

Entropic structure of the Landau equation with Coulomb interaction

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Use of the entropy principle for specific equations

Spatially Homogeneous Kinetic equations:

- 1 Fokker-Planck: [D. Bakry, M. Emery](#)
- 2 Landau: [LD, C. Villani](#)
- 3 Boltzmann (Cercignani's conjecture): [G. Toscani, C. Villani](#)
- 4 Continuous coagulation-fragmentation: [M. Aizenmann, T. Bak](#)
- 5 Discrete coagulation-fragmentation: [P.-E. Jabin, B. Niethammer](#)

Parabolic equations:

- 1 Nonlinear diffusion: [M. Del Pino, J. Dolbeault](#)
- 2 Fourth order equations: [M. Cáceres, J. Carrillo, G. Toscani](#)
- 3 Reaction-Drift-Diffusion: [A. Glitzky, K. Gröger, R. Hünlich](#)

Use of Sobolev and logarithmic Sobolev inequalities; Use of contradiction (compactness) arguments.

Boltzmann operator, abstract form

Unknown: $f := f(v) \geq 0$ density of a rarefied monoatomic gas w.r.t. the velocity $v \in \mathbb{R}^N$ ($N = 2, 3$).

Operator:

$$Q_{1,B}(f, f)(v) = \int \int \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N} \left(f(v')f(v'_*) - f(v)f(v_*) \right)$$

$$\times \delta_{v+v_* = v'+v'_*} \delta_{|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2} B \left(|v - v_*|, \widehat{(v - v_*) \cdot (v' - v'_*)} \right) dv' dv'_* dv_*.$$

Remark: $Q_1(M, M) = 0$ when $M(v) = \exp(a + b \cdot v - c|v|^2)$ is a Maxwellian function of v .

Boltzmann operator, parameter-form when $N = 2$

Boltzmann, Maxwell, ca 1860:

$$Q_{1,B}(f, f)(v) = \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} \left(f(v')f(v'_*) - f(v)f(v_*) \right) B(|v - v_*|, |\theta|) d\theta dv_*,$$

with

$$v' = \frac{v + v_*}{2} + R_{\theta} \left(\frac{v - v_*}{2} \right),$$

$$v'_* = \frac{v + v_*}{2} - R_{\theta} \left(\frac{v - v_*}{2} \right).$$

Grazing collisions asymptotics

Chapman-Cowling, circa 50

At the formal level,

$$Q_{2,\psi}(f, f)(v) = \lim_{\varepsilon \rightarrow 0} Q_{1,B_\varepsilon}(f, f)(v),$$

when B_ε is a rescaled version of B (extended by 0):

$$B_\varepsilon(z, \theta) = \frac{1}{\varepsilon^3} B\left(z, \frac{\theta}{\varepsilon}\right),$$

and

$$\psi(z) = Cst |z|^2 \int B(z, \theta) (1 - \cos \theta) d\theta.$$

Rigorous results in the context of renormalized solutions:

Alexandre-Villani, circa 2000.

Landau, 36: For $f := f(v) \geq 0$ density of charged particles of velocity $v \in \mathbb{R}^N$ in a plasma ($N = 3$),

$$Q_{2,\psi}(f, f)(v) = \nabla \cdot \left\{ \int_{\mathbb{R}^N} a(v-w) \left(f(w) \nabla f(v) - f(v) \nabla f(w) \right) dw \right\},$$

with

$$a_{ij}(z) := \psi(|z|) \Pi_{ij}(z), \quad \Pi_{ij}(z) := \delta_{ij} - \frac{z_i z_j}{|z|^2}.$$

Moreover $\psi(z) = |z|^{-1}$ in the Coulomb case (of charged particles).

Remark: $Q_2(M, M) = 0$ when $M(v) = \exp(a + b \cdot v - c|v|^2)$ is a Maxwellian function of v .

Common formalism for Boltzmann and Landau kernels (presented here in 2D)

Proposition LD, Salvarani 01: We assume that:

- Q is bilinear and continuous from $\mathcal{S}(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2)$ in $C_{\text{temp}}^\infty(\mathbb{R}^2)$;
- Q is invariant by translations and rotations, i.-e. $Q \circ \tau_h = \tau_h \circ Q$, and $Q \circ \Gamma_R = \Gamma_R \circ Q$, where $\Gamma_R \varphi(x) = \varphi(Rx)$;
- $Q(M, M) = 0$, when $M(v) = \exp(a + b \cdot v - c|v|^2)$;
- For all $f, g \in \mathcal{S}(\mathbb{R}^2)$, $v_0 \in \mathbb{R}^2$, such that $f, g \geq 0$ on \mathbb{R}^2 and $f(v_0) = 0$, one has $Q(f, g)(v_0) \geq 0$, i.-e. $e^{tQ(\cdot, g)} \geq 0$.

Common formalism for Boltzmann and Landau kernels (presented here in 2D)

Then there exists a positive measure μ such that

$$Q(f, g)(v) = \left\langle Pf \left(\frac{\mu(|v - v_*|, \theta)}{1 - \cos \theta} \right), \right. \\ \left. f \left(\frac{v + v_*}{2} + R_\theta \left(\frac{v - v_*}{2} \right) \right) g \left(\frac{v + v_*}{2} - R_\theta \left(\frac{v - v_*}{2} \right) \right) \right\rangle_{\theta, v_*}.$$

When the measure μ has an atomic part in 0 (and in other points), one finds the sum of a Boltzmann operator (with or without angular cutoff) and a Landau operator.

Weak formulation of Boltzmann operator

$$\begin{aligned} & \int_{\mathbb{R}^N} Q_{1,B}(f, f)(v) \varphi(v) dv \\ &= \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{N-1}} \left(f(v')f(v'_*) - f(v)f(v_*) \right) \\ & \quad \times \left(\varphi(v) + \varphi(v_*) - \varphi(v') - \varphi(v'_*) \right) \\ & \quad \times \delta_{v+v_* = v'+v'_*} \delta_{|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2} B dv' dv'_* dv_*. \end{aligned}$$

Consequence: conservation of mass, momentum and energy:

$$\int_{\mathbb{R}^N} Q_{1,B}(f, f)(v) \begin{pmatrix} 1 \\ v_j \\ |v|^2/2 \end{pmatrix} dv = 0.$$

Weak formulation of Landau operator

$$\begin{aligned} & \int_{\mathbb{R}^N} Q_{2,\psi}(f, f)(v) \varphi(v) dv \\ &= \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} f(v) f(w) \psi(|v-w|) \left(\frac{\nabla f(v)}{f(v)} - \frac{\nabla f(w)}{f(w)} \right)^T \Pi(v-w) \\ & \quad \left(\nabla \varphi(v) - \nabla \varphi(w) \right) dv dw. \end{aligned}$$

Consequence: conservation of mass, momentum and energy:

$$\int_{\mathbb{R}^N} Q_{2,\psi}(f, f)(v) \begin{pmatrix} 1 \\ v_j \\ |v|^2/2 \end{pmatrix} dv = 0.$$

Entropy inequality (H theorem) for Boltzmann equation

Entropy production: ($f := f(v)$)

$$\begin{aligned} D_{1,B}(f) &:= - \int_{\mathbb{R}^N} Q_{1,B}(f, f)(v) \ln f(v) dv \\ &= \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N} \left(f(v')f(v'_*) - f(v)f(v_*) \right) \\ &\quad \times \left(\ln(f(v')f(v'_*)) - \ln(f(v)f(v_*)) \right) \delta_{\{..\}} \delta_{\{..\}} B dv dv_* dv' dv'_* \geq 0 \end{aligned}$$

Entropy inequality (H theorem) for Landau equation

Entropy production: ($f := f(v)$)

$$D_{2,\psi}(f) := - \int_{\mathbb{R}^N} Q_{2,\psi}(f, f)(v) \ln f(v) dv$$

$$= \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} f(v) f(w) \psi(|v - w|)$$

$$\left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w) \right)^T \Pi(v - w) \left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w) \right) dv dw \geq 0$$

H Theorem, case of equality (1)

When $D_{1,B}(f) = 0$, one has (if $B \neq 0$ a.e.)

$$v + v_* = v' + v'_* \quad \text{and} \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$$

$$\Rightarrow \quad f(v) f(v_*) = f(v') f(v'_*),$$

so that finally

$$f(v) f(v_*) = T(v + v_*, |v|^2 + |v_*|^2).$$

H Theorem, case of equality (2)

We know that

$$f(v) f(v_*) = T(v + v_*, |v|^2 + |v_*|^2).$$

Then we use the operator:

$$L := (v - v_*) \times (\nabla_v - \nabla_{v_*}),$$

i.-e.

$$L_{ij} := (v_i - v_{*i})(\partial_{v_j} - \partial_{v_{*j}}) - (v_j - v_{*j})(\partial_{v_i} - \partial_{v_{*i}}),$$

and get

$$L \left[T(v + v_*, |v|^2 + |v_*|^2) \right] = 0.$$

H Theorem, case of equality (3)

Consequence: When $D_{1,B}(f) = 0$, one has (for $i \neq j$)

$$q_{ij}^f := (v_i - v_{*i}) \left(\frac{\partial_j f(v)}{f(v)} - \frac{\partial_j f(v_*)}{f(v_*)} \right) - (v_j - v_{*j}) \left(\frac{\partial_i f(v)}{f(v)} - \frac{\partial_i f(v_*)}{f(v_*)} \right) = 0,$$

i.-e.

$$\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(v_*) \quad // \quad v - v_*.$$

We recall that for Landau equation:

$$D_{2,\psi}(f) = \frac{1}{2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(v) f(w) \psi(|v-w|) \left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w) \right)^T \Pi(v-w) \left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w) \right) dv dw.$$

Link between entropy dissipations

Proposition (Toscani-Villani 99, Villani 03): When $\psi(z) = |z|^2$ (Maxwell molecules), and $B(|z|, |\theta|) = |z|^2 \beta(|\theta|)$ (super hard spheres), one gets

$$D_{1,B}(f) \geq Cst \int_0^\infty e^{-Cst\tau} D_{2,\psi}(e^{\tau L} f) d\tau,$$

where $e^{\tau L}$ is a semigroup generated by Fokker-Planck's operator:

$$Lf = \Delta f + \nabla \cdot (v f).$$

H Theorem, resolution of the case of equality (method of Boltzmann)

We know that (for $i \neq j$)

$$(v_i - v_{*i}) \left(\frac{\partial_j f(v)}{f(v)} - \frac{\partial_j f(v_*)}{f(v_*)} \right) = (v_j - v_{*j}) \left(\frac{\partial_i f(v)}{f(v)} - \frac{\partial_i f(v_*)}{f(v_*)} \right).$$

Using ∂_{v_i} , one gets

$$\frac{\partial_j f(v)}{f(v)} - \frac{\partial_j f(v_*)}{f(v_*)} + (v_i - v_{*i}) \partial_{ij} \ln f(v) = (v_j - v_{*j}) \partial_{ii} \ln f(v).$$

Using $\partial_{v_{*j}}$, one gets $-\partial_{jj} \ln f(v_*) = -\partial_{ii} \ln f(v)$, and finally using $\partial_{v_{*i}}$, one gets $-\partial_{ij} \ln f(v_*) - \partial_{ij} \ln f(v) = 0$.

At the end

$$f(v) = \exp(a + b \cdot v - c |v|^2).$$

H Theorem, resolution of the case of equality (method using integrals)

We know that (for $i \neq j$)

$$q_{ij}^f(v, w) = \left[v_i \frac{\partial_j f(v)}{f(v)} - v_j \frac{\partial_i f(v)}{f(v)} \right] - \frac{\partial_j f(w)}{f(w)} v_i + \frac{\partial_i f(w)}{f(w)} v_j \\ - w_i \frac{\partial_j f(v)}{f(v)} + w_j \frac{\partial_i f(v)}{f(v)} + \left[w_i \frac{\partial_j f(w)}{f(w)} - w_j \frac{\partial_i f(w)}{f(w)} \right] = 0.$$

Then (for $i \neq j$)

$$\frac{\partial_i f(v)}{f(v)} = \frac{\text{Det} \left(\int_{\mathbb{R}^N} f(w) \begin{bmatrix} 1 & 0 & w_i \\ w_i & v_j - w_j & w_i^2 \\ w_j & -(v_i - w_i) & w_i w_j \end{bmatrix} dw \right)}{\text{Det} \left(\int_{\mathbb{R}^N} f(w) \begin{bmatrix} 1 & w_j & w_i \\ w_i & w_j w_i & w_i^2 \\ w_j & w_j^2 & w_i w_j \end{bmatrix} dw \right)}$$

so that

$$f(v) = \exp(a + b \cdot v - c |v|^2).$$

Cercignani's conjecture

Proposition (cf. [LD-Villani 00](#)): When $f \geq 0$ is such that $\int f \, dv = 1$, $\int f v \, dv = 0$, $\int f |v|^2/2 \, dv = N/2$,

$$\begin{aligned} D_{2,z \rightarrow |z|^2}(f) &\geq C(f) \int f(v) \left| \frac{\nabla f}{f}(v) + v \right|^2 dv \\ &\geq C(f) \left(\int f \ln f - \int M \ln M \right), \end{aligned}$$

where $C(f)$ only depends on (an upper bound of) the entropy $\int f \ln f$ of f , and

$$M(v) = (2\pi)^{-N/2} \exp(-|v|^2/2).$$

Consequence ([Toscani-Villani 99](#), [Villani 03](#)): Cercignani's conjecture for the equation of Boltzmann (with "super hard spheres").

Idea of the proof

Link between $\frac{\partial_i f}{f}$ and the quantity in $D_{2,z \mapsto |z|^2}(f)$ (cf. LD 89):

$$D_{2,z \mapsto |z|^2}(f) = \frac{1}{4} \sum_{i,j=1,\dots,N} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} f(v) f(w) |q_{ij}^f(v, w)|^2 dv dw,$$

$$q_{ij}^f(v, w) = (v_i - w_i) \left(\frac{\partial_j f}{f}(v) - \frac{\partial_j f}{f}(w) \right) - (v_j - w_j) \left(\frac{\partial_i f}{f}(v) - \frac{\partial_i f}{f}(w) \right).$$

Then for $i \neq j$

$$\frac{\partial_i f(v)}{f(v)} = \frac{\text{Det} \left(\int_{\mathbb{R}^N} f(w) \begin{bmatrix} 1 & q_{ij}^f(v, w) & w_i \\ w_i & q_{ij}^f(v, w) w_i + (v_j - w_j) & w_i^2 \\ w_j & q_{ij}^f(v, w) w_j - (v_i - w_i) & w_i w_j \end{bmatrix} dw \right)}{\text{Det} \left(\int_{\mathbb{R}^N} f(w) \begin{bmatrix} 1 & w_j & w_i \\ w_i & w_j w_i & w_i^2 \\ w_j & w_j^2 & w_i w_j \end{bmatrix} dw \right)}$$

Consequence

Estimate of convergence towards thermodynamic equilibrium in large time of the Landau equation with Maxwell molecules, and Boltzmann equation with super hard spheres

Theorem (cf. [Toscani-Villani 99](#), [LD-Villani 00](#), [Villani 03](#)): When $f_{in} \geq 0$ satisfies $\int f_{in}(v) dv = 1$, $\int f_{in}(v) v dv = 0$, $\int f_{in}(v) |v|^2/2 dv = N/2$, and $\int f_{in} \ln f_{in} \leq \bar{H}$, the (unique) solution $f := f(t, v)$ of the (spatially homogeneous) Landau equation

$$\partial_t f = Q_{2,z \mapsto |z|^2}(f, f)$$

with Maxwell molecules, or the (spatially homogeneous) Boltzmann equation

$$\partial_t f = Q_{1,B}(f, f)$$

with super hard spheres, satisfies

$$\|f(t) - M\|_{L^1(\mathbb{R}^3)} \leq Cst e^{-Cst}.$$

where the constants only depend on \bar{H} .

Proof (quantitative version of La Salle's principle)

Method of entropy-entropy dissipation: If

$$\partial_t f = Q_{2,z \mapsto |z|^2}(f, f),$$

then

$$\begin{aligned} -\partial_t \int f \ln f &= - \int Q_{2,z \mapsto |z|^2}(f, f) \ln f \\ &= D_{2,z \mapsto |z|^2}(f), \\ &\geq C(f) \left(\int f \ln f - \int M \ln M \right). \end{aligned}$$

We conclude with Gronwall's lemma and Csiszár-Kullback-Pinsker inequality.

Large time behavior for Landau equation in the Coulomb case

Theorem: Carrapatoso-LD-He 16; improved in Carrapatoso-Mischler 17

Let $f_{in} \in L^1(e^{\kappa|v|^{1/2}}) \cap L \ln L(\mathbb{R}^3)$, with $\kappa \in]0, 2/e[$.

Then there exists a global weak solution f for the (spatially homogeneous) Landau equation in the Coulomb case (with initial data f_{in}) such that

$$\forall t \geq 0, \quad \|f(t, \cdot) - M\|_{L^1(\mathbb{R}^3)} \leq Cst e^{-(1+t)^{\frac{1}{7}}} \log(1+t)^{-\frac{6}{7}}.$$

where Cst is a constant depending on the initial mass, energy, entropy and $\|f_{in}\|_{L^1(e^{\kappa|v|^{1/2}})}$. The rate can be improved under extra conditions on the initial datum.

Idea of the proof

Proposition: LD 14, Carrapatoso-LD-He 16 One can find $C := C(\bar{H}) > 0$ depending only on $\bar{H} > 0$ such that for all $f \geq 0$ satisfying

$$\int_{\mathbb{R}^3} f(v) dv = 1, \quad \int_{\mathbb{R}^3} f(v) v dv = 0, \quad \int_{\mathbb{R}^3} f(v) |v|^2 dv = 3,$$

and also satisfying (an upper bound on the entropy)

$$H(f) := \int_{\mathbb{R}^3} f(v) \ln f(v) dv \leq \bar{H},$$

the following inequality holds:

$$D_{2,z \mapsto |z|^{-1}}(f) \geq C(\bar{H}) \left(\int f |v|^5 dv \right)^{-1} \int_{\mathbb{R}^3} f(v) \left| \frac{\nabla f(v)}{f(v)} + v \right|^2 (1+|v|^2)^{-3/2} dv.$$

Idea of the proof

Sobolev logarithmic inequality (Bakry-Emery 84): For $f \geq 0$ such that

$$\int_{\mathbb{R}^3} f(v) dv = \int_{\mathbb{R}^3} M(v) dv,$$

the following inequality holds:

$$\begin{aligned} & \int_{\mathbb{R}^3} f(v) \left| \frac{\nabla f(v)}{f(v)} + v \right|^2 (1 + |v|^2)^{-3/2} dv \\ & \geq Cst \int_{\mathbb{R}^3} \left\{ f(v) \ln \left(\frac{Z_1}{Z_2(f)} \frac{f(v)}{M(v)} \right) + \frac{Z_2(f)}{Z_1} M(v) - f(v) \right\} (1 + |v|^2)^{-3/2} dv, \end{aligned}$$

with

$$Z_1 := \int_{\mathbb{R}^3} M(v) (1 + |v|^2)^{-3/2} dv, \quad Z_2(f) := \int_{\mathbb{R}^3} f(v) (1 + |v|^2)^{-3/2} dv,$$

and

$$M(v) = (2\pi)^{-3/2} \exp(-|v|^2/2).$$

Idea of the proof

Weak version of **Cercignani's conjecture** for Landau equation with Coulomb potential: For $f \geq 0$ satisfying the normalization and such that $H(f) \leq \bar{H}$,

$$D_{2, z \mapsto |z|^{-1}}(f) \geq C(\bar{H}) \left(\int f |v|^5 dv \right)^{-1} \\ \times \int_{\mathbb{R}^3} \left\{ f(v) \ln \left(\frac{Z_1}{Z_2(f)} \frac{f(v)}{M(v)} \right) + \frac{Z_2(f)}{Z_1} M(v) - f(v) \right\} (1+|v|^2)^{-3/2} dv,$$

with

$$Z_1 := \int_{\mathbb{R}^3} M(v) (1+|v|^2)^{-3/2} dv, \quad Z_2(f) := \int_{\mathbb{R}^3} f(v) (1+|v|^2)^{-3/2} dv,$$

and

$$M(v) = (2\pi)^{-3/2} \exp(-|v|^2/2).$$

Idea of the proof: polynomial moments are propagated

Polynomially weighted moments:

$$M_\ell(f) = \int_{\mathbb{R}^3} f(v) (1 + |v|^2)^{\ell/2} dv.$$

Proposition: Carrapatoso-LD-He 16 Assume that $f_{in} := f_{in}(v) \geq 0$ lies in $L_\ell^1 \cap L \ln L(\mathbb{R}^3)$, $\ell \geq 2$, and that f is an H/weak solution in $L^\infty(\mathbb{R}_+; L_2^1(\mathbb{R}^3))$ (with initial datum f_{in}) of the spatially homogeneous Landau equation in the Coulomb case.

Then there exists $C = C(M_2(0), \int f_{in}(v) \ln f_{in}(v) dv, M_\ell(0), \ell) > 0$ such that

$$\forall t \geq 0, \quad M_\ell(f(t)) \leq C(1 + t).$$

Idea of the proof: stretched exponential moments are propagated

Exponentially weighted moments:

$$M_{s,\kappa}(f) = \int_{\mathbb{R}^3} f(v) \exp(\kappa |v|^s) dv.$$

Proposition: Carrapatoso-LD-He 16 Assume that $f_{in} := f_{in}(v) \geq 0$ lies in $L^1(e^{\kappa|v|^s})$ with $\kappa > 0$ and $s \in]0, 1/2[$ or $\kappa \in]0, 2/e[$ and $s = 1/2$ and in $L \ln L(\mathbb{R}^3)$, and that f is an H-solution in $L^\infty(\mathbb{R}_+; L^1_2(\mathbb{R}^3))$ of the spatially homogeneous Landau equation in the Coulomb case (with initial datum f_{in}). Then, there exists $C = C(M_2(0), \int f_{in}(v) \ln f_{in}(v) dv, M_{s,\kappa}(0)) > 0$ such that

$$\forall t \geq 0, \quad M_{s,\kappa}(f(t)) \leq C(1 + t).$$

Interpolation: For any $\kappa_1 > \kappa$ and $\kappa_2 > 3/2 \kappa$ there holds

$$\begin{aligned} \int e^{\kappa|v|^5} f(v) \ln f(v) dv &\leq Cst (M_{S,\kappa_1}(f) + M_{S,\kappa_2}(f))^{\frac{2}{3}} \|f\|_{L^3_{-3}} + 1) \\ &\leq Cst (M_{S,\kappa_1}(f) + M_{S,\kappa_2}(f))^{\frac{2}{3}} D_{2,z \mapsto |z|^{-1}}(f) + 1). \end{aligned}$$

Use of a previous coercivity estimate

Proposition: LD 14 For all $f \geq 0$,

$$\begin{aligned}\|f\|_{L^3_{-3}} &\leq Cst \int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 (1 + |v|^2)^{-3/2} dv \\ &\leq C(1 + D_{z \mapsto |z|^{-1}}(f)),\end{aligned}$$

where

$$C := C\left(\int_{\mathbb{R}^3} f(v) dv, \int_{\mathbb{R}^3} f(v) v dv, \int_{\mathbb{R}^3} f(v) |v|^2/2 dv, \bar{H}\right),$$

and

$$\bar{H} \geq \int_{\mathbb{R}^3} f(v) \ln f(v) dv.$$

Recent progresses on the subject

- Exponential convergence towards equilibrium for Boltzmann equation with angular cutoff and hard potentials [Mouhot 06](#)
- Exponential convergence towards equilibrium for Landau equation with (very) moderate soft potentials $\delta \in]1, 2[$ [Carrapatoso 13/15](#)
- Exponential convergence towards equilibrium for the (homogeneous or spatially inhomogeneous with initial data close to equilibrium) Boltzmann equation without angular cutoff and hard potentials [Tristani 14](#), [Héraud-Tonon-Tristani 17](#)
- Stretched-exponential convergence towards equilibrium for the spatially inhomogeneous Landau equation with Coulomb potentials and initial data close to equilibrium [Carrapatoso-Mischler 17](#)

Based on estimates of spectral gaps in enlarged spaces

[Gualdani-Mischler-Mouhot 13](#)

Open questions

- Optimal rate of convergence for general (non L^2) initial data
- Better smoothness estimates (possibly giving uniqueness)
- Possible link between the entropy dissipations of Boltzmann and Landau when the Coulomb case or other cases (different from Maxwell molecules) are considered
- Case of equality in the H theorem for generalized Boltzmann equations (relativistic, semiconductor-based, linked to weak turbulence)

Generalized Boltzmann equations, H theorem, case of equality (with M. Breden)

Abstract form of generalized Boltzmann's kernel:

$$Q'_1(f, f)(v) = \int \int \int \left(f(v')f(v'_*) - f(v)f(v_*) \right) \\ \times \delta_{v+v_* = v' + v'_*} \delta_{\varepsilon(v) + \varepsilon(v_*) = \varepsilon(v') + \varepsilon(v'_*)} B dv' dv'_* dv_*,$$

so that

$$D'_1(f) = \int \int \int \int \left(f(v')f(v'_*) - f(v)f(v_*) \right) \\ \times \left(\ln(f(v')f(v'_*)) - \ln(f(v)f(v_*)) \right) \\ \times \delta_{v+v_* = v' + v'_*} \delta_{\varepsilon(v) + \varepsilon(v_*) = \varepsilon(v') + \varepsilon(v'_*)} B dv dv' dv'_* dv_*.$$

Generalized Boltzmann equations, H theorem, case of equality

Consequence: When $D'_1(f) = 0$ (and $B \neq 0$ a.e.), then

$$f(v) f(v_*) = T(v + v_*, \varepsilon(v) + \varepsilon(v_*)).$$

One uses the operator:

$$L' := (\nabla \varepsilon(v) - \nabla \varepsilon(v_*)) \times (\nabla_v - \nabla_{v_*}),$$

and gets

$$L' \left[T(v + v_*, \varepsilon(v) + \varepsilon(v_*)) \right] = 0.$$

Generalized Boltzmann equations, H theorem, case of equality

Consequence: When $D'_1(f) = 0$ (and $B \neq 0$ a.e.), then (for $i \neq j$)

$$\begin{aligned} & (\partial_i \varepsilon(v) - \partial_i \varepsilon(v_*)) \left(\frac{\partial_j f(v)}{f(v)} - \frac{\partial_j f(v_*)}{f(v_*)} \right) \\ &= (\partial_j \varepsilon(v) - \partial_j \varepsilon(v_*)) \left(\frac{\partial_i f(v)}{f(v)} - \frac{\partial_i f(v_*)}{f(v_*)} \right), \end{aligned}$$

that is

$$\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(v_*) \quad // \quad \nabla \varepsilon(v) - \nabla \varepsilon(v_*).$$

Generalized Boltzmann equations, H theorem, case of equality

Proposition: We assume that (true in the relativistic case)

- For all $a, b, c \in \mathbb{R}, i \neq j$,

$$|\{v, \quad a + b \partial_i \varepsilon(v) + c \partial_j \varepsilon(v) = 0\}| = 0;$$

-

$$\{1, \quad \partial_i \varepsilon(v), \quad \partial_i \varepsilon(v) \partial_j \varepsilon(v)\}_{i \leq j}$$

is a linearly independent family;

Then if $D'_1(f) = 0$, one has

$$f(v) = \exp(a + b \cdot v - c \varepsilon(v)).$$