ON THE SPATIALLY HOMOGENEOUS LANDAU EQUATION FOR HARD POTENTIALS PART II : H-THEOREM AND APPLICATIONS

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ABSTRACT. We find a lower bound for the entropy dissipation of the spatially homogeneous Landau equation with hard potentials in terms of the entropy itself. We deduce from this explicit estimates on the speed of convergence towards equilibrium for the solution of this equation. In the case of so-called overmaxwellian potentials, the convergence is exponential. We also compute a lower bound for the spectral gap of the associated linear operator in this setting.

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1. INTRODUCTION AND MAIN RESULT

We recall the spatially homogeneous Landau equation (Cf. [8, 18]),

$$\frac{\partial f}{\partial t}(t,v) = Q(f,f)(t,v), \qquad v \in \mathbb{R}^N, \quad t \geq 0, \tag{1}$$

where f is a nonnegative function and Q is a nonlinear quadratic operator acting on the variable v only,

(2)
$$Q(f,f)(v) = \frac{\partial}{\partial v_i}\left\{\int_{\mathbb{R}^N} dv_*\, a_{ij}(v-v_*)\left(f_*\,\frac{\partial f}{\partial v_j}(v) - f\,\frac{\partial f}{\partial v_j}(v_*)\right)\right\},$$

where $f_* = f(v_*)$, and the convention of Einstein for repeated indices is (and will systematically be) used.

Here, $(a_{ij}(z))_{ij}$ $(z \in \mathbb{R}^N)$ is a nonnegative symmetric matrix function with only one degenerate direction, namely that of z. More precisely,

$$a_{ij}(z) = \Pi_{ij}(z)\,\Psi(|z|), \tag{3}$$

where Ψ is a nonnegative cross section and

$$\Pi_{ij}(z) = \delta_{ij} - \frac{z_i z_j}{|z|^2} \tag{4}$$

is the orthogonal projection onto $z^\perp = \{y \,/\, y \cdot z = 0\}$.

We address the reader to Part I of this work [13] for references on the subject.

The Landau equation is obtained as a limit of the Boltzmann equation when grazing collisions prevail. The terminology concerning the cross section is therefore closely related to that of the Boltzmann equation.

In this paper, we shall deal with different types of cross sections Ψ. We recall the important particular case of Maxwellian molecules (coming out of an inverse power force in $r^{\perp(2N\perp1)}$),

$$\Psi(|z|) = |z|^2. \tag{5}$$

Any cross section Ψ, such that Ψ is locally integrable and satisfying

$$\Psi(|z|) \geq |z|^2 \tag{6}$$

will be called overmaxwellian (of course Maxwellian molecules are overmaxwellian).

The "true" hard potentials cross section (coming out of an inverse power force in $r^{\perp s}$ for $s > 2N-1$) is

$$\Psi(|z|) = |z|^{\gamma+2} \tag{7}$$

for some $\gamma \in (0,1)$. Such a cross section is not overmaxwellian because of its behavior near $z=0$. We therefore define "modified" hard potentials by the requirements that Ψ is of class C^2, overmaxwellian, and

$$\Psi(|z|) \sim |z|^{\gamma+2} \quad \text{as } |z| \to +\infty. \tag{8}$$

Note that multiplication of Ψ by a given strictly positive constant amounts to a simple rescaling of time.

For a given nonnegative initial datum f_{in}, we shall use the notations

$$M(f_{in}) = \int_{\mathbb{R}^N} f_{in}(v)\,dv, \qquad E(f_{in}) = \frac{1}{2}\int_{\mathbb{R}^N} f_{in}(v)|v|^2\,dv,$$

$$H(f_{in}) = \int_{\mathbb{R}^N} f_{in}(v)\log f_{in}(v)\,dv,$$

for the initial mass, energy and entropy. It is classical that if $f_{in} \geq 0$ and $M(f_{in}), E(f_{in}), H(f_{in})$ are finite, then f_{in} belongs to

$$L\log L = \left\{ f \in L^1(\mathbb{R}^N); \quad \int_{\mathbb{R}^N} |f(v)|\,|\log\big(|f(v)|\big)|\,dv < +\infty \right\}.$$

The solutions of the Landau equation satisfy (at least formally, thanks to the change of variables $(v, v_*) \leftrightarrow (v_*, v)$) the conservation of mass, momentum and energy, that is

$$M(f(t,\cdot)) \equiv \int_{\mathbb{R}^N} f(t,v)\,dv = \int_{\mathbb{R}^N} f_{in}(v)\,dv = M(f_{in}), \tag{9}$$

$$\int_{\mathbb{R}^N} f(t,v)\,v\,dv = \int_{\mathbb{R}^N} f_{in}(v)\,v\,dv, \tag{10}$$

$$E(f(t,\cdot)) \equiv \int_{\mathbb{R}^N} f(t,v)\,\frac{|v|^2}{2}\,dv = \int_{\mathbb{R}^N} f_{in}(v)\,\frac{|v|^2}{2}\,dv = E(f_{in}). \tag{11}$$

They also satisfy (at the formal level) the entropy dissipation identity

$$\frac{d}{dt}H(f(t,\cdot)) = -D(f(t,\cdot)), \tag{12}$$

where H is the entropy

$$H(f) \equiv \int_{\mathbb{R}^N} f(v)\log f(v)\,dv, \tag{13}$$

and D is the entropy dissipation functional

$$D(f) = -\int_{\mathbb{R}^N} Q(f,f)(v)\log f(v)\,dv \tag{14}$$

$$= \frac{1}{2}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} a_{ij}(v-v_*)\, f f_* \left(\frac{\partial_i f}{f}(v) - \frac{\partial_i f}{f}(v_*)\right) \left(\frac{\partial_j f}{f}(v) - \frac{\partial_j f}{f}(v_*)\right) \geq 0.$$

Due to the singularities at points where f vanishes, this formula is not very convenient for a mathematical study. Therefore, as in [26], we shall rewrite the entropy dissipation for the Landau equation in a form which makes sense under very little assumptions on f. Since, formally,

$$\sqrt{ff_*}\left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(v_*)\right) = 2\left(\sqrt{f_*}\nabla\sqrt{f}(v) - \sqrt{f}\nabla\sqrt{f}(v_*)\right)$$
$$= 2\,(\nabla_v - \nabla_{v_*})\sqrt{ff_*},$$

the entropy dissipation is

$$2\int\int dv\, dv_*\, a(v-v_*)(\nabla - \nabla_*)\sqrt{ff_*}\,(\nabla - \nabla_*)\sqrt{ff_*}.$$

In other words,

$$D(f) = \frac{1}{2}||K||^2_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}, \tag{15}$$

where

$$K(v,v_*) = 2\,\Pi(v-v_*)\,\Psi^{1/2}(|v-v_*|)\,(\nabla_v - \nabla_{v_*})\sqrt{f(v)\,f(v_*)}.$$

We show in Appendix A that K is well–defined as a distribution on $\mathbb{R}^N\times\mathbb{R}^N$ as soon as Ψ is locally integrable and $f\in L^1(\mathbb{R}^N)$. In particular, as noted in [26], this allows to cover the physical cases where Ψ has a singularity at the origin. Hence, formula (15) enables us to define $D(f)$ as an element of $[0,+\infty]$ in the most general case, and we shall always consider it as the *definition* of the entropy dissipation. Of course, with this convention, formula (14) holds only under suitable regularity assumptions on f (and its logarithm).

The equality $D(f)=0$ holds (at the formal level, and when $f,\Psi>0$ a.e.) only if for all $v,v_*\in\mathbb{R}^N$,

$$\nabla(\log f)(v) - \nabla(\log f)(v_*) = \lambda_{v,v_*}(v-v_*)$$

for some $\lambda_{v,v_*}\in\mathbb{R}$. It is easy to check that this implies that for all $v\in\mathbb{R}^N$, $\nabla f(v) = \lambda v + V$ for some fixed $\lambda\in\mathbb{R}$ and $V\in\mathbb{R}^N$. This ensures in turn that f is a Maxwellian function of v,

$$f(v) = \frac{\rho}{(2\pi T)^{N/2}}\, e^{\perp\frac{(v-u)^2}{2T}} \equiv M_{\rho,u,T}(v), \tag{16}$$

for some $u \in \mathbb{R}^N$, $\rho, T > 0$. A rigorous proof (under suitable assumptions on f) can be found for instance in [23]. Other proofs shall be given in the present paper.

This theorem is the Landau version of Boltzmann's H-theorem, in view of which it is expected that a solution $f(t,\cdot)$ of the Landau equation converges when $t \to +\infty$ towards the Maxwellian function $M^f = M_{\rho^f, u^f, T^f}$ defined by

$$\rho^f = \int_{\mathbb{R}^N} f(v)\, dv, \qquad \rho^f\, u^f = \int_{\mathbb{R}^N} f(v)\, v\, dv,$$

and

$$\int_{\mathbb{R}^N} f(v)\, |v|^2\, dv = \rho^f\, |u^f|^2 + N\, T^f.$$

The purpose of this paper is to study the speed of convergence of $f(t,\cdot)$ towards M^f. Let us summarize briefly the state of the art concerning the asymptotic behavior of the solutions to the spatially homogeneous Boltzmann and Landau equations. The reader will find many references (but unfortunately not the most recent ones) in [12] on the general problem of the behavior when $t \to +\infty$ of the solutions of the Boltzmann equation in various settings, including the full x–dependent equation.

In the homogeneous setting, we are aware of essentially two types of theorems :

- The results by Arkeryd [2] and Wennberg [27] give exponential convergence towards equilibrium for the spatially homogeneous Boltzmann equation with hard (or Maxwellian) potentials in weighted L^p norms, namely

  $$\|f - M^f\| \le C e^{\perp \delta t},$$

  but with a rate $\delta > 0$ (depending on the initial datum), which is obtained by a compactness argument and is therefore not explicit. These results are based on the study of the spectral properties of the linearized Boltzmann operator.

- On the other hand, Carlen and Carvalho obtain in [4, 5] an estimate which gives only at most algebraic decay for the Boltzmann equation (with Maxwellian molecules or hard-spheres), but which is completely explicit (though rather complicated). These results rely on a precise study of the entropy dissipation D_B of the Boltzmann equation. A function Φ (with $\Phi(0) = 0$) is computed in such a way that

$$D_B(f) \geq \Phi\Big(H(f) - H(M^f)\Big).$$

This function Φ is strictly increasing from 0 (but very slowly). As a consequence, it is shown in [5] how, for a given initial datum f_{in} and $\varepsilon > 0$, one can compute $T_\varepsilon(f_{in}) > 0$ such that

$$t \geq T_\varepsilon(f_{in}) \Longrightarrow \|f(t) - M^f\|_{L^1} \leq \varepsilon.$$

The results by Carlen and Carvalho have been applied successfully to several situations, for example in the context of an hydrodynamical–type limit, or in order to study the trend to equilibrium when initial data have infinite entropy.

We also note that the *optimal* rate of convergence for the Boltzmann equation with Maxwellian molecules was recently obtained by Carlen, Gabetta and Toscani in [6], using a completely different approach, which does not seem easily adaptable to other potentials.

We finally mention that a general but somewhat weaker entropy dissipation inequality was established by the first author of this work in [11] (Cf. also [28]) for various collision operators including Landau (under suitable assumptions on Ψ). This estimate would imply a local (in velocity variable) convergence towards equilibrium, in some sense, which is essentially in $O(t^{\perp 1/2})$, if the solutions were known to satisfy certain additional technical assumptions. The results of this paper are not used here. Some of its ideas are however retained (Cf. section 3).

We shall not try here to use the spectral gap of the linearized operator (Cf. [10]), but we shall focus as in [4] and [5] on the entropy dissipation $D(f)$. In fact, as we shall show in section 7, from our work one can recover an *explicit* estimate of the spectral gap of the linearized Landau operator in the case of overmaxwellian molecules. Such an estimate seems difficult to obtain by classical methods using Weyl's theorem (i.e. the property that the essential spectrum is unchanged by compact perturbations).

Our study relies on the use of the Fisher information,

$$I(f) \equiv \int \frac{|\nabla f|^2}{f} = 4 \int |\nabla \sqrt{f}|^2, \tag{17}$$

which has already been successfully used in related problems by Carlen and Carvalho [4] and Toscani [22] (respectively for the Boltzmann and the linear Fokker–Planck equations). In particular, our use of the logarithmic Sobolev inequality was inspired by this last work.

We now state our main result. For $s > 0$, we use the notation

$$L^1_s(\mathbb{R}^N) = \left\{ f \in L^1(\mathbb{R}^N);\ \int |f(v)|(1+|v|^2)^{s/2}\, dv < +\infty \right\}.$$

Theorem 1. *Let f be in $L^1_2(\mathbb{R}^N)$, and Q a be Landau operator with overmaxwellian cross section (i.e. $\Psi(|z|) \geq |z|^2$). Then there exists $\lambda > 0$ explicitly computable and depending only on ρ^f, u^f, and the N^2 scalars*

$$P^f_{ij} = \int_{\mathbb{R}^N} f(v)\, v_i v_j\, dv,$$

such that

$$D(f) \geq \lambda \Big(I(f) - I(M^f) \Big). \tag{18}$$

Theorem 1 will be proven in section 2 by a simple computation, which relies on explicit calculations done in [24] for Maxwellian molecules (that is, $\Psi(|z|) = |z|^2$). The reader may find this proof somewhat unsatisfactory, in that it seems to be heavily dependent on the particular structure of the Landau equation with Maxwellian molecules. This is why we present in section 3 another proof, which has interest in itself, and gives the same result (though with a smaller λ).

Section 4 is devoted to the applications of theorem 1 to the Landau equation with overmaxwellian cross sections. Exponential decay (in L^1 norm) is proven with an explicit rate. An interesting feedback property due to the nonlinearity of the equation is detailed. It allows one to get better constants in the rate of decay than one could expect at first sight.

However, the relaxation times obtained in section 4 are still very large if the initial datum is only assumed to have finite mass, energy and entropy. In section 5, we show how to remove this drawback, and get realistic relaxation times, with a rate which differs from the (optimal) relaxation rate for the linear Fokker-Planck equation only by a factor 1/6 (in dimension 3). At this point, the entropy dissipation is used as a control of the concentration, or equivalently the weak L^1 compactness.

Then, in section 6, we deal with "true" hard potentials. Algebraic decay with explicit constants is proven for the solution of the Landau equation in this case. Thanks to the result in [13], we also prove that the convergence holds in fact in $H^\infty(\mathbb{R}^N)$, and that the solution is globally (in time) stable with respect to its initial datum. As in [13], this global stability also holds with respect to small perturbations of the cross section Ψ.

In section 7, we show how our work can be used to get estimates for the linearized Landau kernel. Namely, inequalities such as (18) enable us to find simple refinements of results of Degond and Lemou (Cf. [10]).

Finally, in section 8, we give a last application of inequality (18). Namely, under rather weak conditions, the weak cluster points f of asymptotically grazing solutions of the Boltzmann equation (Cf. [13]) have automatically a square root belonging to $L^2_{loc}(\mathbb{R}^+_t; H^1(\mathbb{R}^N_v))$.

Useful results concerning the definition and approximations of $D(f)$ are given in Appendix A and B.

To conclude this introduction, we mention that among the numerous remaining open problems in this subject, the possibility of finding estimates in the case of soft potentials (i.e. $\gamma < 0$) seems to be a particularly interesting and difficult question.

2. Entropy dissipation : first method

First proof of theorem 1. In sections 2 and 3, we shall assume that

$$\int_{\mathbb{R}^N} f(v)\,dv = 1, \quad \int_{\mathbb{R}^N} f(v)\,v\,dv = 0, \quad \int_{\mathbb{R}^N} f(v)\,|v|^2\,dv = N, \tag{19}$$

which amounts to a simple change of coordinates of the form

$$(t, v) \longrightarrow (a\,t, b\,v + c), \qquad a, b \in \mathbb{R}, c \in \mathbb{R}^N. \tag{20}$$

Then, M will denote the centered Maxwellian with normalized mass and temperature,

$$M(v) = M^f(v) = \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{N/2}}. \tag{21}$$

We recall that

$$I(M) = N.$$

Next, it is clear that the entropy dissipation depends linearly on Ψ. In particular, if D_k is the entropy dissipation corresponding to the Landau operator with cross section Ψ_k,

$$\Psi_2 \geq \Psi_1 \Longrightarrow D_2(f) \geq D_1(f).$$

Therefore, we only need to prove Theorem 1 for $\Psi(|z|) = |z|^2$. Moreover, thanks to the lemma of Appendix B, it is enough to prove theorem 1 when $f \in \mathcal{S}$ and $|\log f|$ is bounded by $C\,(1 + |v|^2)$ for some $C > 0$. Note that as far as the evolution problem for the Landau equation with reasonable potentials is concerned, we need theorem 1 only for such smooth functions f, thanks to our study in [13]. We shall however need theorem 1 in its full generality when giving applications to the regularity of the weak cluster points to Boltzmann equation (Cf. section 8).

It is shown in [24] that for a Maxwellian cross section and a normalized initial datum,

$$Q(f,f) = N\nabla\cdot(\nabla f + fv) - \Big(P_{ij}\partial_{ij} f + \nabla\cdot(fv)\Big) + \Delta_{\theta\phi} f, \tag{22}$$

where $\Delta_{\theta\phi}$ denotes the Laplace-Beltrami operator of spherical diffusion. We recall that in the physically realistic case $N = 3$, using the usual spherical coordinates (r,θ,ϕ) defined by $v_1 = r\sin\theta\cos\phi$, $v_2 = r\sin\theta\sin\phi$, $v_3 = r\cos\theta$, the action of the Laplace-Beltrami operator is defined by

$$\Delta_{\theta\phi} f = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta f) + \frac{1}{\sin^2\theta}\partial_{\phi\phi} f.$$

We first note that the contribution of $\Delta_{\theta\phi}$ to the entropy dissipation is nonnegative. Indeed (supposing that $N = 3$ for simplicity),

$$\begin{aligned}
&\int \Delta_{\theta\phi} f\ \log f\, r^2 \sin\theta\, dr\, d\theta\, d\phi \\
&= \int \partial_\theta(\sin\theta\partial_\theta f)\log f\, r^2\, dr\, d\theta\, d\phi + \int \partial_{\phi\phi} f\log f \frac{1}{\sin^2\theta} r^2\ \sin\theta dr\, d\theta\, d\phi \\
&= -\int \frac{|\partial_\theta f|^2}{f} r^2\ \sin\theta dr\, d\theta\, d\phi - \int \frac{|\partial_\phi f|^2}{f}\frac{1}{\sin^2\theta} r^2\ \sin\theta dr\, d\theta\, d\phi
\end{aligned}$$

$$= -\int \frac{|\nabla_{\theta\phi} f|^2}{f}, \tag{23}$$

where (in polar coordinates)

$$\nabla_{\theta\phi} f = \begin{pmatrix} 0 \\ \partial_\theta f \\ \dfrac{1}{\sin\theta}\partial_\phi f \end{pmatrix}.$$

A direct proof without spherical coordinates is also obtained very easily.

The quantity (23) vanishes only for radially symmetric functions, and we note that it is very large when $r \to +\infty$ compared with the entropy dissipation induced by a usual Laplace operator. This suggests that solutions to the Landau equation have a tendency to become radial rather fast. However, we shall not use this information in the sequel.

Noting that

$$\int fv\cdot\nabla(\log f) = \int v\cdot\nabla f = -N\int f = -N,$$

we easily see that the entropy dissipation induced by the terms in (22) other than $\Delta_{\theta\phi}$ is

$$N \int \frac{|\nabla f|^2}{f} - P_{ij} \int \frac{\partial_i f \partial_j f}{f} - N(N-1). \tag{24}$$

As in [24], we can always assume that (P_{ij}) is diagonal. Indeed,

$$q : \vec{e} \in \mathbb{R}^N \longmapsto \int f(v)(v,\vec{e})(v,\vec{e})\, dv$$

defines a (nonnegative) quadratic form, so that there exists an orthonormal basis $(\vec{e_1}, \dots, \vec{e_N})$ which is also orthogonal for q. Let us define the "directional temperatures"

$$T_i = \int f v_i^2.$$

With these notations,

$$D(f) \geq \sum_i (N - T_i) \int \frac{(\partial_i f)^2}{f} - N(N-1). \tag{25}$$

We recall the following elementary lemma (Cf. also [25]).

Lemma 1. *Consider $\alpha_1, .., \alpha_N \geq 0$. Under the constraints*

$$f \geq 0, \qquad \int f = 1, \qquad \int f\, v_i = 0, \qquad \int f v_i^2 = T_i,$$

one has

$$\sum_i \alpha_i \int \frac{(\partial_i f)^2}{f} \geq \sum_i \frac{\alpha_i}{T_i}. \tag{26}$$

Proof. First note that, by density, it is sufficient to treat the case when f is smooth and nonvanishing. Then

$$0 \leq \alpha_i \int \left(\frac{\partial_i f}{f} - \frac{\partial_i g}{g} \right)^2 f, \tag{27}$$

where

$$g(v) = \prod_i \frac{e^{-\frac{v_i^2}{2T_i}}}{\sqrt{2\pi T_i}},$$

and $\partial_i g = -(v_i/T_i)\, g$. Expanding (27), we obtain

$$\begin{aligned} 0 &\leq \alpha_i \int \frac{(\partial_i f)^2}{f} + 2\alpha_i \int \partial_i f \frac{v_i}{T_i} + \alpha_i \int \frac{v_i^2}{T_i^2} f \\ &= \alpha_i \int \frac{(\partial_i f)^2}{f} - 2\frac{\alpha_i}{T_i} + \frac{\alpha_i}{T_i}, \end{aligned}$$

which proves the lemma. $\square$

We now come back to the proof of theorem 1. Let us set

$$\lambda = \min(N - T_i), \tag{28}$$

and write

$$D(f) - \lambda\Big(I(f) - I(M)\Big)$$

$$\geq (N - T_i - \lambda)\sum_i \int \frac{(\partial_i f)^2}{f} - N(N-1-\lambda).$$

Applying lemma 1, we get

$$D(f) - \lambda\Big(I(f) - I(M)\Big) \geq \sum_i \frac{N - T_i - \lambda}{T_i} - N(N-1-\lambda)$$

$$= (N-\lambda)\left(\sum_i \frac{1}{T_i} - N\right).$$

This last quantity is nonnegative. Indeed, since $\sum_i T_i = N$, one has

$$\sum_i \frac{1}{T_i} - N = \sum_i T_i \left(\frac{1}{T_i} - 1\right)^2.$$

Finally, noting that

$$\lambda \geq \min T_i,$$

we see that $\lambda > 0$ (since f cannot be concentrated on a single axis), and theorem 1 is proven (note that λ is explicit and depends only on T_i under assumption (19), on ρ_f, u_f and the N^2 scalars $\int f\, v_i v_j$ in general).

We conclude this section by showing how λ can be explicitly estimated (from below) by a function of $M(f)$, $E(f)$ and

$$\tilde{H}(f) = \int_{\mathbb{R}^N} f\,|\log f| dv$$

(note that these quantities are controlled by $M(f_{in})$, $E(f_{in})$ and $H(f_{in})$ when one deals with a solution of the Landau equation).

Proposition 2. *Let $s \geq 2$ and assume that f belongs to $L^1_s \cap L\log L$, and satisfies* (19)*. Then,*

$$\lambda \geq \min T_i(f) \geq \frac{1}{512 \times 16^{(N\perp 1)/s}|B_{N\perp 1}|^2}\;\frac{e^{\perp 16\tilde{H}(f)}}{||f||^{2(N\perp 1)/s}_{L^1_s(\mathbb{R}^N)}}, \tag{29}$$

where $|B_{N\perp 1}|$ *denotes the volume of the unit ball in dimension* $N-1$. *In particular, for* $s=2$,

$$T_i(f) \geq \frac{1}{512 \times 4^{N\perp 1}|B_{N\perp 1}|^2}\,\frac{e^{\perp 16\tilde{H}(f)}}{(N+1)^{N\perp 1}}. \tag{30}$$

Proof. For any $R>0$,

$$\int f(v)1_{|v|\geq R}\,dv \leq \frac{||f||_{L^1_s(\mathbb{R}^N)}}{R^s},$$

hence

$$\int f(v)1_{|v|\leq R}\,dv \geq 1 - \frac{||f||_{L^1_s(\mathbb{R}^N)}}{R^s}.$$

By lemma 6 of Part I of the present work,

$$\int f 1_{\substack{|v|\leq R\\ |v_i|\leq \delta}} \leq \varepsilon,$$

with $2\,\delta\,|B_{N\perp 1}|R^{N\perp 1} = \varepsilon/(2\exp(2\,\tilde{H}(f)/\varepsilon))$.

Therefore,

$$\begin{aligned}\int f(v)\,v_i^2\,dv &\geq \delta^2\left(1 - \frac{||f||_{L^1_s(\mathbb{R}^N)}}{R^s} - \varepsilon\right)\\ &= \frac{\varepsilon^2}{16\,|B_{N\perp 1}|^2 R^{2(N\perp 1)} e^{4\tilde{H}(f)/\varepsilon}}\left(1 - \frac{||f||_{L^1_s(\mathbb{R}^N)}}{R^s} - \varepsilon\right).\end{aligned}$$

Choosing $R = (4\,||f||_{L^1_s(\mathbb{R}^N)})^{1/s}$, $\varepsilon = 1/4$, we obtain the desired inequality. □

It is clear that this estimate is very rough, and that many other choices of R, s and ε are possible, depending on the context. If one applies crudely estimate (30) to a function f such that $\tilde{H}(f)$ is of the order of $\tilde{H}(M^f)$, one finds a rate λ whose order of magnitude is approximately $10^{\perp 36}$. Better estimates can be obtained by different choices of R, s and ε, but the very small factor $e^{\perp 4\tilde{H}(f)/\varepsilon}$ can never be much bigger than $e^{\perp 17}$ (in dimension 3), which is really very bad ! (compare to the linear Fokker-Planck equation, for which the rate corresponding to λ is of order 1, see [22]). The main cause is that the estimate in $L\log L$ is very poor for controlling the concentration of f.

In many cases, we know that in fact $f \in L^\infty([0,+\infty)\times\mathbb{R}^N)$. This is true for example for hard and modified hard potentials, even if the initial datum is not smooth, and this allows better estimates : for instance,

$$\int f 1_{\substack{|v|\leq R \\ |v_i|\leq \delta}} \leq 2\,\delta\,|B_{N-1}|\,\|f\|_{L^\infty}\, R^{N-1}.$$

3. Entropy dissipation : second method

Second proof of theorem 1. Here we shall recover the previous result with a different method. The idea is to try to transform the proof of the case of equality in the H–theorem into an explicit inequality. We know that we can restrict ourselves to the case of Maxwellian molecules. Moreover, we assume $N = 3$ for simplicity and suppose as in the previous section that $f \in \mathcal{S}(\mathbb{R}^3)$ with $|\log f|$ at most quadratically increasing.

For any function $(v, v_*) \longmapsto \lambda_{v,v_*}$, we define

$$R(v, v_*) = \frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(v_*) - \lambda_{v,v_*}(v - v_*).$$

For any circular permutation (i, j, k) of $(1, 2, 3)$, one has

$$\Big[(v - v_*) \wedge R(v, v_*)\Big]_k = (v_j - v_{*j})\left(\frac{\partial_i f}{f}(v) - \frac{\partial_i f}{f}(v_*)\right) - (v_i - v_{*i})\left(\frac{\partial_j f}{f}(v) - \frac{\partial_j f}{f}(v_*)\right).$$

We multiply this identity by the test function $\varphi_\alpha(v_*)f(v_*)$, where φ_α will be chosen later on. Integrating in the variable v_*, we obtain

$$\left(v_j\frac{\partial_i f}{f} - v_i\frac{\partial_j f}{f}\right)\left(\int f\varphi_\alpha\right) - \left(\frac{\partial_i f}{f}\right)\left(\int f v_j\varphi_\alpha\right) + \left(\frac{\partial_j f}{f}\right)\left(\int f v_i\varphi_\alpha\right)$$

$$= A^\alpha + B^\alpha v_j + C^\alpha v_i + \int \Big[(v - v_*) \wedge R(v, v_*)\Big]\Big|_k f_*\varphi_\alpha(v_*)\,dv_*,$$

with

$$\text{(31)} \qquad \begin{cases} A^\alpha = \int v_j\partial_i f\varphi_\alpha - \int v_i\partial_j f\varphi_\alpha, \\ B^\alpha = -\int \partial_i f\varphi_\alpha, \\ C^\alpha = \int \partial_j f\varphi_\alpha. \end{cases}$$

For α in $\{1, 2, 3\}$, this is a linear 3×3 system of equations for the quantities

$$v_j\frac{\partial_i f}{f} - v_i\frac{\partial_j f}{f}, \quad \frac{\partial_i f}{f}, \quad \frac{\partial_j f}{f}.$$

Using Cramer's formulas, we get

$$\frac{\partial_j f}{f} = \frac{\mathrm{Det}\left(\int f\varphi_\alpha, -\int f v_j \varphi_\alpha, Z_\alpha\right)}{\mathrm{Det}\left(\int f\varphi_\alpha, -\int f v_j\varphi_\alpha, \int f v_i \varphi_\alpha\right)}, \tag{32}$$

where

$$Z_\alpha = A^\alpha + B^\alpha v_j + C^\alpha v_i + \int (v-v_*)\wedge R(v,v_*)|_k\, f_*\varphi_\alpha(v_*)\, dv_*.$$

We now choose $\varphi_1 = 1$, $\varphi_2 = -v_j$, $\varphi_3 = v_i$. Assuming as before that $P_{ij} = T_i\,\delta_{ij}$, the denominator in the right–hand side of (32) is

$$D = \begin{vmatrix} \int f & -\int f v_j & \int f v_i \\ -\int f v_j & \int f v_j^2 & -\int f v_i v_j \\ \int f v_i & -\int f v_i v_j & \int f v_i^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & T_j & 0 \\ 0 & 0 & T_i \end{vmatrix} = T_i\, T_j.$$

Moreover, we immediately compute

$$A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

and

$$\mathrm{Det}\left(\int f\varphi_\alpha, -\int f v_j\varphi_\alpha, A^\alpha + B^\alpha v_j + C^\alpha v_i\right) = T_j v_j.$$

Thus, eq. (32) simply becomes

$$\frac{\partial_j f}{f} - \frac{v_j}{T_i} = \frac{\mathrm{Det}\left(\int f\varphi_\alpha, -\int f v_j\varphi_\alpha, \int (v-v_*)\wedge R(v,v_*)|_k\, f_*\varphi_\alpha(v_*)\, dv_*\right)}{T_i T_j}.$$

Taking the square of this expression, multiplying by f and integrating with respect to v, we obtain

$$\int \left(\frac{\partial_j f}{f} - \frac{v_j}{T_i}\right)^2 f \le \frac{1}{T_i^2\, T_j^2}$$

$$\times \int f \left|\mathrm{Det}\left(\int f\varphi_\alpha, -\int f v_j\varphi_\alpha, \int (v-v_*)\wedge R(v,v_*)|_k\, f_*\varphi_\alpha(v_*)\, dv_*\right)\right|^2 dv$$

$$\le \frac{1}{T_i^2\, T_j^2} \int f\, ||f||^2_{L^1_2(\mathbb{R}^3)} \left|\int (v-v_*)\wedge R(v,v_*)|_k\, f_*\varphi_\alpha(v_*)\, dv_*\right|_2^2 dv$$

$$\le \frac{(1+3)^2}{T_i^2\, T_j^2} \int dv\, f \left(\int dv_*\, |v-v_*|^2 R(v,v_*)^2 f_*\right)\left(\int dv_*\, f_*(1+v_*^2)\right)$$

$$\le \frac{(1+3)^3}{T_i^2\, T_j^2} \int\int dv\, dv_*\, f f_* |v-v_*|^2 R(v,v_*)^2.$$

$(1+3$ is here the sum of the moments of order 0 and 2 of $f)$.

On the other hand, since $\Pi(v-v_*)$ is the orthogonal projection onto $(v-v_*)^\perp$,

$$\frac{1}{2}\int\int \Pi(v-v_*)\left(\frac{\nabla f}{f}(v)-\frac{\nabla f}{f}(v_*)\right)\left(\frac{\nabla f}{f}(v)-\frac{\nabla f}{f}(v_*)\right)|v-v_*|^2\, ff_*$$

$$=\frac{1}{2}\int\int \left|\frac{\nabla f}{f}(v)-\frac{\nabla f}{f}(v_*)-\lambda_{v,v_*}(v-v_*)\right|^2 |v-v_*|^2 ff_*,$$

for a suitable function $(v,v_*)\longmapsto \lambda_{v,v_*}$. Therefore, noting that

$$\mu \equiv \frac{1}{2\,(1+3)^3}\inf_{i,j} T_i^2\, T_j^2 \geq \frac{1}{2\,(1+3)^3}\inf_i T_i^4 > 0$$

(thanks to the definition of T_j), we obtain for all i, j,

$$D(f)\geq \mu \int \left(\frac{\partial_j f}{f}-\frac{v_j}{T_i}\right)^2 f.$$

Expanding the square, we find that the integral in the right-hand side is simply

$$\int \frac{(\partial_j f)^2}{f}+\frac{T_j}{T_i^2}-\frac{2}{T_i}.$$

Summing up over i,j and dividing by 3^2, we get

$$D(f)\geq \frac{\mu}{3}\left(I(f)+\sum_i \frac{1}{T_i^2}-\sum_i\frac{2}{T_i}\right).$$

But since

$$\sum_i \frac{1}{T_i^2}-\sum_i\frac{2}{T_i}+3=\sum_i\left(\frac{1}{T_i}-1\right)^2\geq 0,$$

we finally find

$$D(f)\geq \frac{\mu}{3}(I(f)-3).$$

Note that the coefficient $\mu/3$ found with this method is much worse than the coefficient λ found in the previous one.

4. The trend towards equilibrium : overmaxwellian case

¿From now on, we shall consider classical solutions of the Landau equation on an arbitrary open time interval. We begin with a precise definition of these.

Definition 1: *A solution f on $\mathbb{R}_t^+\times\mathbb{R}_v^N$ of the Landau equation is said to be smooth if it belongs to $C^1((0,+\infty)_t;\mathcal{S}(\mathbb{R}_v^N))$ and if $|\log f|$ is bounded (for all $t_0,T>0$) by $C_{t_0,T}\,(1+|v|^2)$ when $t\in[t_0,T]$.*

As we said before, these assumptions are always satisfied in the case of hard potentials, modified or not, under very weak assumptions on the initial datum (see [13] for complete proofs in the case of hard potentials).

It is now easy to obtain an explicit result of convergence. We use the notation $H(f|M) = H(f) - H(M)$.

Theorem 3. *Let f be a smooth solution (in the sense of definition 1) of the Landau equation with initial datum f_{in} and overmaxwellian cross section (i.e. $\Psi(|z|) \geq |z|^2$). Then, for all time $t > 0$,*

$$H(f(t,\cdot)|M^{f_{in}}) \leq e^{\perp 2\lambda_0 t} H(f_{in}|M^{f_{in}}), \tag{33}$$

$$\|f(t,\cdot) - M^{f_{in}}\|_{L^1(\mathbb{R}^N)} \leq e^{\perp \lambda_0 t} \sqrt{2H(f_{in}|M^{f_{in}})}, \tag{34}$$

where $\lambda_0 > 0$ depends only on $M(f_{in}), E(f_{in})$ and $H(f_{in})$, and can be explicitly estimated from below.

Proof. We first note that (34) is deduced from (33) by the well-known Csiszar-Kullback inequality [9, 17, 22]. To obtain the theorem, it suffices, as in [22], to use the logarithmic Sobolev inequality of [15], in the form

$$I(f) - I(M^f) \geq \frac{2}{T^f} H(f|M^f),$$

which implies, thanks to theorem 1 (and proposition 2),

$$-\frac{d}{dt} H(f(t,\cdot)|M^{f_{in}}) \geq 2\,\lambda_0\, H(f(t,\cdot)|M^{f_{in}}).$$

Finally, even if f does not belong to $C([0,+\infty)_t, L\log L(\mathbb{R}^N_v))$, estimate (34) follows from the observation that for all time $\theta > 0$,

$$H(f(\theta,\cdot)|M^{f_{in}}) \leq H(f_{in}|M^{f_{in}}).$$

□

Remark. An explicit lower bound on λ_0 can be computed thanks to theorem 1 (including proposition 2), the rescaling (20), and the classical relations between $\tilde{H}$, $H(f_{in})$, $E(f_{in})$ and $M(f_{in})$. Of course, λ_0 is equal to λ when $M^f = M$.

As mentioned in the introduction, we now show that the return to equilibrium entails better entropy dissipation bounds, which in turn improve the convergence towards equilibrium. One can prove for example the

Theorem 4. *Let f be a smooth solution (in the sense of definition 1) of the Landau equation with initial datum f_{in} and overmaxwellian cross section. Then for all time $t > 0$ and $\varepsilon > 0$, there exists $s > 0$, and C depending only on* $\sup_t ||f(t,\cdot)||_{L^1_s(\mathbb{R}^N)}$, *such that*

$$
\begin{aligned}
-\frac{d}{dt}&H(f(t,\cdot)|M^f) \geq \\
&\max\Big\{2\,(N-1)\Big(1 - C\,H(f(t,\cdot)|M^f)^{\frac{1-\varepsilon}{2}}\Big), 2\,\lambda_0\Big\}\ H(f(t,\cdot)|M^f).
\end{aligned}
$$

Proof. We prove that for all direction i,

$$\int f(v)\, v_i^2\, dv \leq 1 + C\,H(f|M^f)^{\frac{1-\varepsilon}{2}}.$$

The conclusion will follow by (28).

Clearly,

$$\int f(v)\, v_i^2\, dv \leq \int M^f(v)\, v_i^2\, dv + \int |f - M^f|\, |v|^2.$$

But for all $R > 0$,

$$\int |f - M^f|\, |v|^2 \leq R^2\, \|f - M^f\|_{L^1(\mathbb{R}^N)} + \frac{||f||_{L^1_s(\mathbb{R}^N)}}{R^{s\perp 2}}.$$

Choosing $R = (||f||_{L^1_s(\mathbb{R}^N)}/\|f - M^f\|_{L^1(\mathbb{R}^N)})^{1/s}$, we get

$$\int |f - M|\, |v|^2 \leq C\, \|f - M\|^{1\perp 2/s}_{L^1(\mathbb{R}^N)} \leq CH(f|M^f)^{1/2\perp 1/s}.$$

□

Remark.

1. As a consequence of the appearance and (global in time) propagation of all moments in the case of modified hard potentials, we see that for sufficiently large times, the rate of exponential convergence in L^1 norm in that situation can be taken bounded from below by $(2-\varepsilon)(N-1)$ for $\varepsilon > 0$ as small as desired.
2. In the case of Maxwellian molecules (i.e. $\Psi(|z|) = |z|^2$), the second-order moments are explicitly computable and return exponentially fast to their equilibrium value [24]. This implies that the rate of the trend to equilibrium is much better.

5. Improved results

In the preceding section, we used the entropy estimate to control the concentration of f. This method is quite natural, but, as we saw, leads to rather poor results. It turns out, surprisingly, that much more realistic times are obtained when one controls this concentration by the use of the entropy dissipation itself.

To this purpose, we consider$f \in L^1_2(\mathbb{R}^N)$ a nonnegative function with $M(f) = 1$, $E(f) = N$, and we limit ourselves to the overmaxwellian case. Let us start again from

$$D(f) \geq \sum_i (N - T_i) \int \frac{(\partial_i f)^2}{f} - N(N-1),$$

and apply Lemma 1 to find

$$D(f) \geq \sum_i \frac{N - T_i}{T_i} - N(N-1) = N \sum_i \frac{1}{T_i} - N^2,$$

hence

$$\sum_i \frac{1}{T_i} \leq N + \frac{D(f)}{N}.$$

Since we chose an orthonormal basis in which $\int f v_i v_j = T_i \delta_{ij}$, this proves the

Proposition 5. *Let $f \in L^1_2(\mathbb{R}^N)$ be a nonnegative function with $M(f) = 1$, $E(f) = N$, and let $T_f = \inf_{e \in S^{N-1}} \int f(v)(v \cdot e)^2 \, dv$. Then (under the overmaxwellian assumption $\Psi(|z|) \geq |z|^2$),*

$$T_f \geq \frac{1}{N + \frac{D(f)}{N}}. \tag{35}$$

Then we introduce this estimate in the (first) proof of theorem 1, and we get

$$D(f) \geq T_f \big[I(f) - I(M) \big] \geq \frac{2N}{D(f) + N^2} H(f|M).$$

Thus

$$D(f)^2 + N^2 D(f) - 2N H(f|M) \geq 0.$$

Since $D(f) \geq 0$, this inequality implies the

Proposition 6. *Let $f \in L^1_2(\mathbb{R}^N)$ be a nonnegative function with $M(f) = 1$, $E(f) = N$. Then (under the overmaxwellian assumption $\Psi(|z|) \geq$*

$|z|^2)$,

$$D(f) \geq \sqrt{2N\,H(f|M) + \frac{N^4}{4}} - \frac{N^2}{2}. \tag{36}$$

Now, let f be a classical solution of the Landau equation, with overmaxwellian potential, satisfying the previous normalizations of mass and energy. The proposition above implies that $H(f(t)|M) \leq y(t)$, where y is the solution of

$$\begin{cases} -\dot{y} = \sqrt{2Ny + \frac{N^4}{4}} - \frac{N^2}{2} \\ \\ y(0) = y_0 = H(f_{in}|M). \end{cases} \tag{37}$$

By standard computations, y is implicitly defined by

$$t = \int_y^{y_0} \frac{dz}{\sqrt{2Nz + \frac{N^4}{4}} - \frac{N^2}{2}}$$

$$\begin{aligned} &= \frac{1}{N}\sqrt{2Ny_0 + \frac{N^4}{4}} + \frac{N}{2}\ln\left(\sqrt{2Ny_0 + \frac{N^4}{4}} - \frac{N^2}{2}\right) \\ &\quad - \frac{1}{N}\sqrt{2Ny + \frac{N^4}{4}} - \frac{N^2}{2}\ln\left(\sqrt{2Ny + \frac{N^4}{4}} - \frac{N^2}{2}\right). \end{aligned} \tag{38}$$

Let $C_0 = \sqrt{2Ny_0 + N^4/4} - N^2/2$: since $y \geq 0$, (38) entails

$$t \leq \frac{C_0}{N} + \frac{N}{2}\ln\frac{C_0}{\sqrt{2Ny + \frac{N^4}{4}} - \frac{N^2}{2}},$$

then

$$\begin{aligned} y &\leq \frac{1}{2N}\left\{\left(C_0 e^{\frac{2}{N^2}C_0} e^{\perp\frac{2}{N}t} + \frac{N^2}{2}\right)^2 - \frac{N^4}{4}\right\} \\ &= \frac{C_0^2}{2N} e^{\frac{4}{N^2}C_0} e^{\perp\frac{4}{N}t} + \frac{NC_0}{2} e^{\frac{2}{N^2}C_0} e^{\perp\frac{2}{N}t}. \end{aligned}$$

As a conclusion, using again the Csiszar-Kullback inequality, we have proven the following refinement of theorem 3.

Theorem 7. *Let f be a classical solution of the Landau equation with overmaxwellian potential in $\mathbb{R}^N$, and such that $M(f_{in}) = 1$, $E(f_{in}) = N$, $H(f_{in}) < \infty$. Let $M(v) = e^{\perp|v|^2/2}/(2\pi)^{N/2}$, and*

$$C_0 = \sqrt{2NH(f_{in}|M) + \frac{N^4}{4}} - \frac{N^2}{2}.$$

Then, for all time t,

$$\|f(t) - M\|_{L^1} \le \left(\sqrt{NC_0}e^{\frac{1}{N^2}C_0}\right) e^{\perp \frac{t}{N}} + \left(\frac{C_0}{\sqrt{N}} e^{\frac{2}{N^2}C_0}\right) e^{\perp \frac{2t}{N}}. \tag{39}$$

Remarks.

1. The rate obtained, $1/N$, differs from the rate of the linear Fokker-Planck equation $\partial_t f = (N-1)\nabla \cdot (\nabla f + fv)$ by a multiplicative factor $1/N(N-1)$. Of course, it is possible to combine this estimate with theorem 4 to obtain other estimates (more complicated) with an almost optimal rate.
2. The idea of using the entropy dissipation to control the concentration of f is apparently due to Arkeryd and Nouri [3]; their motivation was to get L^1 compactness estimates for solutions of the stationary Boltzmann equation in a (one-dimensional) slab, in the absence of entropy estimate. In the present case, from Proposition 6 one immediately deduces that (in the overmaxwellian case), if (f^n) is a sequence of nonnegative functions such that $\sup_n[M(f^n) + E(f^n) + D(f^n)] < \infty$, then (f^n) is weakly relatively compact in L^1.

6. The trend towards equilibrium : the case of true hard potentials

For true hard potentials, (that is $\Psi(z) = |z|^{2+\gamma}$), a new difficulty arises from the fact that if $|z|$ is close to 0, then $\Psi(|z|) = |z|^{2+\gamma}$ is negligible in front of $|z|^2$. Because of that, when f is close to M, we recover an algebraic decay instead of an exponential one.

Theorem 8. *If f is a smooth solution (in the sense of definition 1) of the Landau equation with initial datum f_{in} and true hard potentials (that is $\Psi(|z|) = |z|^{2+\gamma}$ with $\gamma \in (0,1)$), then there exists $\lambda_1, \lambda_2 > 0$ (depending only on N, γ and λ of theorem 1), such that for all time $t > 0$,*

$$-\frac{d}{dt} H(f(t,\cdot)|M^{f_{in}}) \ge \min\Big(\lambda_1 H(f(t,\cdot)|M^{f_{in}}), \lambda_2 H(f(t,\cdot)|M^{f_{in}})^{1+\frac{\gamma}{2}}\Big).$$

In particular, for all time $t > 0$, there is a constant C depending only on γ, λ_1, λ_2 and $H(f_{in})$, such that

$$H(f(t,\cdot)|M^{f_{in}}) \le C(1+t)^{\perp 2/\gamma}.$$

Proof. We consider only the case when f satisfies the normalization (19). For all $\varepsilon > 0$, we have

$$\Psi(|z|) = |z|^{\gamma+2} \ge \varepsilon^\gamma z^2 - \varepsilon^{\gamma+2} 1_{|z| \le \varepsilon}. \tag{40}$$

Estimating the entropy dissipation for $\Psi \equiv 1$ from above, we get (using the convolution structure)

$$\frac{1}{2}\int\int dv\, dv_*\, \Pi(v-v_*) f f_* \left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(v_*)\right)\left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(v_*)\right)$$
$$= \int\int dv\, dv_*\, \Pi(v-v_*) f_* \frac{\nabla f\, \nabla f}{f} - \int\int dv\, dv_*\, \Pi(v-v_*) \nabla f(v_*)\, \nabla f(v)$$
$$= \int dv\, \overline{\Pi}(v) \frac{\nabla f\, \nabla f}{f}(v) + \int \nabla^2 \overline{\Pi}\, f,$$

where $\overline{\Pi} = \Pi * f$, and

$$\nabla^2 \overline{\Pi}(z) = \left(\sum_{ij} \partial_{ij} \Pi_{ij}\right) * f = -\frac{(N-1)(N-2)}{|z|^2} * f \le 0.$$

Since, in the sense of matrices,

$$\|\overline{\Pi}\|_{L^\infty} \le \|\Pi\|_{L^\infty} \|f\|_{L^1} \le 1,$$

we obtain

(41)
$$\frac{1}{2}\int\int dv\, dv_*\, \Pi(v-v_*) f f_* \left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(v_*)\right)\left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(v_*)\right) \le I(f).$$

Therefore, in view of (40), and thanks to theorem 1,

$$-\frac{d}{dt} H(f(t,\cdot)|M^{f_{in}}) \ge \lambda_0\, \varepsilon^\gamma \Big(I(f(t,\cdot)) - I(M^{f_{in}}) \Big) - \varepsilon^{\gamma+2} I(f(t,\cdot))$$
$$\ge \lambda_0 \left(\varepsilon^\gamma - \frac{\varepsilon^{\gamma+2}}{\lambda_0}\right) \Big(I(f(t,\cdot)) - I(M^{f_{in}}) \Big) - N \varepsilon^{\gamma+2},$$

where we have used $I(M^{f_{in}}) = N$. Choosing

$$\varepsilon^2 = \min\left\{ \frac{\lambda_0}{2}, \frac{\lambda_0}{4N} \Big(I(f(t,\cdot)) - I(M^{f_{in}}) \Big) \right\}, \tag{42}$$

we obtain

$$-\frac{d}{dt} H(f(t,\cdot)|M^{f_{in}}) \ge \frac{\lambda_0}{4} \Big(I(f(t,\cdot)) - I(M^{f^{in}}) \Big) \varepsilon^\gamma. \tag{43}$$

Separating the cases when $I(f(t,\cdot)) - I(M^{f_{in}}) \le 2N$ or not, we get the result by (42) and the logarithmic Sobolev inequality. □

Remarks.

1. The convergence in theorem 8 actually holds in $H^\infty(\mathbb{R}^N)$ as soon as $f_{in} \in L^1_{s_0}(\mathbb{R}^N)$ for some $s_0 > \gamma^2/2 + \gamma + 4$. This is easily proved by induction, using the bounds found in [13], and elementary interpolation inequalities such as
$$\|f-M\|^2_{L^2(\mathbb{R}^N)} \le \|f-M\|_{L^1(\mathbb{R}^N)} \|f-M\|_{L^\infty(\mathbb{R}^N)}$$
$$\le C_N \|f-M\|_{L^1(\mathbb{R}^N)} \|f-M\|_{H^{\frac{N+2}{2}}(\mathbb{R}^N)}$$
and
$$\|f-M\|^2_{H^k(\mathbb{R}^N)} \le C_N \int |\widehat{f}-\widehat{M}|^2 (1+|\xi|^2)^k$$
$$\le C_N \left(\int |\widehat{f}-\widehat{M}|^2 \right)^{1/2} \left(\int |\widehat{f}-\widehat{M}|^2 (1+|\xi|^2)^{2k} \right)^{1/2}$$
$$\le C_N \|f-M\|_{L^2(\mathbb{R}^N)} \|f-M\|_{H^{2k}(\mathbb{R}^N)}.$$
2. In this theorem as well as in those of the previous section, the assumption that the solution of the Landau equation is smooth is not absolutely necessary. As a general rule, entropy inequalities are usually true even with very weak solutions, due to the convex character of the entropy production.

We now turn to another problem, namely the uniform (in time) stability of the solutions of the Landau equation with true hard potentials. The local (in time) stability is obtained in [13], as a consequence of results of Gronwall type which also entail the uniqueness of the solution. Once the study of the long-time behaviour is done, we are now able to prove the following result.

Theorem 9. *Let $f_{in}, \tilde{f}_{in} \in L^1_r \cap L^2_s(\mathbb{R}^N)$ with $r > 3\gamma + 4 + N$, $s > 2\gamma + 8 + N$, and f, $\tilde{f}$ be the unique weak solutions of the Landau equation for true hard potentials (that is, with $\Psi(z) = |z|^{2+\gamma}$, $\gamma \in (0,1)$) and initial data $f_{in}, \tilde{f}_{in}$. Then, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|f_{in} - \tilde{f}_{in}\|_{L^1_r \cap L^2_s} \le \delta$, then* $\sup_{t>0} ||\tilde{f}(t) - f(t)||_{L^1(\mathbb{R}^N)} \le \varepsilon$.

Proof. Note first that $f \to M^f$ is continuous from $L^1_2(\mathbb{R}^N)$ to $L^1(\mathbb{R}^N)$. Hence we take $\delta > 0$ in such a way that
$$||M^{f_{in}} - M^{\tilde{f}_{in}}||_{L^1(\mathbb{R}^N)} \le \varepsilon/3.$$
Then, one of the consequences of theorem 8 is the convergence in L^1 of $f(t,\cdot)$ and $\tilde{f}(t,\cdot)$ to $M^{f_{in}}$ and $M^{\tilde{f}_{in}}$ respectively, with rates that are bounded by below explicitly, depending only on all second-order moments of f_{in}, $\tilde{f}_{in}$, and the associated entropies. Thus, one can find

$T_\varepsilon > 0$ such that when $t > T_\varepsilon$,

$$||f(t,\cdot) - M^{f_{in}}||_{L^1(\mathbb{R}^N)} \leq \varepsilon/3, \quad ||\tilde{f}(t,\cdot) - M^{\tilde{f_{in}}}||_{L^1(\mathbb{R}^N)} \leq \varepsilon/3.$$

Finally, thanks to the property of local (in time) stability of our equation (Cf. [13]), there exists $\delta > 0$ such that

$$||f_{in} - \tilde{f}_{in}||_{L^1_r \cap L^2_s} \leq \delta \Longrightarrow ||f(t,\cdot) - \tilde{f}(t,\cdot)||_{L^1(\mathbb{R}^N)} \leq \varepsilon/3.$$

for all $t \in [0, T_\varepsilon]$. $\square$

Remark. As in [13], one can also obtain the stability with respect to the cross section, in the sense that Ψ may be replaced by

$$\tilde{\Psi}(z) = |z|^{\gamma+2}(1 + \eta(|z|)),$$

where η as well as its derivatives up to order 2 are small enough in $L^\infty([0, +\infty[)$.

7. Poincaré-type inequalities and applications

The aim of this section is to give two applications of the techniques previously developed in the context of the linearized Landau equation. These techniques enable us to precise estimates proven in [10] by Degond and Lemou and to make some of the constants which appear in those estimates more explicit.

First, we recall the following Poincaré–type inequality of [10]:

Proposition 10. *For all $r > 0$, one can find $C(r) > 0$ such that for any $h \in \mathcal{D}(\mathbb{R}^N)$ verifying $\int M\, h = 0$,*

$$\int (1 + |v|^2)^{r/2}\, M(v)\, |\nabla h(v)|^2\, dv \geq C(r) \int M(v)\, |h(v)|^2\, dv.$$

Thanks to a linearization of the logarithmic Sobolev inequality, one can in fact prove the

Proposition 11. *For any $h \in \mathcal{D}(\mathbb{R}^N)$ verifying $\int M\, h = 0$, one has*

$$\int M(v)\, |\nabla h(v)|^2\, dv \geq \int M(v)\, |h(v)|^2\, dv. \tag{44}$$

Proof. Let us write the logarithmic Sobolev inequality in the form

$$\int f_\varepsilon^2 \log f_\varepsilon^2\, M \leq 2 \int |\nabla f_\varepsilon|^2\, M + \left(\int f_\varepsilon^2\, M \right) \log \left(\int f_\varepsilon^2\, M \right), \tag{45}$$

where $f_\varepsilon = 1 + \varepsilon\, h$ and $\varepsilon > 0$ is small enough.

Developing (45) with respect to ε when $\varepsilon \to 0$, and keeping only the main order terms (that is, those in ε^2), we exactly recover proposition 11. $\square$

Remark.

1. The assumption $h \in \mathcal{D}(\mathbb{R}^N)$ can be replaced in propositions 10 and 11 by a much weaker assumption. In fact, inequalities (44) and (45) hold as soon as they are well–defined.
2. In a slightly more abstract setting, this result is known as the Rothaus-Simon mass gap theorem (Cf. [20, 21, 16]).

Let us now define $Lh = 2\,M^{-1}\,Q(M, M\,h)$, the usual linearized Landau operator. Here Q is the symmetric bilinear form related to the quadratic Landau operator (2), namely

$$Q(f,g) = \frac{1}{2}\frac{\partial}{\partial v_i}\Big\{ \int_{\mathbb{R}^N} dv_*\, a_{ij}(v-v_*) \left(f_* \frac{\partial g}{\partial v_j}(v) + g_* \frac{\partial f}{\partial v_j}(v) \right. \\ \left. - g\, \frac{\partial f}{\partial v_j}(v_*) - f\,\frac{\partial g}{\partial v_j}(v_*) \right) \Big\}. \tag{46}$$

We denote by (,) the (weigthed L^2) scalar product

$$(f,g) = \int_{\mathbb{R}^N} f\,g\,M\,dv.$$

It is proven in [10] in the case of true hard potentials that L is coercitive (up to a known finite dimensional vector space) with respect to this scalar product. The sketch of the proof is the following : the operator is decomposed as $L = L_1 + L_2$, where L_1 is a diffusion operator, and L_2 is a compact perturbation. The coercitiveness of L_1 is a consequence of proposition 10, and that of L a consequence of Weyl's theorem on compact perturbations of operators. The constant of coercitiveness of L is therefore not explicitly known.

We show here how, in the simpler case of modified hard potentials (or more generally for any overmaxwellian cross section), the following explicit estimate of coercitiveness holds,

Proposition 12. *In the case of overmaxwellian cross section (that is, when $\Psi(z) \geq |z|^2$), one has for any $h \in \mathcal{D}(\mathbb{R}^N)$ satisfying the assumption $\int M\,h \begin{pmatrix} 1 \\ v_i \\ v^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (for $i = 1,..,N$),*

$$-(Lh,h) \geq 2\,(N-1)\,(h,h). \tag{47}$$

Proof. Thanks to theorem 1 and eq. (28) applied to $f_\varepsilon = M\,(1+\varepsilon\,h)$ (with $\varepsilon > 0$ small enough), one has

$$-D(f_\varepsilon) \geq \lambda\,(H(f_\varepsilon) - H(M)) = \min_{i=1,..,N}(N - T_i^\varepsilon)\,(H(f_\varepsilon) - H(M)), \tag{48}$$

where $T_i^\varepsilon = \int f_\varepsilon\, v_i^2$.

Developing the terms of (48) with respect to ε up to order 2, one finds

$$H(f_\varepsilon) - H(M) = \varepsilon^2 \int h^2\, M + O(\varepsilon^3),$$

$$D(f_\varepsilon) = 2\,\varepsilon^2 \int Q(M, M\, h)\, h + O(\varepsilon^3),$$

and

$$T_i^\varepsilon = 1 + O(\varepsilon).$$

Letting ε go to 0, we get eq. (47). □

This theorem gives a lower bound on the size of the spectral gap of L in the case of overmaxwellian molecules. This should be compared to the frame of the Boltzmann equation, in which the spectrum of the linearized operator can be explicitly computed only for true Maxwellian molecules [7].

8. Entropy dissipation and regularity estimates

We now show that under very weak assumptions, any solution f of the Landau equation for Maxwellian or hard potentials which is a weak cluster point of asymptotically grazing solutions of the Boltzmann equation (Cf. [13] for a precise definition) satisfies $\sqrt{f} \in L^2_{loc}([0,+\infty)_t; H^1(\mathbb{R}^N_v))$ (Here the important point is that the initial time is included in the estimate).

Many different assumptions on the type of the considered Boltzmann equation are possible. We restrict ourselves here to a typical example.

Theorem 13. *Let $f_{in} \in L^1_2 \cap L\log L(\mathbb{R}^N)$, and let (f^ε) be a family of asymptotically grazing solutions of the Boltzmann equation with (cut-offed or uncutoffed) hard potentials and initial datum f_{in} (Cf.* [13] *for a precise definition, and the following notations).*

Assume in addition that all f^ε satisfy an entropy inequality, i.e.

$$(49)\quad \frac{1}{4}\int_0^T dt \int dv\, dv_*\, d\theta\, d\phi\, |v-v_*|^\gamma \zeta_\varepsilon(\theta)\left(f'^\varepsilon f'^\varepsilon_* - f^\varepsilon f^\varepsilon_*\right)\log\frac{f'^\varepsilon f'^\varepsilon_*}{f^\varepsilon f_*^\varepsilon} \le H(f_{in}) - H(f^\varepsilon(T,\cdot)),$$

Then, any cluster point f in $L^1(\mathbb{R}^+_t; L^1(\mathbb{R}^N_v) - w)$ satisfies

$$\nabla\sqrt{f} \in L^2_{loc}([0,+\infty)_t; H^1(\mathbb{R}^N_v)).$$

Remarks.

1. For any initial datum $f_{in} \in L^1_{2+\delta}(\mathbb{R}^N)$ (with $\delta > 0$), such solutions f^ε of Boltzmann equation always exist, and in addition f is a weak solution of the Landau equation with $\Psi(z) = |z|^{\gamma+2}$ (Cf. [26]).
2. If there is uniqueness in the Cauchy problem for the Boltzmann equation with cross section B_ε, then all solutions satisfy an entropy inequality. This holds in particular for cutoffed cross sections, i.e. $\zeta_\varepsilon \in L^1(0,\pi)$, in view of the results in [19] for instance.
3. We think that most probably the conclusion of Theorem 13 fails if one does not assume that f_{in} belongs to $L\log L$. Think for example of the fundamental solution E of the heat equation, which satisfies only $\|\nabla\sqrt{E}\|^2_{L^2(\mathbb{R}^N)} \in L^{1,\infty}(0,T)$.

Proof. The following result is proven in [26],

$$\frac{1}{2}\int_0^T dt \int dv\, dv_*\, a(v-v_*)(\nabla - \nabla_*)\sqrt{ff_*}\,(\nabla-\nabla_*)\sqrt{ff_*}$$
$$\le \underline{\lim}\frac{1}{4}\int_0^T dt \int dv\, dv_*\, d\theta\, d\phi\, \zeta_\varepsilon(\theta)\left(f'^\varepsilon f'^\varepsilon_* - f^\varepsilon f^\varepsilon_*\right)\log\frac{f'^\varepsilon f'^\varepsilon_*}{f^\varepsilon f^\varepsilon_*}.$$

In short,

$$D(f) \le \underline{\lim} D_{B_\varepsilon}(f^\varepsilon),$$

where D_{B_ε} is the entropy dissipation for the Boltzmann equation with cross section B_ε. According to (49), the right-hand side is bounded by $\underline{\lim}[H(f_{in}) - H(f^\varepsilon(T,\cdot))]$, which is in turn bounded from above by $H(f_{in}) - H(M^{f_{in}})$.

Using estimate (43) (which does not require any smoothness assumption if the left-hand side is defined as the entropy production in the sense of appendix A) we get

$$\int_0^T dt\, I(f(t,\cdot)) < +\infty,$$

which is enough to conclude.

Note that theorem 1 is used here (and only here in this paper) for non smooth solutions of the Landau equation. □

Appendix A. Definition of the entropy dissipation

We have defined the entropy dissipation of a function $f \in L^1(\mathbb{R}^N)$ as the nonnegative quantity (possibly infinite) $(1/2)\|K\|^2_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}$, where

$$K(v,v_*) = 2\,\Pi(v-v_*)\,\Psi^{1/2}(|v-v_*|)\,(\nabla-\nabla_*)\sqrt{ff_*}.$$

In order that this be meaningful, we must of course check that K defines a distribution on $\mathbb{R}^N_v \times \mathbb{R}^N_{v_*}$. Since $\sqrt{ff_*}$ belongs only to $L^2(\mathbb{R}^N \times \mathbb{R}^N)$, it would seem that one needs to impose some smoothness assumptions on Ψ for the product of distributions to make sense. But, following remarks done in [26] (see also [23]), we shall show that this is not the case, and that Ψ only needs to be locally integrable. To that purpose, we shall use two key points of the structure of the Landau equation, namely the symmetry and the orthogonal projection.

Let $\varphi \in C_0^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ be a smooth test function. We shall use the notations

$$\varphi = \varphi(v, v_*), \qquad \varphi_* = \varphi(v_*, v).$$

First of all, by symmetry, since $K(v_*, v) = -K(v, v_*)$,

$$\int\int K(v,v_*)\,\varphi(v,v_*)\,dv\,dv_* = \frac{1}{2}\int\int K(v,v_*)\,(\varphi - \varphi_*)$$

$$= \int\int \Pi(v-v_*)\Psi^{1/2}(|v-v_*|)(\nabla - \nabla_*)\sqrt{ff_*}\,(\varphi - \varphi_*).$$

Next, integrating by parts,

$$\begin{aligned}\int\int K\varphi = &-\int\int (\nabla - \nabla_*)\cdot\Pi(v-v_*)\,\Psi^{1/2}(|v-v_*|)\sqrt{ff_*}\,(\varphi-\varphi_*)\\ &-\int\int \Pi(v-v_*)\,(\nabla-\nabla_*)\Psi^{1/2}(|v-v_*|)\,\sqrt{ff_*}\,(\varphi-\varphi_*)\\ &-\int\int \Pi(v-v_*)\,\Psi^{1/2}(|v-v_*|)\,\sqrt{ff_*}(\nabla-\nabla_*)(\varphi-\varphi_*).\end{aligned}$$

Now, we note that the integral in the second line vanishes, because $(\nabla - \nabla_*)\Psi^{1/2}(|v-v_*|)$ is proportional to $v - v_*$, and lies therefore in the kernel of Π. Since

$$(\nabla - \nabla_*)\cdot\Pi(v-v_*) = -2\,(N-1)\,(v-v_*)/|v-v_*|^2,$$

we get

$$\begin{aligned}\int K\varphi =&(N-1)\int\int \frac{v-v_*}{|v-v_*|^2}\,\Psi^{1/2}(|v-v_*|)\,\sqrt{ff_*}\,(\varphi-\varphi_*)\\ &-\int\int \Pi(v-v_*)\,\Psi^{1/2}(|v-v_*|)\,\sqrt{ff_*}\,(\nabla-\nabla_*)(\varphi-\varphi_*).\end{aligned}$$

Since $\sqrt{ff_*} \in L^2$ and $\varphi - \varphi_* = O(|v-v_*|)$ (as a C^∞ function vanishing when $v - v_* = 0$), we see that this definition is meaningful as soon as $\Psi^{1/2}(|v-v_*|) \in L^2_{\mathrm{loc}}(dv\,dv_*)$, i.e.

$$z \longmapsto \Psi(|z|) \in L^1_{loc}(\mathbb{R}^N), \tag{50}$$

which is more than sufficient to cover all the physically interesting cases (see [26] for related ideas).

Appendix B. Approximation of the entropy dissipation

To prove rigorously theorem 1 in the case when f is not smooth, we need the following lemma (we note $\Pi = \Pi(v - v_*)$).

Lemma 2. *Let f be a nonnegative function belonging to $L^1_2(\mathbb{R}^N)$, and let us consider the Landau operator in the case of Maxwellian molecules (that is, when $\Psi(|z|) = |z|^2$). Then there exists a sequence of smooth functions f^ε such that $f^\varepsilon \longrightarrow f$ in $L^1(\mathbb{R}^N)$ and $D(f^\varepsilon) \longrightarrow D(f) \in [0, +\infty]$. More precisely, f^ε can be chosen in $\mathcal{D}(\mathbb{R}^N)$, or in $\mathcal{S}(\mathbb{R}^N)$ with $|\log f^\varepsilon| \leq C_\varepsilon\,(1 + |v|^2)$.*

Remarks. This lemma also holds for hard potentials, and in fact for a very large class of cross sections, provided that f lies in a restricted class of functions (for example $L^1_{2+\gamma}(\mathbb{R}^N)$ for hard potentials).

Proof. First note that the result is clear if $D(f) = +\infty$. Indeed, if there existed a sequence f^ε converging towards f without $D(f^\varepsilon)$ going to infinity, one could extract a subsequence, still denoted f^ε, with $D(f^\varepsilon) \leq C$, and by standard convexity arguments $D(f)$ would be finite (at least if $\sqrt{f^\varepsilon}$ converges towards $\sqrt{f}$ in $L^2(\mathbb{R}^N)$).

Hence, let us assume that $D(f) < +\infty$.

We define

$$f^\eta = f\,\chi_\eta, \qquad f^{\eta,\delta} = \left(\sqrt{f\,\chi_\eta} * \rho_\delta\right)^2,$$

where $\chi_\eta(v) = \chi(\eta\, v)$ is a smooth truncation function which is identically 1 for $|v| \leq \eta^{-1}$, and identically 0 for $|v| \geq 2\,\eta^{-1}$, and $\rho_\delta(v) = \delta^{-N}\rho(\delta^{-1}\, v)$ is an approximation of the identity for the convolution. The function ρ is chosen either compactly supported, or a centered normalized Maxwellian, in order to get the two versions of the lemma.

For a given $\eta > 0$, the convergence of $D(f^{\eta,\delta})$ towards $D(f^\eta)$ is a simple consequence of the following classical result (in the case when ρ is compactly supported, Cf. [14] for instance),

$$b \in W^{1,\infty}, \quad g \in L^2 \quad \Longrightarrow b\,(\nabla g) * \rho_\delta - (b\,\nabla g) * \rho_\delta \longrightarrow 0 \quad \text{in } L^2_{\mathrm{loc}},$$

applied with $b = \Pi\,|v - v_*|$, and $g = \sqrt{f^{\eta,\delta}\, f^{\eta,\delta}_*}$. When ρ is a Maxwellian, this result also holds with a slightly different proof.

It remains to check the convergence of $D(f^\eta)$ towards $D(f)$. It holds because by assumption $D(f)$ is finite and $f \in L^1_2(\mathbb{R}^N)$.

Finally, we choose $f^\varepsilon = f^{\eta(\varepsilon),\varepsilon}$, with $\eta(\varepsilon)$ defined in a convenient way, to get the theorem. Note that $\log(f^\varepsilon)$ is at most quadratically increasing when ρ is a Maxwellian thanks to standard estimates on the heat semigroup. $\square$

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