3$), the most representative and complex term (for $|\alpha| + |\beta| \leq 3$) is $B_{3,0}$. The other terms $B_{\alpha,\beta}$ can be treated analogously. Since $3 > \frac{3}{2} + 1$, $H^3(\mathbb{R}^3)$ and $H^2(\mathbb{R}^3)$ are embedded in $L^\infty(\mathbb{R}^3)$. Moreover, $H^1(\mathbb{R}^3)$ is embedded in $L^6(\mathbb{R}^3)$ (the inclusions being continuous). We recall that on $[0, T_{k+1}[$, $\|U^k - U_0\|_3 \le R$. Therefore,

$$\begin{aligned} \|U^k\|_{L^\infty, T_{k+1}} &\le C(R, U_0), \\ \|\nabla_x U^k\|_{L^\infty, T_{k+1}} &\le C(R, U_0), \\ \|\nabla_{xx} U^k\|_{L^6, T_{k+1}} &\le C(R, U_0). \end{aligned}$$

In the same way, since $H^1(\mathbb{R}^6)$ is embedded in $L^3(\mathbb{R}^6)$, and f^{k+1} satisfies (4.9) and (4.10), we have:

$$\|\nabla_{xv} f^{k+1}\|_{L^3, T_{k+1}} \le C(f_0).$$

Moreover, thanks to (4.12), f^{k+1} has a compact support in v, given by V_M^{k+1}. We now examine each term appearing in (4.41), for some $t \in [0, T[$ (t is not explicitly written down in the sequel).

The terms containing derivatives of first order of U^k are the simplest:

$$\begin{aligned} &\|3\nabla_x \rho^k \nabla_{xx} f^{k+1} - 3(\nabla_x(\rho^k u^k) - v\nabla_x \rho^k)\nabla_{xxv} f^{k+1}\|_0 \\ &\quad \le \|\nabla_x U^k\|_{L^\infty}(\|\nabla_{xx} f^{k+1}\|_0 + (1 + V_M^{k+1})\|\nabla_{xxv} f^{k+1}\|_0) \\ &\quad \le C(R, U_0, f_0). \end{aligned}$$

Then, we treat the terms containing derivatives of second order of U^k:

$$\begin{aligned} \|\nabla_{xx} U^k \nabla_{xv} f^{k+1}\|_0 &= \left\{ \int (\nabla_{xx} U^k)^2 \left(\int (\nabla_{xv} f^{k+1})^2 dv \right) dx \right\}^{1/2} \\ &\le \left(\int (\nabla_{xx} U^k)^6 dx \right)^{1/6} \left\{ \int \left(\int (\nabla_{xv} f^{k+1})^2 dv \right)^{3/2} dx \right\}^{1/3} \\ &\le \|\nabla_{xx} U^k\|_{L^6} \left\{ \int \left(\int (\nabla_{xv} f^{k+1})^3 dv \right) \right. \\ &\quad \left. \times \left(\int 1_{Supp_v f^{k+1}} dv \right)^{1/2} dx \right\}^{1/3} \\ &\le C(R, U_0) C(f_0) (V_M^{k+1})^{1/6} \\ &\le C(R, U_0, f_0), \end{aligned} \tag{4.47}$$

and similarly

$$\|v\nabla_{xx} U^k \nabla_{xv} f^{k+1}\|_0, \quad \|\nabla_{xx} U^k \nabla_x f^{k+1}\|_0 \le C(R, U_0, f_0). \tag{4.48}$$

The last term contains derivatives of third order of U^k. We use here the L^∞ norm of f^{k+1} or $\nabla_v f^{k+1}$ as obtained in (4.45), for example:

$$\|\nabla_{xxx} U^k \nabla_v f^{k+1}\|_0 \le \|\nabla_v f^{k+1}\|_{L^\infty} \|\nabla_{xxx} U^k\|_0 \tag{4.49}$$

$$\le C(s, R, f_0, G_2) C(R, U_0). \tag{4.50}$$

Finally, we use estimates (4.47)–(4.49) in order to obtain

$$\|B_{3,0}\|_{0,T} \le C(s, R, f_0, G_2, U_0). \tag{4.51}$$

Using (4.42)–(4.44) and (4.51), we end up with the estimate (for $t \in [0, T[$):

$$\|f^{k+1}\|_{s,t}^2 \le 2e^{C(G_2)t}\|f_0\|_s^2 + 2t^2 e^{C(G_2)t} C(s, R, f_0, G_2, U_0). \tag{4.52}$$

Remembering that $T \le 1$, we get (4.33).

We now conclude the proof of Proposition 4.1. We see that if $T_* \in \,]0,1]$ satisfies $T_* c(G_2)^{-1} C(s, R, G_2, U_0, f_0) \le R$ (in (4.22)), $e^{C(G_2)T_*} \le 2$ (in (4.32)), $\sqrt{2}e^{C(G_2)T_*} \le \frac{3}{2}$ (in (4.33)), $T_* C(s, R, f_0, G_2, U_0) \le \frac{1}{2}\|f_0\|_s$ (in (4.33)), $e^{C(G_2)T_*} \le \frac{3}{2}$ (in (4.34)), $C(G_2)T_* \le \frac{1}{2}V_M(0)$ (in (4.34)), $C(G_2, f_0)T_* \le X_M(0)$ (in (4.35)), then $T_{k+1} \ge T_*$.

5. Passing to the Limit

We now pass to the limit when $k \to \infty$ in (4.4) and (4.6). As suggested in [10], we study $\|U^{k+1} - U^k\|_{0,T_{**}}$ for some $T_{**} \in \,]0, T_*[$. We show the

Proposition 5.1. *We consider initial data such that* (4.3) *holds and define the sequence* θ_k, U^k, f^k *by* (4.4)–(4.7), *and* T_* *thanks to Proposition* 4.1.

Then one can find $T_{**} \in \,]0, T_*[$, *such that* (*for* $k \ge 2$)

$$\|U^{k+1} - U^k\|_{0,T_{**}} \le \frac{1}{4}\|U^k - U^{k-1}\|_{0,T_{**}} + \frac{1}{4}\|U^{k-1} - U^{k-2}\|_{0,T_{**}}, \tag{5.1}$$

$$\|f^k - f^{k-1}\|_{0,T_{**}} \le C(G_2, s, R, f_0)\|U^{k-1} - U^{k-2}\|_{0,T**}. \tag{5.2}$$

Proof. Note first that (for $k \ge 2$), the function $U^{k+1} - U^k$ is solution of the system:

$$S(U^k)\partial_t(U^{k+1} - U^k) + \sum_i (SA_i)(U^k)\partial_{x_i}(U^{k+1} - U^k)$$
$$= b(U^k, f^k) - b(U^{k-1}, f^{k-1}) + F_k,$$

where

$$F_k = (S(U^{k-1}) - S(U^k))\partial_t U^k + \sum_i ((SA_i)(U^{k-1}) - (SA_i)(U^k))\partial_{x_i} U^k.$$

Moreover, $U^{k+1}(0,x) - U^k(0,x) = 0$.

Thanks to Lemma 4.2 (formula (4.25)), we can write (when $t \in [0, T_*[$)

$$\|U^{k+1} - U^k\|_0(t)$$
$$\le c(G_2)^{-1}\left(\frac{1}{2}\|\partial_t S(U^k) + \sum_i \partial_{x_i}(SA_i)(U^k)\|_{L^\infty, T_*}\int_0^t \|U^{k+1} - U^k\|_0(\tau)d\tau\right.$$
$$\left. + \int_0^t (\|F_k\|_0(\tau) + \|b(U^k, f^k) - b(U^{k-1}, f^{k-1})\|_0(\tau))d\tau\right).$$

We use then Gronwall's lemma. For all $t \in [0, T_*[$, according to (4.17),

$$\begin{aligned}\|U^{k+1} - U^k\|_0(t) &\leq c(G_2)^{-1} e^{\frac{c(G_2)^{-1}}{2}\|\partial_t S(U^k) + \sum_i \partial_{x_i}(SA_i)(U^k)\|_{L^\infty, T_*} T_*} \\ &\quad \times \int_0^t (\|F_k\|_0(\tau) + \|b(U^k, f^k) - b(U^{k-1}, f^{k-1})\|_0(\tau)) d\tau \\ &\leq c(G_2)^{-1} e^{c(G_2)^{-1} C(s, G_2, R, U_0, f_0) T_*} T_* \\ &\quad \times (\|F_k\|_{0,T_*} + \|b(U^k, f^k) - b(U^{k-1}, f^{k-1})\|_{0,T_*}). \end{aligned} \tag{5.3}$$

Using estimates (4.8), (4.13) and the fact that S and SA_i are smooth on G_2 (more precisely, their derivatives are bounded by a constant $C(G_2)$), we get

$$\|F_k\|_{0,T_*} \leq C(s, G_2, R, U_0) \|U^k - U^{k-1}\|_{0,T_*}. \tag{5.4}$$

Moreover, for $t \in [0, T_*[$ (and without writing t explicitly)

$$\begin{aligned}&\|b(U^k, f^k) - b(U^{k-1}, f^{k-1})\|_0 \\ &\quad \leq \left\|(\rho^k u^k - \rho^{k-1} u^{k-1}) \int f^k dv - (\rho^k - \rho^{k-1}) \int f^k v dv\right\|_0 \\ &\qquad + \left\|(\rho^{k-1} u^{k-1}) \int (f^k - f^{k-1}) dv - \rho^{k-1} \int (f^k - f^{k-1}) v dv\right\|_0 \\ &\quad \leq \left\|\int f^k dv\right\|_{L^\infty} \|\rho^k u^k - \rho^{k-1} u^{k-1}\|_0 + \left\|\int f^k v dv\right\|_{L^\infty} \|\rho^k - \rho^{k-1}\|_0 \\ &\qquad + \|\rho^{k-1} u^{k-1}\|_{L^\infty} \left\|\int (f^k - f^{k-1}) dv\right\|_0 + \|\rho^{k-1}\|_{L^\infty} \left\|\int (f^k - f^{k-1}) v dv\right\|_0.\end{aligned}$$

Then,

$$\begin{aligned}\left\|\int (f^k - f^{k-1}) dv\right\|_0 &\leq (4V_M(0))^{N/2} \|f^k - f^{k-1}\|_0 \\ &\leq C(f_0) \|f^k - f^{k-1}\|_0, \\ \left\|\int (f^k - f^{k-1}) v dv\right\|_0 &\leq C(f_0) \|f^k - f^{k-1}\|_0.\end{aligned}$$

Finally,

$$\begin{aligned}\|b(U^k, f^k) - b(U^{k-1}, f^{k-1})\|_0 &\leq C(f_0) \|U^k - U^{k-1}\|_0 \\ &\quad + C(G_2, f_0) \|f^k - f^{k-1}\|_0.\end{aligned} \tag{5.5}$$

Then, we note that

$$\begin{aligned}&\partial_t (f^k - f^{k-1}) + v \cdot \nabla_x (f^k - f^{k-1}) + (\rho^{k-1} u^{k-1} - \rho^{k-1} v) \cdot \nabla_v (f^k - f^{k-1}) \\ &\quad = N \rho^{k-1} (f^k - f^{k-1}) + \nabla_v \cdot (f^{k-1}((\rho^{k-2} u^{k-2} - \rho^{k-2} v) - (\rho^{k-1} u^{k-1} - \rho^{k-1} v))).\end{aligned}$$

Moreover, at $t = 0$, $f^k(x, v, 0) = f^{k-1}(x, v, 0) = f_0(x, v)$. So

$$\begin{aligned}(f^k - f^{k-1})(t, x, v) = \int_0^t & e^{\int_\tau^t N \rho^{k-1}(X^{k-1}(s; x, v, t), s) ds} \\ & \times B(X^{k-1}(\tau; x, v, t), V^{k-1}(\tau; x, v, t), \tau) d\tau,\end{aligned}$$

with

$$\begin{aligned}B &= \nabla_v \cdot (f^{k-1}((\rho^{k-2}u^{k-2} - \rho^{k-2}v) - (\rho^{k-1}u^{k-1} - \rho^{k-1}v)))\\ &= (\rho^{k-2}u^{k-2} - \rho^{k-1}u^{k-1}) \cdot \nabla_v f^{k-1} - (\rho^{k-2} - \rho^{k-1})(Nf^{k-1} + v \cdot \nabla_v f^{k-1}).\end{aligned}$$

Using Cauchy–Schwarz' inequality and the change of variables $(x, v) \mapsto (X^{k-1}, V^{k-1})$, we get for all $t \in [0, T_*[$,

$$\begin{aligned}\iint ((f^k - f^{k-1})(t, x, v))^2 dxdv &= \iint \left(\int_0^t e^{\int_\tau^t N\rho^{k-1}(X^{k-1}(s;x,v,t),s)ds}\right.\\ &\qquad \left.\times B(X^{k-1}(\tau; x, v, t), V^{k-1}(\tau; x, v, t), \tau)d\tau\right)^2 dxdv\\ &\le \iint \left(\int_0^t e^{\int_\tau^t N\rho^{k-1}(X^{k-1}(s;x,v,t),s)ds} d\tau\right)\\ &\qquad \times \int_0^t e^{\int_\tau^t N\rho^{k-1}(X^{k-1}(s;x,v,t),s)ds}\\ &\qquad \times B^2(X^{k-1}(\tau; x, v, t), V^{k-1}(\tau; x, v, t), \tau)d\tau dxdv\\ &\le te^{C(G_2)t} \int_0^t \iint B^2(x, v, \tau)dxdvd\tau.\end{aligned}$$

In order to bound B in L^2, we use the L^∞ bound on f^{k-1} and its derivative with respect to v obtained in (4.45).

Then,

$$\begin{aligned}\iint B^2(x, v, \tau)dxdv &\le (2V_M^{k-1}(t))^N(\|\nabla_v f^{k-1}\|_{L^\infty}\\ &\qquad + \|f^{k-1}\|_{L^\infty} + V_M^{k-1}(\tau)\|\nabla_v f^{k-1}\|_{L^\infty})^2\|U^{k-1} - U^{k-2}\|_0^2(\tau)\\ &\le C(s, R, f_0, G_2)\|U^{k-1} - U^{k-2}\|_0^2(\tau).\end{aligned}$$

Finally, for $t \in [0, T_*[$ (and remembering that $T_* \le 1$),

$$\begin{aligned}\|f^k - f^{k-1}\|_0^2(t) &\le te^{C(G_2)t}C(s, R, f_0, G_2)\int_0^t \|U^{k-1} - U^{k-2}\|_0^2(\tau)d\tau\\ &\le t^2C(s, R, f_0, G_2)\|U^{k-1} - U^{k-2}\|_{0,t}^2. \end{aligned} \tag{5.6}$$

Then, thanks to (5.5) and (5.6),

$$\begin{aligned}&\|b(U^k, f^k) - b(U^{k-1}, f^{k-1})\|_{0,T_*}\\ &\quad \le C(s, R, f_0, G_2)(\|U^k - U^{k-1}\|_{0,T_*} + \|U^{k-1} - U^{k-2}\|_{0,T_*}).\end{aligned} \tag{5.7}$$

Using now (5.3), (5.4) and (5.7), we end up (for $t \in [0, T_*]$, and $T_* \le 1$) with

$$\begin{aligned}\|U^{k+1} - U^k\|_0(t) &\le C(s, R, f_0, U_0, G_2)T_*\\ &\qquad \times (\|U^k - U^{k-1}\|_{0,T_*} + \|U^{k-1} - U^{k-2}\|_{0,T_*}).\end{aligned}$$

We choose $T_{**} \in]0, T_*[$ in such a way that

$$C(s, R, f_0, U_0, G_2) T_{**} < \frac{1}{4}.$$

We get in this way the estimate (5.1). Then, estimate (5.2) is a simple consequence of (5.6). □

We now conclude the proof of Theorem 2.1. Thanks to (5.1), we see that

$$\sum_k \|U^{k+1} - U^k\|_{0,T_{**}} < +\infty. \tag{5.8}$$

Then, the sequence U^k is a Cauchy sequence and converges towards some limit U in $L^\infty([0, T_{**}], L^2(\mathbb{R}^N))$. It is clear that $U \in C([0, T_{**}[, L^2(\mathbb{R}^N))$ (remember that for all $k \geq 0$, U^k is smooth). Moreover, thanks to (5.2) and (5.8),

$$\sum_k \|f^{k+1} - f^k\|_{0,T_{**}} < +\infty. \tag{5.9}$$

Then, the sequence f^k is a Cauchy sequence and converges in the space $L^\infty([0, T_{**}], L^2(\mathbb{R}^N \times \mathbb{R}^N))$ towards some function $f \in C([0, T_{**}[, L^2(\mathbb{R}^N \times \mathbb{R}^N)$ (remember that for all $k \geq 0$, f^k is smooth).

Since we know that $\|U^k - U_0\|_{s,T_{**}} \leq R$, we also have that U lies in $L^\infty([0, T_{**}], H^s(\mathbb{R}^N))$, and that $\|U - U_0\|_{s,T_{**}} \leq R$. In particular, $U = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}$ lies in $L^\infty([0, T_{**}], C^1(\mathbb{R}^N))$ and takes its values in G_2. For the same reason, f lies in $L^\infty([0, T_{**}], H^s(\mathbb{R}^N \times \mathbb{R}^N))$ and $\|f\|_{s,T_{**}} \leq 2\|f_0\|_s$. Since the support in v of f^k is uniformly contained in a compact set, we see that the sequences $\int f^k dv$ and $\int v f^k dv$ are bounded in $L^\infty([0, T_{**}], H^s(\mathbb{R}^N))$ and converge in $L^\infty([0, T_{**}], L^2(\mathbb{R}^N))$ towards $\int f dv$ and $\int v f dv$. Therefore, $\int f dv$ and $\int v f dv$ lie in $L^\infty([0, T_{**}], C^1(\mathbb{R}^N))$. Then, $b(U^k, f^k) = \begin{pmatrix} 0 \\ \rho^k u^k \int f^k dv - \rho^k \int v f^k dv \end{pmatrix}$ converges in $L^\infty([0, T_{**}], L^1(\mathbb{R}^N))$ towards $b(U, f)$.

By interpolation, for all $s' \in]\frac{N}{2} + 1, s[$

$$\begin{aligned} \|U^{k+1} - U^k\|_{s',T_{**}} &\leq C(s) \|U^{k+1} - U^k\|_{0,T_{**}}^{1-\frac{s'}{s}} \\ &\quad \times \left(\|U^{k+1} - U_0\|_{s,T_{**}}^{\frac{s'}{s}} + \|U^k - U_0\|_{s,T_{**}}^{\frac{s'}{s}} \right) \\ &\leq C(s, s', R) \|U^{k+1} - U^k\|_{0,T_{**}}^{1-\frac{s'}{s}}. \end{aligned}$$

So U^k converges to U in $L^\infty([0, T_{**}], H^{s'}(\mathbb{R}^N))$ for $s' \in]\frac{N}{2} + 1, s[$, and $U \in C([0, T_{**}[, H^{s'}(\mathbb{R}^N)) \subset C([0, T_{**}[, C^1(\mathbb{R}^N))$. Then, $\partial_{x_i} U^k$ converges to $\partial_{x_i} U$ in $C([0, T_{**}[\times \mathbb{R}^N)$, and $\sum_i A_i(U^k) \partial_{x_i} U^{k+1}$ converges to $\sum_i A_i(U) \partial_{x_i} U$ in $C([0, T_{**}[\times \mathbb{R}^N)$. Still by interpolation, $\int f^k dv$ and $\int v f^k dv$ converge in $L^\infty([0, T_{**}], H^{s'}(\mathbb{R}^N))$ towards $\int f dv$ and $\int v f dv$, and therefore they also converge in $C([0, T_{**}[\times \mathbb{R}^N)$. Finally, $b(U^k, f^k)$ converges towards $b(U, f)$ in $C([0, T_{**}[\times \mathbb{R}^N)$.

Passing to the limit in the sense of distributions in

$$\partial_t U^{k+1} = -\sum_i A_i(U^k)\partial_{x_i} U^{k+1} + b(U^k, f^k),$$

we get the equation

$$\partial_t U = -\sum_i A_i(U)\partial_{x_i} U + b(U, f), \tag{5.10}$$

and $\partial_t U \in C([0, T_{**}] \times \mathbb{R}^N)$, so that (5.10) is satisfied in the classical sense.

We now pass to the limit in (4.6) in the sense of distributions. We recall that ρ^k and $\rho^k u^k$ converge in $L^\infty([0, T_{**}[, L^2(\mathbb{R}^N))$ towards ρ and ρu, and that f^{k+1} converges in $L^\infty([0, T_{**}[, L^2(\mathbb{R}^N \times \mathbb{R}^N))$ towards f. We get at the end Eq. (1.9). Remembering now that the characteristics are C^1 (because $U \in C^1$) and using Eq. (3.3), we see that $f \in C^1_c([0, T_{**}[\times \mathbb{R}^N \times \mathbb{R}^N)$, so that (4.6) is satisfied in the classical sense. Passing then in the limit in (4.5), (4.7), we conclude the proof of existence of Theorem 2.1 in the case when the initial data are smooth.

6. Uniqueness, Initial Data and Higher Dimensions

6.1. *Uniqueness*

We note that uniqueness in Theorem 2.1 is a consequence of the estimates proven in Sec. 5 (more precisely, (5.1) and (5.2). Namely, if we consider two solutions $(\rho^1, \rho^1 u^1, f^1)$ and $(\rho^2, \rho^2 u^2, f^2)$ which are smooth in $[0, T]$ (and such that U^1, U^2 take their values in some compact subset of G), then we can prove that for some $T_{**} \in]0, T[$,

$$\|U^1 - U^2\|_{0,T_{**}} \le \frac{1}{4}\|U^1 - U^2\|_{0,T_{**}} + \frac{1}{4}\|U^1 - U^2\|_{0,T_{**}},$$
$$\|f^1 - f^2\|_{0,T_{**}} \le C(G_2, s, R, f_0)\|U^1 - U^2\|_{0,T_{**}}.$$

As a consequence $U_1 = U_2$ and $f_1 = f_2$ on this time interval, and (by considering the maximal interval where this identity holds) $U_1 = U_2$ and $f_1 = f_2$ on $[0, T]$.

6.2. *General initial data*

We now prove Theorem 2.1 without assuming that the initial data are in C^∞_c. We recall that our assumption is instead:

$\tilde{\rho}_0 \in H^s(\mathbb{R}^N)$, $u_0 \in H^s(\mathbb{R}^N)$ (i.e. $U_0 - \overline{U_0} \in H^s(\mathbb{R}^N)$) and $f_0 \in C^1_c(\mathbb{R}^N \times \mathbb{R}^N) \cap H^s(\mathbb{R}^N \times \mathbb{R}^N)$. We still assume that U_0 takes its values in a compact subset G_1 of G.

We begin by introducing a smoothing sequence: we choose $j \in C^\infty(\mathbb{R}^N)$ a non-negative function with support in $B(0,1)$ such that $\int j = 1$. Then, we consider $j_{\epsilon_k}(x) = (\epsilon_k)^N j(\frac{x}{\epsilon_k})$, where $\epsilon_k = 2^{-k}\epsilon_0$, and ϵ_0 will be chosen later. Finally, we define $U_0^k = j_{\epsilon_k} * U_0$, and $f_0^k = (j_{\epsilon_k} \otimes j_{\epsilon_k}) * f_0$. With such a choice, one has the estimate:

$$\sum_k \|U_0^{k+1} - U_0^k\|_0 < +\infty, \qquad \sum_k \|f_0^{k+1} - f_0^k\|_0 < +\infty. \tag{6.1}$$

The solution is then obtained as the limit of the inductive sequence (4.4)–(4.6) together with:

$$U^{k+1}(x,0) = U_0^{k+1}(x), \qquad f^{k+1}(0,x,v) = f_0^{k+1}(x,v).$$

The proof is then very close to the one presented in sections 4 and 5. We only indicate the two main (small) modifications: first, one has to prove estimate on $U^k - U_0^0$ instead of $U^k - U_0$: in order to be able to do so, one takes ε_0 small enough. Secondly, estimates (5.1) and (5.2) are replaced by

$$\begin{aligned}\|U^{k+1} - U^k\|_{0,T_{**}} &\le \frac{1}{4}\|U^k - U^{k-1}\|_{0,T_{**}} \\ &\quad + \frac{1}{4}\|U^{k-1} - U^{k-2}\|_{0,T_{**}} + C\|U_0^{k+1} - U_0^k\|_0, \\ \|f^k - f^{k-1}\|_{0,T_{**}} &\le C(G_2, s, R, f_0)\|U^{k-1} - U^{k-2}\|_{0,T**} + \|f_0^k - f_0^{k-1}\|_0.\end{aligned}$$

6.3. *Extension to dimension $N > 3$*

We explain here briefly how to prove estimate (4.33) in the case when $N \neq 3$ (or $s \neq 3$). The only estimate that we did not prove already in this general setting is (4.46) (with $|\alpha| + |\beta| \le s$). In order to do so, we notice that all terms appearing in $B_{\alpha,\beta}$ (still with $|\alpha| + |\beta| \le s$) are of the form $v^p D^m U^k D^l f^{k+1}$, with D denoting any derivative, $p \in \{0,1\}$, and $m, l \in \mathbb{N}$, $m + l \le s + 1$, $m \le s$.

First, we consider the case $l \in \{0,1\}$. Then, using (4.10) and (4.45), we have

$$\|D^l f^{k+1}\|_{L^\infty, T_{k+1}} \le C(s, R, f_0, G_2).$$

So,

$$\begin{aligned}\|v^p D^m U^k D^l f^{k+1}\|_0 &\le \|D^l f^{k+1}\|_{L^\infty, T_{k+1}} \\ &\quad \times \left(\int (D^m U^k)^2 \left(\int (v^p)^2 1_{Supp_v f^{k+1}} dv\right) dx\right)^{1/2} \\ &\le C(s, R, f_0, G_2) C(f_0) \|D^m U^k\|_0,\end{aligned}$$

and it is bounded since $\|U^k - U_0\|_s \le R$ (on $[0, T_{k+1}]$) and $m \le s$.

Secondly, we consider the case $l \ge 2$, so that $m \le s - 1$. But $\|U^k - U_0\|_s \le R$, so thanks to Sobolev inequalities,

$$\|D^m U^k\|_{L^{\frac{2N}{N-2(s-m)}}, T_{k+1}} \le C(R, U_0).$$

Then, using (4.9) and (4.10), and still thanks to Sobolev inequalities,

$$\|D^l f^{k+1}\|_{L^{\frac{2N}{N-s+l}},T_{k+1}} \le C(f_0). \tag{6.2}$$

In the previous equation, one has to replace the exponent of the Lebesgue space by ∞ if it is nonpositive.

Therefore, using the compact support (in v) of f^{k+1} given by (4.12),

$$\begin{aligned}
\|v^p\, D^m U^k D^l f^{k+1}\|_0 &= \left\{\int (D^m U^k)^2 \left(\int (v^p D^l f^{k+1})^2 dv\right) dx\right\}^{1/2} \\
&\le \left(\int (D^m U^k)^{\frac{2N}{N-2(s-m)}} dx\right)^{\frac{N-2(s-m)}{2N}} \\
&\quad \times \left\{\int \left(\int (v^p D^l f^{k+1})^2 dv\right)^{\frac{N}{2(s-m)}} dx\right\}^{\frac{s-m}{N}} \\
&\le \|D^m U^k\|_{L^{\frac{2N}{N-2(s-m)}}} \left\{\int \left(\int (D^l f^{k+1})^{\frac{N}{s-m}} dv\right)\right. \\
&\quad \times \left.\left(\int (|v^p| 1_{Supp_v f^{k+1}})^{\frac{2N}{N-2(s-m)}} dv\right)^{\frac{N}{2(s-m)}-1} dx\right\}^{\frac{s-m}{N}} \\
&\le C(R, U_0) C(f_0) \|D^l f^{k+1}\|_{L^{\frac{N}{s-m}}}.
\end{aligned}$$

Then, according to estimate (6.2), the norm in this last equation is bounded as soon as $\frac{N}{s-m} \le \frac{2N}{N-s+l}$, or, equivalently, $N + l + 2m \le 3s$. Remembering that this has to hold for m, $l \in \mathbb{N}$ such that $m + l \le s + 1$, $m \le s - 1$, we see that this is true as soon as $s \ge N$. In this way, we can prove that all the terms appearing in $B_{\alpha,\beta}$ are bounded in L^2, so that (4.46) holds.

References

[1] A. A. Amsden, P. J. O'Rourke and T. D. Butler, Kiva-II, a computer program for chemical reactive flows with sprays, Tech. report Los Alamos National Laboratory (1989).

[2] R. Caflisch and G. Papanicolaou, Dynamic theory of suspensions with brownian effects, *SIAM J. Appl. Math.* **43** (1983).

[3] J. F. Clouet and K. Domelevo, Solutions of a kinetic stochastic equation modeling a spray in a turbulent gas flow, *Math. Models Methods Appl. Sci.* **7** (1997) 239–263.

[4] K. Domelevo, Long time behaviour for a kinetic modelling of two-phase flows with thin sprays and point particles, Preprint de TMR-project "Asymptotics Methods in Kinetic Theory" (2001).

[5] K. Domelevo and J.-M. Roquejoffre, Existence and stability of travelling waves solutions in a kinetic model of two phase flows, *Commun. Partial Differential Equations* **24**(1–2) (1999) 61–108.

[6] K. Domelevo and M.-H. Vignal, Limites visqueuses pour des systèmes de type Fokker-Planck-Burgers unidimensionnels, *C. R. Acad. Sci. Paris Sér. I Math.* **332**(9) (2001) 863–868.

[7] T. Goudon, Asymptotic problems for a kinetic model of two-phase flow, *Proc. Roy. Soc. Edinburgh Sect. A* **131**(6) (2001) 1371–1384.

[8] K. Hamdache, Global existence and large time behaviour of solutions for the Vlasov-Stokes equations, *Japan J. Indust. Appl. Math.* **15**(1) (1998) 51–74.

[9] B. L. Keyfitz, R. Sanders and M. Sever, Lack of hyperbolicity in the two-fluid model for two-phase incompressible flow, *Discrete Contin. Dynam. Syst.* **B3** (2003) 541–563.

[10] A. Majda, *Compressible Fluid Flows and Systems of Conservation Laws in Several Space Variables* (Springer-Verlag, 1984).

[11] R. Motte, A numerical method for solving particle-fluid equations, in *Proc. Conf. Trends in Numerical and Physical Modeling for Industrial Multiphase Flows* (Cargèse, France, 2000).

[12] M. Ndjinga, A. Kumbaro, F. De Vuyst and P. Laurent-Gencoux, Influence of the interfacial forces on the hyperbolicity of the two-fluid model, in *5th Int. Symp. Multiphase Flow, Heat Mass Transfer and Energy Conversion* (Xi'an China, July 3–6 2005).

[13] P. J. O'Rourke, Collective drop effects on vaporizing liquid sprays. PhD thesis, Los Alamos National Laboratory (1981).

[14] K. PfaffelMoser, Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data, *J. Differential Equations* **95**(2) (1992) 281–303.

[15] D. Ramos, Quelques résultats mathématiques et simulations num ériques d'écoulements régis par des modèles bifluides, PhD thesis, Ecole Normale Supérieure de Cachan (2000).

[16] W. E. Ranz and W. R. Marshall, Evaporization from drops, part I-II, *Chem. Eng. Prog.* **48**(3) (1952) 141–180.

[17] D. Serre, *Systèmes de lois de Conservation. I* (Fondations. Diderot Editeur, Paris, 1996).

[18] F. A. Williams *Combustion Theory* (Benjamin Cummings, 1985).