ON THE CHAPMAN-ENSKOG ASYMPTOTICS FOR A MIXTURE OF MONOATOMIC AND POLYATOMIC RAREFIED GASES

CÉLINE BARANGER, MARZIA BISI, STÉPHANE BRULL,
AND LAURENT DESVILLETTES

Abstract. In this paper, we propose a formal derivation of the Chapman-Enskog asymptotics for a mixture of monoatomic and polyatomic gases. We use a direct extension of the model devised in [8, 16] for treating the internal energy with only one continuous parameter. This model is based on the Borgnakke-Larsen procedure [6]. We detail the dissipative terms related to the interaction between the gradients of temperature and the gradients of concentrations (Dufour and Soret effects), and present a complete explicit computation in one case when such a computation is possible, that is when all cross sections in the Boltzmann equation are constants.

1. Introduction

In the computations of the flow around a shuttle in the context of reentry in the upper atmosphere, it is necessary to use a kinetic description (that is, Boltzmann equations) since the Knudsen number Kn (defined as the mean free path of a molecule of the gas divided by a characteristic length of the shuttle) is of order 1 (or larger) at high altitude. It is also necessary to couple this kinetic description with a coherent macroscopic description used at lower altitudes where the Knudsen number becomes much smaller than 1.

Such a coupling is well understood for one monotamic gas thanks to the establishment of the Chapman-Enskog asymptotics, which clarifies (at the formal level, cf. [2], [11], and, in a perturbative context, also at the rigorous level, cf. [22]) the relationships between the Boltzmann equation and the compressible Navier-Stokes(-Fourier) equations of one perfect monoatomic gas. The link between the cross section in the Boltzmann equation and the dependence of the transport coefficients (viscosity and heat conductivity) w.r.t. temperature is related to the resolution of a specific linear Boltzmann equation (cf. [15] for example), which can be solved in some specific situations, including the case of Maxwell molecules (cf. [11]).

It is however important to perform the Chapman-Enskog asymptotics in situations much more complicated than the ones in which is considered one single monoatomic gas. Indeed, the main chemical species found in the upper atmosphere of the earth are the molecular oxygen (O_2) and the molecular nitrogen (N_2), which are both diatomic. Moreover, due to the chemical (dissociation/recombination) reactions taking place in the heated air surrounding a shuttle, one should also (at least) take into account the atomic oxygen O, the atomic nitrogen N (both are obviously monoatomic) and the diatomic nitrogen monoxide NO. As a consequence, it is important to be able to treat mixtures of several monoatomic and polyatomic gases with different masses (note that it is possible to approximate the masses of N_2, O_2 and NO by a common value, but this cannot be generalized if one takes into account the (atomic) argon Ar, whose concentration in the upper atmosphere is not unsignificant).

Our goal is to present in detail the Chapman-Enskog asymptotics in a model as simple as possible fulfilling the assumptions described above (that is, taking into account a mixture of several monoatomic and polyatomic gases with different masses), and which enables to recover at the macroscopic level a set of compressible Navier-Stokes equations for perfect gases with general energy laws. The model proposed in [8, 16] almost fulfills those assumptions. It uses as unknowns the number densities $f^{(i)}(t, x, v, I)$ of particles of the i-th species which at time t and point x move with velocity v and have a one-dimensional internal energy parameter $I > 0$. The choice of one parameter in the model enables to get quite general energy equations, but unfortunately not the energy equation of monoatomic gases (which can be recovered only as a limit of the model). In order to integrate the possibility of having mixtures of monoatomic and polyatomic species, we introduce therefore in the model of [16] collision kernels for monoatomic-diatomic collisions (these kernels are described in section 2). For some applications of such models we refer to [23], [27], [19]. In particular in [23], the authors highlight different types of shock profiles which are specific to the polyatomic setting by using the model given in [1], [10], [24]. In [17], a numerical model for polyatomic gases using the reduced distribution technique is derived.

In order to test the compatibility of numerical (usually DSMC) codes used at the kinetic level with fluid mechanics codes used at the macroscopic level, it is useful to have one example in which the transport coefficients can be explicitly derived from the cross sections used in the Boltzmann equation. We provide in this paper such an explicit computation (that is, when the cross sections are constants). This computation can be seen as an extension of classical computations of transport coefficients for monoatomic gases with a cross section of Maxwell molecules type (cf. [11]).

We notice that in [20], [18], the authors describe the internal energy variable with a discrete parameter. This way of modelling has been adopted in [21], [4], where kinetic equations of Boltzmann or BGK–type are built up for mixtures of gases undergoing also a bimolecular reversible chemical reaction. In [4] the hydrodynamic limit of the BGK model for a fast reactive mixture of monatomic gases is derived, at both Euler and Navier-Stokes levels, by a Chapman-Enskog procedure in terms of the relevant hydrodynamic variables. This BGK model has been recently generalized in [3] to a mixture of polyatomic gases (inert or reacting), each one having a set of discrete energy levels; the relevant asymptotic limit is available only for a single gas, and its comparison with phenomenological results obtained in the frame of Extended Thermodynamics seems to be promising [5]. Suitable fluid–dynamic closures for a single polyatomic gas have been achieved in the case of a continuous internal energy [25], and the state of the art on the matter may be found in the book [26]. However, for the reasons explained above, in view of practical applications, it is important to provide a complete Navier–Stokes description for a mixture involving monoatomic and polyatomic species, and this is the aim of our work.

The paper is organised as follows. In section 2, the kinetic model for mixtures of monoatomic and polyatomic gases is introduced, Boltzmann kernels are written down together with the corresponding linear operators, and conservations laws associated to the kernels are recalled. In section 3, the asymptotic expansion is performed, and the various transport terms appearing in the Navier Stokes system are described and linked to the cross sections of the Boltzmann kernels. Then, section 4 is devoted to the complete treatment of the case when all cross sections are constant: in this case all transport terms can be explicitly computed. Some basic integrals widely used in the procedure are finally listed in a short Appendix.

2. Boltzmann kernels for a mixture of rarefied monoatomic and polyatomic gases

In this section, we present a direct extension of the model devised in [16] to the case of a mixture of monoatomic and polyatomic gases.

2.1. **General definitions.** We consider a mixture of A monoatomic gases and B polyatomic gases. The distribution function (at time t, point x and velocity v) of each monoatomic species $i \in \{1,\dots,A\}$ writes $f^{(i)}(t,x,v)$, where $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$. Then, we introduce for the polyatomic species $i \in \{A+1,\dots,A+B\}$ a unique continuous energy variable $I \in \mathbb{R}_+$, collecting rotational and vibrational energies. Therefore the distribution function of each polyatomic species writes $f^{(i)}(t,x,v,I)$, where $(t,x,v,I) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+$. Following [8] and [16], we introduce (for each poyatomic species $i = A+1,\dots,A+B$) a function $\varphi_i(I) > 0$, which is a parameter of the model. This function is related to the energy law obtained at the macroscopic level for the considered species i (cf. [14]), for example $\varphi_i(I) = 1$ for the energy law of diatomic gases $e = \frac{5}{2}T$ (e being the macroscopic internal energy by unit of mass, and T being the temperature, computed in a unit such that the constant of perfect gases is 1).

Finally we define the mass m_i of a molecule of species i, and recall the definition of macroscopic quantities:

The (macroscopic) mass of monoatomic species $i \in \{1,\dots,A\}$ (at time t and point x):

$$m_i\, n^{(i)}(t,x) := \int_{\mathbb{R}^3} f^{(i)}(t,x,v)\, m_i\, dv.$$

The (macroscopic) mass of polyatomic species $i \in \{A+1,\dots,A+B\}$ (at time t and point x):

$$m_i\, n^{(i)}(t,x) := \int_{\mathbb{R}^3}\int_0^\infty f^{(i)}(t,x,v)\, m_i\, \varphi_i(I)\, dI dv.$$

The momentum of monoatomic species $i \in \{1,\dots,A\}$ (at time t and point x):

$$m_i\, n^{(i)}(t,x)\, u_i(t,x) := \int_{\mathbb{R}^3} f^{(i)}(t,x,v)\, m_i\, v\, dv.$$

The momentum of polyatomic species $i \in \{A+1,\dots,A+B\}$ (at time t and point x):

$$m_i\, n^{(i)}(t,x)\, u_i(t,x) := \int_{\mathbb{R}^3}\int_0^\infty f^{(i)}(t,x,v)\, m_i\, v\, \varphi_i(I)\, dI dv.$$

The (macroscopic, internal) energy of monoatomic species $i \in \{1, \dots, A\}$ (at time t and point x):

$$m_i\, n^{(i)}(t,x)\, e_i(t,x) := \int_{\mathbb{R}^3} f^{(i)}(t,x,v)\, m_i\, \frac{|v-u_i(t,x)|^2}{2}\, dv.$$

The (macroscopic, internal) energy of polyatomic species $i \in \{A+1, \dots, A+B\}$ (at time t and point x):

$$m_i\, n^{(i)}(t,x)\, e_i(t,x) := \int_{\mathbb{R}^3} \int_0^\infty f^{(i)}(t,x,v) \left(m_i\, \frac{|v-u_i(t,x)|^2}{2} + I \right) \varphi_i(I)\, dI dv.$$

2.2. **Collision operators.** In this subsection, we define the collision operators enabling to treat the collisions between the various types of gases (monoatomic and polyatomic).

2.2.1. *Collision Operator for monoatomic species.* We write here the usual Boltzmann kernel, for collisions between species i and j ($i, j \in \{1, \dots, A\}$).

We define (for $f := f(v) \geq 0, g := g(v) \geq 0$ number densities of the considered species):

(1)
$$Q_{ij}(f,g)(v) = \int_{\mathbb{R}^3} \int_{S^2} \Big\{ f(v')\, g(v'_*) - f(v)\, g(v_*) \Big\}\, B_{ij} \left(|v-v_*|, \frac{v-v_*}{|v-v_*|} \cdot \sigma \right) d\sigma dv_*,$$

with

$$v' = \frac{m_i\, v + m_j\, v_*}{m_i + m_j} + \frac{m_j}{m_i + m_j}\, |v - v_*|\, \sigma, \tag{2}$$

$$v'_* = \frac{m_i\, v + m_j\, v_*}{m_i + m_j} - \frac{m_i}{m_i + m_j}\, |v - v_*|\, \sigma. \tag{3}$$

The cross section B_{ij} satisfies the symmetry constraint $B_{ij} = B_{ji}$. As a consequence, the operator satisfies the following weak formulation: For $\psi_i := \psi_i(v)$, $\psi_j := \psi_j(v)$,

$$\int_{\mathbb{R}^3} Q_{ij}(f,g)(v)\, \psi_i(v)\, dv + \int_{\mathbb{R}^3} Q_{ji}(g,f)(v)\, \psi_j(v)\, dv$$

$$= -\frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} \Big\{ f(v')\, g(v'_*) - f(v)\, g(v_*) \Big\} \times \Big(\psi_i(v') + \psi_j(v'_*) - \psi_i(v) - \psi_j(v_*) \Big)$$
$$\times B_{ij} \left(|v-v_*|, \frac{v-v_*}{|v-v_*|} \cdot \sigma \right) d\sigma dv_* dv.$$

This weak formulation implies the conservation of momentum and kinetic energy:

$$\int_{\mathbb{R}^3} Q_{ij}(f,g)(v) \begin{pmatrix} m_i\, v \\ m_i \frac{|v|^2}{2} \end{pmatrix} dv + \int_{\mathbb{R}^3} Q_{ji}(g,f)(v) \begin{pmatrix} m_j\, v \\ m_j \frac{|v|^2}{2} \end{pmatrix} dv = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

together with the entropy inequality:

$$\int_{\mathbb{R}^3} Q_{ij}(f,g)(v)\, \ln f(v)\, dv + \int_{\mathbb{R}^3} Q_{ji}(g,f)(v)\, \ln g(v)\, dv \leq 0.$$

2.2.2. *Collision operators between monoatomic and polyatomic molecules.* We write here the asymmetric operator enabling to treat the collisions between a polyatomic molecule (of mass m_i, with $i \in \{A+1, \dots, A+B\}$), and a monoatomic one (of mass m_j, with $j \in \{1, \dots, A\}$). This operator is inspired from the operators presented in [8], [14], [16].

We define (for $f := f(v, I)$ number densities for a polyatomic species, and $g := g(v)$ for a monoatomic one):

$$Q_{ij}(f,g)(v,I) = \int_{\mathbb{R}^3} \int_{S^2} \int_0^1 \Big\{ f(v',I')\, g(v'_*) - f(v,I)\, g(v_*) \Big\} \tag{4}$$

$$\times B_{ij}\left(\sqrt{E}, R^{1/2}\, |v - v_*|, \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) R^{1/2}\, \varphi_i(I)^{-1} dR d\sigma dv_*,$$

with

$$v' = \frac{m_i\, v + m_j\, v_*}{m_i + m_j} + \frac{m_j}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma, \tag{5}$$

$$v'_* = \frac{m_i\, v + m_j\, v_*}{m_i + m_j} - \frac{m_i}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma, \tag{6}$$

$$I' = (1 - R)\, E, \tag{7}$$

where $\mu_{ij} = \frac{m_i\, m_j}{m_i + m_j}$ is the reduced mass, $E = \frac{1}{2}\, \mu_{ij}\, |v - v_*|^2 + I$ is the total energy of the two molecules in the center of mass reference frame, and the parameter R lies in $[0, 1]$.

We also define the symmetric operator (with the same cross section)

$$Q_{ji}(g,f)(v) = \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} \int_0^1 \Big\{ g(v')\, f(v'_*, I'_*) - g(v)\, f(v_*, I_*) \Big\}$$

$$\times B_{ij}\left(\sqrt{E}, R^{1/2}\, |v - v_*|, \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) R^{1/2}\, dR d\sigma dv_* dI_*,$$

with

$$v' = \frac{m_j\, v + m_i\, v_*}{m_i + m_j} + \frac{m_i}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma,$$

$$v'_* = \frac{m_j\, v + m_i\, v_*}{m_i + m_j} - \frac{m_j}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma,$$

$$I'_* = (1 - R)\, E,$$

where $\mu_{ij} = \frac{m_i\, m_j}{m_i + m_j}$ and $E = \frac{1}{2}\, \mu_{ij}\, |v - v_*|^2 + I_*$.

These operators satisfy the following weak formulation (note that by symmetry, the same cross section B_{ij} appears in Q_{ij} and Q_{ji}): for $\psi_i := \psi_i(v, I) \geq 0$, $\psi_j := \psi_j(v) \geq 0$,

$$\int_{\mathbb{R}^3} \int_0^\infty Q_{ij}(f,g)(v,I)\, \psi_i(v,I)\, \varphi_i(I) dv dI + \int_{\mathbb{R}^3} Q_{ji}(g,f)(v)\, \psi_j(v)\, dv$$

$$= -\frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} \int_0^1 \Big\{ f(v', I')\, g(v'_*) - f(v, I)\, g(v_*) \Big\}$$

$$\times\Big(\psi_i(v',I')+\psi_j(v'_*)-\psi_i(v,I)-\psi_j(v_*)\Big)\, B_{ij}\Big(\sqrt{E}, R^{1/2}\,|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\, R^{1/2}\, dR d\sigma dv_* dI dv.$$

The weak formulation implies the conservation of momentum and total energy:

$$\int_{\mathbb{R}^3}\int_0^\infty Q_{ij}(f,g)(v,I)\,\varphi_i(I)\left(\begin{array}{c} m_i\,v \\ m_i\frac{|v|^2}{2}+I\end{array}\right) dIdv + \int_{\mathbb{R}^3} Q_{ji}(g,f)(v)\left(\begin{array}{c} m_j\,v \\ m_j\frac{|v|^2}{2}\end{array}\right) dv = \left(\begin{array}{c} 0 \\ 0\end{array}\right),$$

together with the entropy inequality:

$$\int_{\mathbb{R}^3}\int_0^\infty Q_{ij}(f,g)(v,I)\,\ln f(v,I)\,\varphi_i(I)\,dIdv + \int_{\mathbb{R}^3} Q_{ji}(g,f)(v)\,\ln g(v)\,dv \le 0.$$

2.2.3. *Collision operators for polyatomic molecules.* We finally present the operator enabling to treat the collisions between two polyatomic molecules of respective mass m_i and m_j $(i,j \in \{A+1,\dots,A+B\})$.

We define (for $f := f(v,I) \ge 0, g := g(v,I) \ge 0$):

$$Q_{ij}(f,g)(v,I) = \int_{\mathbb{R}^3}\int_0^\infty\int_{S^2}\int_0^1\int_0^1 \Big\{f(v',I')\,g(v'_*,I'_*) - f(v,I)\,g(v_*,I_*)\Big\} \tag{8}$$
$$\times B_{ij}\Big(\sqrt{E}, R^{1/2}\,|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\,(1-R)\,R^{1/2}\,drdRd\omega dI_* dv_*,$$

with

$$v' = \frac{m_i\,v + m_j\,v_*}{m_i+m_j} + \frac{m_j}{m_i+m_j}\sqrt{\frac{2R\,E}{\mu_{ij}}}\,\sigma, \tag{9}$$

$$v'_* = \frac{m_i\,v + m_j\,v_*}{m_i+m_j} - \frac{m_i}{m_i+m_j}\sqrt{\frac{2R\,E}{\mu_{ij}}}\,\sigma, \tag{10}$$

$$I' = r\,(1-R)\,E, \qquad I'_* = (1-r)\,(1-R)\,E, \tag{11}$$

where $\mu_{ij} = \frac{m_i\,m_j}{m_i+m_j}$ is the reduced mass, $E = \frac{1}{2}\,\mu_{ij}\,|v-v_*|^2 + I + I_*$ is the total energy of the two molecules in the center of mass reference frame, and r, R lie in $[0,1]$.

Using the symmetry constraints $B_{ij} = B_{ji}$, one can show that these operators satisfy the following weak formulation: for $\psi_i := \psi_i(v,I)$, $\psi_j := \psi_j(v,I)$,

$$\int_{\mathbb{R}^3}\int_0^\infty Q_{ij}(f,g)(v,I)\,\psi_i(v,I)\,\varphi_i(I)\,dvdI + \int_{\mathbb{R}^3}\int_0^\infty Q_{ji}(g,f)(v)\,\psi_j(v,I)\,\varphi_j(I)\,dvdI$$
$$= -\frac{1}{2}\int_{\mathbb{R}^3}\int_0^\infty\int_{\mathbb{R}^3}\int_0^\infty\int_{S^2}\int_0^1\int_0^1 \Big\{f(v',I')\,g(v'_*,I'_*) - f(v,I)\,g(v_*,I_*)\Big\}$$
$$\times\Big(\psi_i(v',I') + \psi_j(v'_*,I'_*) - \psi_i(v,I) - \psi_j(v_*,I_*)\Big)$$
$$\times B_{ij}(\sqrt{E}, R^{1/2}\,|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma)\,(1-R)\,R^{1/2}\,drdRd\omega dI_* dv_* dI dv.$$

This weak formulation implies the conservation of momentum and total energy:

$$\int_{\mathbb{R}^3}\int_0^\infty Q_{ij}(f,g)(v,I)\left(\begin{array}{c} m_i\,v \\ m_i\frac{|v|^2}{2}+I\end{array}\right)\varphi_i(I)\,dIdv$$

$$+\int_{\mathbb{R}^3}\int_0^\infty Q_{ji}(g,f)(v,I)\begin{pmatrix} m_j\, v \\ m_j\frac{|v|^2}{2}+I\end{pmatrix}\varphi_j(I)\,dIdv = \begin{pmatrix} 0\\ 0\end{pmatrix},$$

together with the entropy inequality:

$$\int_{\mathbb{R}^3}\int_0^\infty Q_{ij}(f,g)(v,I)\,\ln f(v,I)\,\varphi_i(I)\,dIdv \tag{12}$$
$$+\int_{\mathbb{R}^3}\int_0^\infty Q_{ji}(g,f)(v,I)\,\ln g(v,I)\,\varphi_j(I)\,dIdv \le 0.$$

2.3. **Linearized operators.** We now introduce the Maxwellian distributions

$$M^{(i)} := \frac{n^{(i)}}{(2\pi\, T/m_i)^{3/2}\, q_i(T)}\,\exp\bigg(-\frac{m_i\,|v-u|^2+2\,r_i\,I}{2\,T}\bigg), \tag{13}$$

with $r_i=0$ for $i=1,\dots,A$ and $r_i=1$ for $i=A+1,\dots,A+B$.

In the formula above, $q_i(T)=1$ for $i=1,\dots,A$ and

$$q_i(T) := \int_0^\infty \varphi_i(I)\,e^{-I/T}\,dI$$

for $i=A+1,\dots,A+B$. We refer to [14] and [16] for those formulas in the case when $r_i=1$. In the framework of [20], [18], this term is considered as an internal energy of species i.

For any family of functions $g^{(i)} := g^{(i)}(v,I)$ with $i=A+1,\dots,A+B$, one can write

$$\Big[[M^{(i)}]^{-1}\,Q_{ij}(M^{(i)},M^{(j)}\,g^{(j)}) + [M^{(i)}]^{-1}\,Q_{ij}(M^{(i)}\,g^{(i)},M^{(j)})\Big](\sqrt{T}\,V+u,J\,T) \tag{14}$$
$$= n^{(j)}\,K_{ij}(g^{(i)}(\cdot\sqrt{T}+u,\cdot\, T),g^{(j)}(\cdot\sqrt{T}+u,\cdot\, T))(V,J),$$

where K_{ij} is defined below. Formulas very close to (14) can be written down when at least one of the molecules is monoatomic (the only difference being that the dependence w.r.t. the second variable of $g^{(i)}$ and/or $g^{(j)}$ does not appear).

We now write down the linearized operators K_{ij} (around a centered reduced Maxwellian, and with a rescaled cross section), which will play an important role in the study of the Chapman-Enskog asymptotics described in next section.

We start with the monoatomic-monoatomic case: For $i=1,\dots,A$, $j=1,\dots,A$,

$$K_{ij}(h^{(i)},h^{(j)})(v) = \int_{\mathbb{R}^3}\int_{S^2}\frac{e^{-\frac{m_j}{2}\,|v_*|^2}}{(2\pi/m_j)^{3/2}} \tag{15}$$
$$\times\bigg[h^{(j)}\bigg(\frac{m_i\,v+m_j\,v_*}{m_i+m_j}-\frac{m_i}{m_i+m_j}\,|v-v_*|\,\sigma\bigg)+h^{(i)}\bigg(\frac{m_i\,v+m_j\,v_*}{m_i+m_j}+\frac{m_j}{m_i+m_j}\,|v-v_*|\,\sigma\bigg)$$
$$-h^{(j)}(v_*)-h^{(i)}(v)\bigg]\,B_{ij}(\sqrt{T}\,|v-v_*|,\frac{v-v_*}{|v-v_*|}\cdot\sigma)\,d\sigma\,dv_*.$$

We then turn to the monoatomic-polyatomic case: For $i=1,\dots,A$, $j=A+1,\dots,A+B$,

$$K_{ij}(h^{(i)},h^{(j)})(v) = \int_{\mathbb{R}^3}\int_0^\infty\int_{S^2}\int_0^1 \frac{e^{-\frac{m_j}{2}\,|v_*|^2-I_*}}{(2\pi/m_j)^{3/2}} \tag{16}$$

$$\times \Bigg[h^{(j)}\Bigg(\frac{m_i\,v+m_j\,v_*}{m_i+m_j}-\frac{m_i}{m_i+m_j}\sqrt{\frac{2R\,E}{\mu_{ij}}}\,\sigma,(1-R)\,E\Bigg)$$

$$+h^{(i)}\Bigg(\frac{m_i\,v+m_j\,v_*}{m_i+m_j}+\frac{m_j}{m_i+m_j}\sqrt{\frac{2R\,E}{\mu_{ij}}}\,\sigma\Bigg)$$

$$-h^{(j)}(v_*,I_*)-h^{(i)}(v)\Bigg]\,\frac{T}{q_j(T)}\,B_{ij}(\sqrt{T}\,\sqrt{E},\sqrt{T}\,R^{1/2}\,|v-v_*|,\frac{v-v_*}{|v-v_*|}\cdot\sigma)\,R^{1/2}\,dR\,d\sigma\,dI_*\,dv_*,$$

with $E=\frac{1}{2}\,\mu_{ij}\,|v-v_*|^2+I_*$.

Symmetrically, we write down the polyatomic-monoatomic case: For $i=A+1,\dots,A+B$, $j=1,\dots,A$,

$$K_{ij}(h^{(i)},h^{(j)})(v,I)=\int_{\mathbb{R}^3}\int_{S^2}\int_0^1\frac{e^{-\frac{m_j}{2}\,|v_*|^2}}{(2\pi/m_j)^{3/2}} \tag{17}$$

$$\times \Bigg[h^{(j)}\Bigg(\frac{m_i\,v+m_j\,v_*}{m_i+m_j}-\frac{m_i}{m_i+m_j}\sqrt{\frac{2R\,E}{\mu_{ij}}}\,\sigma\Bigg)$$

$$+h^{(i)}\Bigg(\frac{m_i\,v+m_j\,v_*}{m_i+m_j}+\frac{m_j}{m_i+m_j}\sqrt{\frac{2R\,E}{\mu_{ij}}}\,\sigma,(1-R)\,E\Bigg)$$

$$-h^{(j)}(v_*)-h^{(i)}(v,I)\Bigg]\,B_{ij}(\sqrt{T}\,\sqrt{E},\sqrt{T}\,R^{1/2}\,|v-v_*|,\frac{v-v_*}{|v-v_*|}\cdot\sigma)\,R^{1/2}\,\varphi_i(I)^{-1}\,dR\,d\sigma\,dv_*,$$

with $E=\frac{1}{2}\,\mu_{ij}\,|v-v_*|^2+I$.

Finally, we consider the polyatomic-polyatomic case. For $i=A+1,\dots,A+B$, $j=A+1,\dots,A+B$,

$$K_{ij}(h^{(i)},h^{(j)})(v,I)=\int_{\mathbb{R}^3}\int_0^\infty\int_{S^2}\int_0^1\int_0^1\frac{e^{-\frac{m_j}{2}\,|v_*|^2-I_*}}{(2\pi/m_j)^{3/2}}$$

$$\times \Bigg[h^{(j)}\Bigg(\frac{m_i\,v+m_j\,v_*}{m_i+m_j}-\frac{m_i}{m_i+m_j}\sqrt{\frac{2R\,E}{\mu_{ij}}}\,\sigma,(1-r)\,(1-R)\,E\Bigg)$$

$$+h^{(i)}\Bigg(\frac{m_i\,v+m_j\,v_*}{m_i+m_j}+\frac{m_j}{m_i+m_j}\sqrt{\frac{2R\,E}{\mu_{ij}}}\,\sigma,r\,(1-R)\,E\Bigg)$$

$$-h^{(j)}(v_*,I_*)-h^{(i)}(v,I)\Bigg]\,\frac{T}{q_j(T)}\,B_{ij}(\sqrt{T}\,\sqrt{E},\sqrt{T}\,R^{1/2}\,|v-v_*|,\frac{v-v_*}{|v-v_*|}\cdot\sigma)$$

$$\times(1-R)\,R^{1/2}\,\varphi_i(I)^{-1}\,dR\,dr\,d\sigma\,dI_*\,dv_*,$$

with $E=\frac{1}{2}\,\mu_{ij}\,|v-v_*|^2+I+I_*$.

We emphasize the fact that those operators (like the quadratic operators Q_{ij}) are Galilean-invariant. In particular, for all isometric transformation $\mathcal{R}$ in $O(3,\mathbb{R})$, one has (denoting by o the composition w.r.t the velocity variable only),

$$K_{ij}(h^{(i)}\,o\,\mathcal{R},h^{(j)}\,o\,\mathcal{R})(v,I)=K_{ij}(h^{(i)},h^{(j)})(\mathcal{R}v,I). \tag{18}$$

This property will be useful for the description of the transport coefficients in the Navier-Stokes systems obtained in next section.

3. Chapman-Enskog expansion for a mixture of mono- and poly-atomic gases

We perform in this section the Chapman-Enskog expansion for a mixture of mono- and poly- atomic gases, when the collision operators are defined by the formulas developed in the previous section of this paper. The expansion is done at the formal level, we do not try here to present a functional setting which would be adapted for obtaining a rigorous expansion. We recall nevertheless that such a setting exists in the case of one single monoatomic gas (cf. [22]).

3.1. **Principle of the expansion.** We present in this subsection the basic ideas underlying the Chapman-Enskog expansion. As in the previous section, we introduce a mixture of A monoatomic gases and B polyatomic gases. We systematically use the notations of subsection 2.1.

We start by writing the Hilbert expansion for our mixture, that is the rescaled (w.r.t the Knudsen number) system of Boltzmann equations:

$$\partial_t f^{(i)} + v \cdot \nabla_x f^{(i)} = \frac{1}{\varepsilon} \sum_{j=1}^{A+B} Q_{ij}(f^{(i)}, f^{(j)}), \tag{19}$$

where the operators Q_{ij} are defined by formulas (1), (4), (8).

We look for solutions of the Boltzmann equation (19) under the form

$$f^{(i)} = M_\varepsilon^{(i)}\,(1 + \varepsilon\, g_\varepsilon^{(i)}), \tag{20}$$

where $M_\varepsilon^{(i)}$ is a Maxwellian distribution of (number) density $n_\varepsilon^{(i)} := n_\varepsilon^{(i)}(t,x) \ge 0$, macroscopic velocity $u_\varepsilon := u_\varepsilon(t,x) \in \mathbb{R}^3$, and temperature $T_\varepsilon := T_\varepsilon(t,x) \ge 0$. It writes (cf. (13))

$$M_\varepsilon^{(i)} = \frac{n_\varepsilon^{(i)}}{(2\pi\, T_\varepsilon/m_i)^{3/2}\, q_i(T_\varepsilon)}\, \exp\left(- \frac{m_i\,|v-u_\varepsilon|^2 + 2\, r_i\, I}{2\, T_\varepsilon} \right), \tag{21}$$

with $r_i = 0$ for $i = 1, \dots, A$ and $r_i = 1$ for $i = A+1, \dots, A+B$. We also assume (this is done without loss of generality, since one can perform a modification of the parameters of the Maxwellian distribution by adding terms of order ε, cf. [13] for example) that the functions $g_\varepsilon^{(i)} := g_\varepsilon^{(i)}(t,x,v) \in \mathbb{R}$ for $i = 1, \dots, A$, and $g_\varepsilon^{(i)} := g_\varepsilon^{(i)}(t,x,v,I) \in \mathbb{R}$ for $i = A+1, \dots, A+B$, satisfy

$$\forall i = 1, \dots, A, \quad \int_{\mathbb{R}^3} M_\varepsilon^{(i)}\, g_\varepsilon^{(i)}\, dv = 0, \tag{22}$$

$$\forall i = A+1, \dots, A+B, \quad \int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M_\varepsilon^{(i)}\, g_\varepsilon^{(i)}\, \varphi_i(I)\, dI dv = 0, \tag{23}$$

$$\sum_{i=1}^{A} \int_{\mathbb{R}^3} M_\varepsilon^{(i)}\, g_\varepsilon^{(i)}\, m_i\, v\, dv + \sum_{i=A+1}^{A+B} \int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M_\varepsilon^{(i)}\, g_\varepsilon^{(i)}\, m_i\, v\, \varphi_i(I)\, dI dv = 0, \tag{24}$$

$$\sum_{i=1}^{A} \int_{\mathbb{R}^3} M_\varepsilon^{(i)}\, g_\varepsilon^{(i)}\, m_i\, \frac{|v|^2}{2}\, dv + \sum_{i=A+1}^{A+B} \int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M_\varepsilon^{(i)}\, g_\varepsilon^{(i)} \left(m_i\, \frac{|v|^2}{2} + I \right) \varphi_i(I)\, dI dv = 0. \tag{25}$$

Introducing (20) in equation (19), we get the (approximated) system of linear equations satisfied by $g_\varepsilon^{(i)}$ for $i = 1, \dots, A+B$:

$$\begin{aligned}(M_\varepsilon^{(i)})^{-1}\left(\partial_t M_\varepsilon^{(i)} + v\cdot\nabla_x M_\varepsilon^{(i)}\right) &= (M_\varepsilon^{(i)})^{-1}\sum_{j=1}^{A+B}[Q_{ij}(M_\varepsilon^{(i)}, M_\varepsilon^{(j)}\,g_\varepsilon^{(j)}) + Q_{ij}(M_\varepsilon^{(i)}\,g_\varepsilon^{(i)}, M_\varepsilon^{(j)})] \\ &+ O(\varepsilon). \end{aligned} \tag{26}$$

Then for $i = 1, \dots, A$, thanks to (14),

(27)

$$\begin{aligned}(M_\varepsilon^{(i)})^{-1}\left(\partial_t M_\varepsilon^{(i)} + v\cdot\nabla_x M_\varepsilon^{(i)}\right) &= \sum_{j=1}^{A} n_\varepsilon^{(j)}\,K_{ij}(g^{(i)}(\cdot\sqrt{T_\varepsilon}+u_\varepsilon), g^{(j)}(\cdot\sqrt{T_\varepsilon}+u_\varepsilon)) \\ &+ \sum_{j=A+1}^{A+B} n_\varepsilon^{(j)}\,K_{ij}(g^{(i)}(\cdot\sqrt{T_\varepsilon}+u_\varepsilon), g^{(j)}(\cdot\sqrt{T_\varepsilon}+u_\varepsilon, \cdot\, T_\varepsilon)),\end{aligned}$$

and for $i = A+1, \dots, A+B$, thanks to (14) again,

(28)

$$\begin{aligned}(M_\varepsilon^{(i)})^{-1}\left(\partial_t M_\varepsilon^{(i)} + v\cdot\nabla_x M_\varepsilon^{(i)}\right) &= \sum_{j=1}^{A} n_\varepsilon^{(j)}\,K_{ij}(g^{(i)}(\cdot\sqrt{T_\varepsilon}+u_\varepsilon, \cdot\, T_\varepsilon), g^{(j)}(\cdot\sqrt{T_\varepsilon}+u_\varepsilon)) \\ &+ \sum_{j=A+1}^{A+B} n_\varepsilon^{(j)}\,K_{ij}(g^{(i)}(\cdot\sqrt{T_\varepsilon}+u_\varepsilon, \cdot\, T_\varepsilon), g^{(j)}(\cdot\sqrt{T_\varepsilon}+u_\varepsilon, \cdot\, T_\varepsilon)),\end{aligned}$$

where the linear operators K_{ij} are defined by (15), (16) and (17).

We can at this level write down the compressible Navier-Stokes equations (neglecting terms of order ε^2) of the mixture under the following abstract form:

- Mass conservation for each monoatomic species: $i = 1, \dots, A$,

$$\partial_t \int_{\mathbb{R}^3} M_\varepsilon^{(i)}\, m_i\, dv + \nabla_x\cdot\int_{\mathbb{R}^3} M_\varepsilon^{(i)}\, m_i\, v\, dv = -\varepsilon\,\nabla_x\cdot\int_{\mathbb{R}^3} M_\varepsilon^{(i)}\, g_\varepsilon^{(i)}\, m_i\, v\, dv; \tag{29}$$

- Mass conservation for each polyatomic species: $i = A+1, \dots, A+B$,

$$\begin{aligned}\partial_t \int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M_\varepsilon^{(i)}\, m_i\,\varphi_i(I)\, dIdv + \nabla_x\cdot\int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M_\varepsilon^{(i)}\, m_i\, v\,\varphi_i(I)\, dIdv \\ = -\varepsilon\,\nabla_x\cdot\int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M_\varepsilon^{(i)}\, g_\varepsilon^{(i)}\, m_i\, v\,\varphi_i(I)\, dvdI;\end{aligned} \tag{30}$$

- Momentum conservation of the mixture (we consider the components $k = 1, \dots, 3$):

$$\begin{aligned}&\partial_t\Bigg(\sum_{i=1}^{A}\int_{\mathbb{R}^3} M_\varepsilon^{(i)}\, m_i\, v_k\, dv + \sum_{i=A+1}^{A+B}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M_\varepsilon^{(i)}\, m_i\, v_k\,\varphi_i(I)\, dIdv\Bigg) \\ &+\nabla_x\cdot\Bigg(\sum_{i=1}^{A}\int_{\mathbb{R}^3} M_\varepsilon^{(i)}\, m_i\, v_k\, v\, dv + \sum_{i=A+1}^{A+B}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M_\varepsilon^{(i)}\, m_i\, v_k\, v\,\varphi_i(I)\, dIdv\Bigg) \\ &= -\varepsilon\nabla_x\cdot\Bigg(\sum_{i=1}^{A}\int_{\mathbb{R}^3} M_\varepsilon^{(i)}\, g_\varepsilon^{(i)}\, m_i\, v_k\, v\, dv + \sum_{i=A+1}^{A+B}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M_\varepsilon^{(i)}\, g_\varepsilon^{(i)}\, m_i\, v_k\, v\,\varphi_i(I)\, dIdv\Bigg);\end{aligned} \tag{31}$$

- Total energy conservation of the mixture:

$$\partial_t\Big(\sum_{i=1}^{A}\int_{\mathbb{R}^3} M_\varepsilon^{(i)}\, m_i\, \frac{|v|^2}{2}\, dv + \sum_{i=A+1}^{A+B}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M_\varepsilon^{(i)} \left(m_i\, \frac{|v|^2}{2} + I\right) \varphi_i(I)\, dI dv\Big) \tag{32}$$
$$+\nabla_x\cdot\Big(\sum_{i=1}^{A}\int_{\mathbb{R}^3} M_\varepsilon^{(i)}\, m_i\, \frac{|v|^2}{2}\, v\, dv + \sum_{i=A+1}^{A+B}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M_\varepsilon^{(i)} \left(m_i\, \frac{|v|^2}{2} + I\right) v\, \varphi_i(I)\, dI dv\Big)$$
$$= -\varepsilon\nabla_x\cdot\Big(\sum_{i=1}^{A}\int_{\mathbb{R}^3} M_\varepsilon^{(i)}\, g_\varepsilon^{(i)}\, m_i\, \frac{|v|^2}{2}\, v\, dv + \sum_{i=A+1}^{A+B}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M_\varepsilon^{(i)}\, g_\varepsilon^{(i)} \left(m_i\, \frac{|v|^2}{2} + I\right) v\, \varphi_i(I)\, dI dv\Big).$$

Next subsections are devoted to computations enabling to write these abstract equations in such a way that they appear as a system of compressible Navier-Stokes equations for our mixture (with dissipative terms of order ε, as always when Chapman-Enskog expansions are concerned). In subsection 3.2, we compute the l.h.s. of equations (29) – (32), which amounts to identifying the terms of order 0 in the expansion, corresponding to the system of compressible Euler equations for the mixture. Then subsection 3.3 is devoted to the computation of the r.h.s, of equations (29) – (32), which amounts to identifying the terms of order ε in the expansion, corresponding to the dissipative terms in the system of compressible Navier-Stokes equations for our mixture.

3.2. **Euler system.** We present here as announced the computations for the l.h.s. of equations (29) – (32). We denote

$$\eta_i(T) = \int_0^\infty I\, \varphi_i(I)\, e^{-I/T}\, dI,$$

and do not write anymore the dependence w.r.t. ε of the various considered terms. In the formalism of [20], [18], the term $\eta_i(T)/q_i(T)$ appearing in (40) corresponds to the average internal energy of i^{th} species. We first compute moments relations for Maxwellian distributions:

$$\forall i = 1,\dots,A, \quad \int_{\mathbb{R}^3} M^{(i)}\, m_i\, dv = m_i\, n^{(i)}, \tag{33}$$

$$\forall i = A+1,\dots,A+B, \quad \int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M^{(i)}\, m_i\, \varphi_i(I)\, dI dv = m_i\, n^{(i)}, \tag{34}$$

$$\forall i = 1,\dots,A, \quad \int_{\mathbb{R}^3} M^{(i)}\, m_i\, v_k\, dv = m_i\, n^{(i)}\, u_k, \tag{35}$$

$$\forall i = A+1,\dots,A+B, \quad \int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M^{(i)}\, m_i\, v_k\, \varphi_i(I)\, dI dv = m_i\, n^{(i)}\, u_k, \tag{36}$$

$$\forall i = 1,\dots,A, \quad \int_{\mathbb{R}^3} M^{(i)}\, m_i\, v_k\, v_l\, dv = m_i\, n^{(i)}\, u_k\, u_l + n^{(i)}\, T\, \delta_{kl}, \tag{37}$$

(38)
$$\forall i = A+1,\dots,A+B, \quad \int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M^{(i)}\, m_i\, v_k\, v_l\, \varphi_i(I)\, dI dv = m_i\, n^{(i)}\, u_k\, u_l + n^{(i)}\, T\, \delta_{kl},$$

$$\forall i = 1,\dots,A, \quad \int_{\mathbb{R}^3} M^{(i)}\, m_i\, \frac{|v|^2}{2}\, dv = m_i\, n^{(i)}\, \frac{|u|^2}{2} + \frac{3}{2}\, n^{(i)}\, T, \tag{39}$$

$$\forall i = A+1, \dots, A+B, \quad \int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M^{(i)} \left(m_i \frac{|v|^2}{2} + I \right) \varphi_i(I)\, dI dv$$
$$= m_i\, n^{(i)} \frac{|u|^2}{2} + n^{(i)} \left[\frac{3}{2} T + \frac{\eta_i(T)}{q_i(T)} \right], \tag{40}$$

$$\forall i = 1, \dots, A, \quad \int_{\mathbb{R}^3} M^{(i)}\, m_i \frac{|v|^2}{2} v_k dv = m_i\, n^{(i)} \frac{|u|^2}{2} u_k + \frac{5}{2} n^{(i)}\, T\, u_k, \tag{41}$$

$$\forall i = A+1, \dots, A+B, \quad \int_{\mathbb{R}^3}\int_{\mathbb{R}_+} M^{(i)} \left(m_i \frac{|v|^2}{2} + I \right) v_k\, \varphi_i(I)\, dI dv$$
$$= m_i\, n^{(i)} \frac{|u|^2}{2} u_k + n^{(i)}\, u_k \left[\frac{5}{2} T + \frac{\eta_i(T)}{q_i(T)} \right]. \tag{42}$$

Using identities (33) - (42), we get as announced the Euler system in conservative form (up to terms of order ε) (remember that we use for the components the notation $k = 1, \dots, 3$):

$$i = 1, \dots, A+B, \qquad \partial_t (m_i\, n^{(i)}) + \nabla_x \cdot (m_i\, n^{(i)}\, u) = O(\varepsilon), \tag{43}$$

$$\partial_t \Big(\sum_{i=1}^{A+B} m_i\, n^{(i)}\, u_k \Big) + \sum_l \partial_{x_l} \left(\sum_{i=1}^{A+B} [m_i\, n^{(i)}\, u_k\, u_l + n^{(i)}\, T\, \delta_{kl}] \right) = O(\varepsilon), \tag{44}$$

$$\partial_t \Big(\sum_{i=1}^{A} [m_i\, n^{(i)} \frac{|u|^2}{2} + \frac{3}{2} n^{(i)}\, T] + \sum_{i=A+1}^{A+B} [m_i\, n^{(i)} \frac{|u|^2}{2} + n^{(i)} \left[\frac{3}{2} T + \frac{\eta_i(T)}{q_i(T)} \right]] \Big) \tag{45}$$
$$+ \sum_l \partial_{x_l} \left(\sum_{i=1}^{A} [m_i\, n^{(i)} \frac{|u|^2}{2} u_l + \frac{5}{2} n^{(i)}\, T\, u_l] \right.$$
$$\left. + \sum_{i=A+1}^{A+B} [m_i\, n^{(i)} \frac{|u|^2}{2} u_l + n^{(i)}\, u_l \left[\frac{5}{2} T + \frac{\eta_i(T)}{q_i(T)} \right]] \right) = O(\varepsilon).$$

These equations can be rewritten under the following non conservative form, which is useful for the computation of the dissipative terms (of order ε) appearing in the Chapman-Enskog asymptotics:

$$i = 1, \dots, A+B, \qquad \partial_t n^{(i)} + (u \cdot \nabla_x)\, n^{(i)} + n^{(i)}\, \nabla_x \cdot u = O(\varepsilon), \tag{46}$$

$$k = 1, \dots, 3, \qquad \partial_t u_k + (u \cdot \nabla_x) u_k + \frac{\sum_{i=1}^{A+B} \partial_{x_k} (n^{(i)}\, T)}{\sum_{i=1}^{A+B} m_i\, n^{(i)}} = O(\varepsilon), \tag{47}$$

$$\partial_t T + (u \cdot \nabla_x)\, T + 2\, \Lambda(T)\, T\, \nabla_x \cdot u = O(\varepsilon), \tag{48}$$

with

$$\Lambda(T) = \frac{\sum_{j=1}^{A+B} n^{(j)}}{3 \sum_{j=1}^{A+B} n^{(j)} + 2 \sum_{j=A+1}^{A+B} n^{(j)} \left(\frac{\eta_j}{q_j} \right)' (T)}. \tag{49}$$

3.3. **Navier-Stokes system.** In this subsection, we provide the dissipative terms (viscosity, Soret and Dufour terms, etc.) of order ε which are typical of the Chapman-Enskog asymptotics.

3.3.1. *Computation of the l.h.s of the linear equations (27), (28).* We start with the computation of the quantity

$$(M^{(i)})^{-1}\,[\partial_t M^{(i)} + v\cdot\nabla_x M^{(i)}],$$

which appears in the l.h.s. of (27), (28):

$$\begin{aligned}
(M^{(i)})^{-1}\,[\partial_t M^{(i)} + v\cdot\nabla_x M^{(i)}] &= \left[\frac{\partial_t n^{(i)}}{n^{(i)}} - \left(\frac{3}{2} + r_i\,T\,\frac{q_i'(T)}{q_i(T)}\right)\frac{\partial_t T}{T}\right] \\
&+ u\cdot\left[\frac{\nabla_x n^{(i)}}{n^{(i)}} - \left(\frac{3}{2} + r_i\,T\,\frac{q_i'(T)}{q_i(T)}\right)\frac{\nabla_x T}{T}\right] \\
+ (v-u)\cdot&\left[\frac{\nabla_x n^{(i)}}{n^{(i)}} - \left(\frac{3}{2} + r_i\,T\,\frac{q_i'(T)}{q_i(T)}\right)\frac{\nabla_x T}{T} + \frac{m_i}{T}\,\partial_t u + \frac{m_i}{T}\,(u\cdot\nabla_x)u\right] \\
&+ \sum_k\sum_l (v_k-u_k)\,(v_l-u_l)\,\frac{m_i}{T}\,\partial_{x_k} u_l \\
&+ \left(\frac{m_i}{2}\,|v-u|^2 + r_i\,I\right)\left(\frac{\partial_t T}{T^2} + u\cdot\frac{\nabla_x T}{T^2}\right) \\
&+ \left(\frac{m_i}{2}\,|v-u|^2 + r_i\,I\right)(v-u)\cdot\frac{\nabla_x T}{T^2}\,.
\end{aligned}$$

Using identities (46) – (48), we get

$$(M^{(i)})^{-1}\,[\partial_t M^{(i)} + v\cdot\nabla_x M^{(i)}] \tag{50}$$

$$\begin{aligned}
&= \frac{v-u}{\sqrt{T}}\cdot\left[\sqrt{T}\left(\frac{\nabla_x n^{(i)}}{n^{(i)}} - \frac{m_i\sum_{j=1}^{A+B}\nabla_x n^{(j)}}{\sum_{j=1}^{A+B} m_j\,n^{(j)}}\right) + \left(1 - \frac{m_i\sum_{j=1}^{A+B} n^{(j)}}{\sum_{j=1}^{A+B} m_j\,n^{(j)}}\right)\frac{\nabla_x T}{\sqrt{T}}\right] \\
&+ P\left(\frac{v-u}{\sqrt{T}}\right) : m_i\left(\frac{\nabla_x u + \nabla_x u^T}{2}\right) \\
&+ \left(\frac{m_i}{T}\,|v-u|^2\left(\frac{1}{3} - \Lambda(T)\right) - 2\,\frac{I}{T}\,r_i\,\Lambda(T) + 2\left(\frac{3}{2} + r_i\,T\,\frac{q_i'(T)}{q_i(T)}\right)\Lambda(T) - 1\right)(\nabla_x\cdot u) \\
&+ \left(\frac{m_i}{2}\,\frac{|v-u|^2}{T} + r_i\,\frac{I}{T} - \left(\frac{5}{2} + r_i\,T\,\frac{q_i'(T)}{q_i(T)}\right)\right)\frac{v-u}{\sqrt{T}}\cdot\frac{\nabla_x T}{\sqrt{T}},
\end{aligned}$$

with

$$P(v) = v\otimes v - \frac{1}{3}\,|v|^2\,Id.$$

We now wish to point out the specificities of the formulas above. First, the term in $P\left(\frac{v-u}{\sqrt{T}}\right)$ is identical to the same term in the case of one monoatomic gas. The term in $\frac{v-u}{\sqrt{T}}$ in the second term of identity (50) is typical of mixtures, it does not appear when only one gas is considered. The term involving $\nabla_x\cdot u$ appears only when at least one polyatomic gas is part of the mixture (since in a mixture of monoatomic gase, one has $\Lambda(T) = \frac{1}{3}$). Finally, the last term has a shape which depends on the monoatomic or polyatomic character of the species i. When $i\in\{1,\dots,A\}$, we recover the usual term (sometimes denoted by Q) $\left(\frac{m_i}{2}\,\frac{|v-u|^2}{T} - \frac{5}{2}\right)\frac{v-u}{\sqrt{T}}$ typical of monoatomic gases.

3.3.2. *Orthogonality properties.* In order to solve the linear system (27), (28) taking into account (50), we need to use orthogonality properties.

We first define the scalar product (for functions $k^1, \dots, k^A$; $l^1, \dots, l^A$ of V, and for functions $k^{A+1} \dots k^{A+B}$; $l^{A+1} \dots l^{A+B}$ of V, J):

$$\langle k^{(1)}, \dots, k^{(A+B)} | \, l^{(1)}, \dots, l^{(A+B)} \rangle := \sum_{i=1}^{A} n^{(i)} \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}} \, k^{(i)}(V) \, l^{(i)}(V) \, dV$$

$$+ \sum_{i=A+1}^{A+B} n^{(i)} \int_0^\infty \int_{\mathbb{R}^3} \frac{e^{-(m_i \frac{|V|^2}{2} + J)}}{(2\pi/m_i)^{3/2}} \, k^{(i)}(V,J) \, l^{(i)}(V,J) \, \frac{T \, \varphi_i(J\,T)}{q_i(T)} \, dV dJ.$$

We then introduce the following families (indexed by $i = 1, \dots, A+B$) (we also indicate the dependencies w.r.t. the components $p = 1, \dots, 3$, and sometimes $q = 1, \dots, 3$):

$$\begin{pmatrix} k_1^{P,p,q} \\ \cdot \\ \cdot \\ k_{A+B}^{P,p,q} \end{pmatrix} = \begin{pmatrix} P_{pq}(V) \, m_1 \\ \cdot \\ \cdot \\ P_{pq}(V) \, m_{A+B} \end{pmatrix},$$

$$\begin{pmatrix} k_1^{Q,p} \\ \cdot \\ \cdot \\ k_{A+B}^{Q,p} \end{pmatrix} = \begin{pmatrix} V_p \, \left(\frac{m_1}{2} \, V^2 + r_1 \, J - (\frac{5}{2} + r_1 \, T \, \frac{q_1'(T)}{q_1(T)}) \right) \\ \cdot \\ \cdot \\ V_p \, \left(\frac{m_{A+B}}{2} \, V^2 + r_{A+B} \, J - (\frac{5}{2} + r_{A+B} \, T \, \frac{q_{A+B}'(T)}{q_{A+B}(T)}) \right) \end{pmatrix},$$

$$\begin{pmatrix} k_1^{W,p} \\ \cdot \\ \cdot \\ k_{A+B}^{W,p} \end{pmatrix} = \begin{pmatrix} s_p^{(1)} \, V_p \\ \cdot \\ \cdot \\ s_p^{(A+B)} \, V_p \end{pmatrix},$$

for all family $(s^{(i)})_{i \in \{1, A+B\}} \in \mathbb{R}^3$ such that for any $p \in \{1, \dots, 3\}$,

$$s_p^{(1)} \, n^{(1)} + \dots + s_p^{(A+B)} \, n^{(A+B)} = 0, \tag{51}$$

and

$$\begin{pmatrix} k_1^{D} \\ \cdot \\ \cdot \\ k_{A+B}^{D} \end{pmatrix} = \begin{pmatrix} m_1 \, V^2 \, (\frac{1}{3} - \Lambda(T)) - 2 r_1 \, J \, \Lambda(T) + (3 + 2 r_1 \, T \, \frac{q_1'(T)}{q_1(T)}) \, \Lambda(T) - 1 \\ \cdot \\ \cdot \\ m_{A+B} \, V^2 \, (\frac{1}{3} - \Lambda(T)) - 2 r_{A+B} \, J \, \Lambda(T) + (3 + 2 r_{A+B} \, T \, \frac{q_{A+B}'(T)}{q_{A+B}(T)}) \, \Lambda(T) - 1 \end{pmatrix},$$

where $\Lambda(T)$ has been defined in (49). One can check that in these families, the first A components only depend on V (and not on J). Note also that the families $k_i^{Q,p}$ and k_i^D depend on T.

We finally introduce the following families (indexed by $i = 1, \dots, A+B$) (for $z = 1, \dots, 3$, $j = 1, \dots, A+B$):

$$\begin{pmatrix} l_1^{\Delta,j} \\ . \\ . \\ . \\ l_j^{\Delta,j} \\ . \\ . \\ . \\ l_{A+B}^{\Delta,j} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ . \\ 0 \\ 1 \\ 0 \\ . \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} l_1^{U,z} \\ . \\ . \\ l_{A+B}^{U,z} \end{pmatrix} = \begin{pmatrix} m_1\, V_z \\ . \\ . \\ m_{A+B}\, V_z \end{pmatrix},$$

$$\begin{pmatrix} l_1^{E} \\ . \\ . \\ l_{A+B}^{E} \end{pmatrix} = \begin{pmatrix} m_1\, \frac{V^2}{2} + r_1\, J \\ . \\ . \\ m_{A+B}\, \frac{V^2}{2} + r_{A+B}\, J \end{pmatrix}.$$

One can check that the subspace

$$Vect((k^{P,p,q})_{i=1,\dots,A+B}, (k^{Q,p})_{i=1,\dots,A+B}, (k^{W,p})_{i=1,\dots,A+B}, (k^{D})_{i=1,\dots,A+B})$$

is orthogonal to the subspace

$$Vect((l^{\Delta,j})_{i=1,\dots,A+B}, (l^{U,z})_{i=1,\dots,A+B}, (l^{E})_{i=1,\dots,A+B})$$

(for the scalar product $\langle\ |\ \rangle$).

In the case of $k^{P,p,q}$, it is a direct consequence of the oddness properties and of changes of variables of the type $(V_1, V_2, V_3) \to (V_1, V_3, V_2)$.

For $k^{Q,p}$ and $k^{W,p}$, the properties of evenness enable to consider only $l^{U,z}$, and for $p = z$ only. This last case can be treated by a direct computation.

Finally for k^D, one needs to perform a direct computation for $l^{\Delta,j}$ and l^E, the case of $l^{U,z}$ being treated by evenness properties.

We observe that the operator

$$\mathcal{K} : \begin{pmatrix} h^{(1)} \\ . \\ . \\ h^{(A+B)} \end{pmatrix} \mapsto \begin{pmatrix} \sum_j n^{(j)}\, K_{1j}(h^{(1)}, h^{(j)}) \\ . \\ . \\ \sum_j n^{(j)}\, K_{A+B\,j}(h^{(A+B)}, h^{(j)}) \end{pmatrix}$$

is symmetric w.r.t. the scalar product $\langle\ |\ \rangle$, so that (admitting that it satisfies Fredholm's property, which we do here since we work at the formal level), its image is the orthogonal of its kernel.

We refer to [7], [12] and [9] for the Fredholm property in the case of a mixture of monoatomic gases (with the same or different masses).

The kernel of $\mathcal{K}$ can easily be found (provided that all cross sections B_{ij} are strictly positive). We refer for this to a computation done in [16].

It is constituted of

$$\begin{pmatrix} l_1^{\Delta,j} \\ \cdot \\ \cdot \\ l_{A+B}^{\Delta,j} \end{pmatrix}, \quad j = 1 \dots A+B, \qquad \begin{pmatrix} l_1^{U,z} \\ \cdot \\ \cdot \\ l_{A+B}^{U,z} \end{pmatrix}, \quad z = 1 \dots 3 \qquad \begin{pmatrix} l_1^{E} \\ \cdot \\ \cdot \\ l_{A+B}^{E} \end{pmatrix}.$$

The families $(k^{P,p,q}, k^{Q,p}, k^{W,p}, k^D)$ belong to the image of $\mathcal{K}$, so that it is possible to find points such that their image by $\mathcal{K}$ is one of the functions of the concerned families. Moreover such a point is unique if we also impose that it belongs to the orthogonal of the kernel of $\mathcal{K}$.

In other words, for all family of tridimensional vectors $s^{(1)}, \dots, s^{(A+B)}$ such that the relation (51) is satisfied, we can find the functions

$$h^{(i),W}_{s^{(1)},\dots,s^{(A+B)}},\ h^{(i),P,p,q},\ h^{(i),D},\ h^{(i),Q,p}, \qquad i = 1, \dots, A+B, \qquad p, q = 1 \dots 3,$$

which depend on V for $i = 1, \dots, A$ and on V, J for $i = A+1, \dots, A+B$, satisfying the linear integral equations

$$\begin{aligned} \mathcal{K}(h^{(i),W}_{s^{(1)},\dots,s^{(A+B)};i=1,\dots,A+B}) &= k^{(i),W}_{i;s^{(1)},\dots,s^{(A+B)};\,i=1,\dots,A+B}, \\ \mathcal{K}(h^{(i),P,p,q}_{i=1,\dots,A+B}) &= k^{P,p,q}_{i;\,i=1,\dots,A+B}, \\ \mathcal{K}(h^{(i),D}_{i=1,\dots,A+B}) &= k^{D}_{i;\,i=1,\dots,A+B}, \\ \mathcal{K}(h^{(i),Q,p}_{i=1,\dots,A+B}) &= k^{Q,p}_{i;\,i=1,\dots,A+B}, \end{aligned}$$

and (with a generic notation, that is for $h^{(i)} = h^{(i),P,p,q}$, $h^{(i),Q,p}_{i=1,..,A+B}$, $h^{(i),W,p}_{s_p^{(1)},..s_p^{(A+B)};i=1,..,A+B}$, $h^{(i),D}_{i=1,..,A+B}$), the orthogonality relations:

$$j = 1, \dots, A, \qquad \int_{\mathbb{R}^3} \frac{e^{-m_j \frac{|V|^2}{2}}}{(2\pi/m_j)^{3/2}}\, h^{(j)}(V)\, dV = 0,$$

$$j = A+1, \dots, A+B, \qquad \int_0^\infty \int_{\mathbb{R}^3} \frac{e^{-m_j \frac{|V|^2}{2} - J}}{(2\pi/m_j)^{3/2}}\, h^{(j)}(V,J)\, \varphi_j(J\,T)\, dV\, dJ = 0,$$

$$\sum_{i=1}^{A} n^{(i)} \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}}\, h^{(i)}(V) \begin{pmatrix} m_i\, V_k \\ m_i \frac{|V|^2}{2} \end{pmatrix} dV$$

$$+ \sum_{i=A+1}^{A+B} n^{(i)} \int_0^\infty \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2} - J}}{(2\pi/m_i)^{3/2}}\, h^{(i)}(V,J) \begin{pmatrix} m_i\, V_k \\ m_i \frac{|V|^2}{2} + J \end{pmatrix} \frac{T\, \varphi_i(J\,T)}{q_i(T)}\, dV\, dJ = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

3.3.3. *Galilean invariance and computation of $g^{(i)}$.* We now notice that thanks to the Galilean invariance (18), we can write (cf. [15]) for $i = 1, \dots, A$ (we do not explicitly write the components):

$$h^{(i),P}(V) = \tilde{h}^{(i),P}(|V|)\, P(V), \quad h^{(i),Q}(V) = \tilde{h}^{(i),Q}(|V|)\, V, \quad h^{(i),D}(V) = \tilde{h}^{(i),D}(|V|),$$

and for $i = A+1, \dots, A+B$:

$$h^{(i),P}(V,J) = \tilde{h}^{(i),P}(|V|, J)\, P(V), \qquad h^{(i),Q}(V,J) = \tilde{h}^{(i),Q}(|V|, J)\, V,$$
$$h^{(i),D}(V,J) = \tilde{h}^{(i),D}(|V|, J).$$

Thanks to (27), (28), and the computations (50), we see that the previous definitions lead to the following formula for $g^{(i)}$:

$$i = 1 \dots A \qquad g^{(i)}(V\sqrt{T}+u) = \tilde{h}^{(i),P}(|V|)\, P(V) : \left(\frac{\nabla_x u + \nabla_x u^T}{2}\right) \tag{52}$$

$$+\tilde{h}^{(i),D}(|V|)\, \nabla_x \cdot u + \tilde{h}^{(i),Q}(|V|)\, V \cdot \frac{\nabla_x T}{\sqrt{T}} + \sqrt{T}\, h^{(i),W}_{s^{(1)},\dots,s^{(A+B)}}(V),$$

(53)

$$i = A+1 \dots A+B \qquad g^{(i)}(V\sqrt{T}+u, J\,T) = \tilde{h}^{(i),P}(|V|, J)\, P(V) : \left(\frac{\nabla_x u + \nabla_x u^T}{2}\right)$$

$$+\tilde{h}^{(i),D}(|V|, J)\, \nabla_x \cdot u + \tilde{h}^{(i),Q}(|V|, J)\, V \cdot \frac{\nabla_x T}{\sqrt{T}} + \sqrt{T}\, h^{(i),W}_{s^{(1)},\dots,s^{(A+B)}}(V, J)$$

with

$$s^{(i)} = \frac{\nabla_x n^{(i)}}{n^{(i)}} - \frac{m_i \sum_{j=1}^{A+B} \nabla_x n^{(j)}}{\sum_{j=1}^{A+B} m_j\, n^{(j)}} + \left(1 - \frac{m_i \sum_{j=1}^{A+B} n^{(j)}}{\sum_{j=1}^{A+B} m_j\, n^{(j)}}\right) \frac{\nabla_x T}{T}. \tag{54}$$

Remark that the terms $(s^{(i)})_{i \in \{1;A+B\}}$ satisfy the relation (51).

3.3.4. *Computation of the dissipative terms.* We can then make explicit the computation of the diffusion terms in the Chapman-Enskog expansion, that is the quantities appearing as derivatives in the r.h.s of (29) – (32).

We begin by considering, for $i = 1, \dots, A$ and $k = 1, \dots, 3$:

$$D^{(i)}_k := \int_{\mathbb{R}^3} M^{(i)}\, g^{(i)}\, m_i\, v_k\, dv.$$

Hence by using a change of variables, we get

$$D^{(i)}_k = \sqrt{T}\, n^{(i)} \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}}\, \tilde{h}^{(i),Q}(|V|)\, V \cdot \frac{\nabla_x T}{\sqrt{T}}\, m_i\, V_k\, dV$$

$$+\sqrt{T}\, n^{(i)} \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}}\, h^{(i),W}_{s^{(1)},\dots,s^{(A+B)}}(V)\, \sqrt{T}\, m_i\, V_k\, dV.$$

Hence according to eveness properties, it comes that

$$D^{(i)}_k = n^{(i)} \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2}}}{(2\pi)^{3/2}}\, \tilde{h}^{(i),Q}\left(\frac{|V|}{\sqrt{m_i}}\right) V_1^2\, dV\, \partial_{x_k} T$$

$$+T\, n^{(i)}\, m_i \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}}\, h^{(i),W}_{s^{(1)},\dots,s^{(A+B)}}(V)\, V_k\, dV. \tag{55}$$

In the same way, for $i = A+1, \dots, A+B$ and $k = 1, \dots, 3$:

$$D^{(i)}_k := \int_0^{+\infty} \int_{\mathbb{R}^3} M^{(i)}\, g^{(i)}\, m_i\, v_k\, \varphi_i(I)\, dv dI$$

$$= n^{(i)} \int_0^{+\infty} \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2} - J}}{(2\pi)^{3/2}}\, \tilde{h}^{(i),Q}\left(\frac{|V|}{\sqrt{m_i}}, J\right) V_1^2\, \frac{\varphi_i(JT)}{q_i(T)}\, dV\, dJ\, \partial_{x_k} T$$

$$+T\, n^{(i)}\, m_i \int_0^{+\infty} \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2} - J}}{(2\pi/m_i)^{3/2}}\, h^{(i),W}_{s^{(1)},\dots,s^{(A+B)}}(V, J)\, V_k\, \frac{\varphi_i(JT)}{q_i(T)}\, dV dJ. \tag{56}$$

In the previous relation, the term $h^{(i),W}_{s^{(1)},\ldots,s^{(A+B)}}$ depends on a linear combination of the terms $s^{(i)}$, $i \in \{1; A+B\}$ defined in (54). Hence, we recover in this way the interspecies diffusion terms (Fick) and the terms corresponding to the Soret effect.

We then compute, for $k,l = 1,\ldots,3$,

$$F_{kl} := \sum_{i=1}^{A} \int_{\mathbb{R}^3} M^{(i)}\, g^{(i)}\, m_i\, v_k\, v_l\, dv + \sum_{i=A+1}^{A+B} \int_0^\infty \int_{\mathbb{R}^3} M^{(i)}\, g^{(i)}\, m_i\, v_k\, v_l\, \varphi_i(I)\, dvdI.$$

By using again a change of variable, it holds that

$$F_{kl} = \sum_{i=1}^{A} T\, n^{(i)} \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}}\, \tilde{h}^{(i),P}(|V|)\, P(V) : \left(\frac{\nabla_x u + \nabla_x u^T}{2}\right) m_i\, V_k\, V_l\, dV$$

$$+ \sum_{i=A+1}^{A+B} T\, n^{(i)} \int_0^\infty \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2} - J}}{(2\pi/m_i)^{3/2}}\, \tilde{h}^{(i),P}(|V|, J)\, P(V) : \left(\frac{\nabla_x u + \nabla_x u^T}{2}\right) m_i\, V_k\, V_l\, \frac{\varphi_i(JT)}{q_i(T)}\, dV dJ$$

$$+ \sum_{i=1}^{A} T\, n^{(i)} \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}}\, \tilde{h}^{(i),D}(|V|)\, \nabla_x \cdot u\ m_i\, V_k\, V_l\, dV$$

$$+ \sum_{i=A+1}^{A+B} T\, n^{(i)} \int_0^\infty \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2} - J}}{(2\pi/m_i)^{3/2}}\, \tilde{h}^{(i),D}(|V|, J)\, \nabla_x \cdot u\ m_i\, V_k\, V_l\, \frac{\varphi_i(JT)}{q_i(T)}\, dV dJ.$$

Then according to evenness properties and the fact that for any function $a := a(|V|)$,

$$\int_{\mathbb{R}^3} a(|V|)\, (V_1^4 - V_1^2\, V_2^2)\, dV = 2 \int_{\mathbb{R}^3} a(|V|)\, V_1^2\, V_2^2\, dV,$$

we get

$$F_{kl} = \Bigg(\sum_{i=1}^{A} T\, \frac{n^{(i)}}{m_i} \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2}}}{(2\pi)^{3/2}}\, \tilde{h}^{(i),P}\left(\frac{|V|}{\sqrt{m_i}}\right) \frac{2}{3}\, V_1^4\, dV$$

$$+ \sum_{i=A+1}^{A+B} T\, \frac{n^{(i)}}{m_i} \int_0^\infty \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2} - J}}{(2\pi)^{3/2}}\, \tilde{h}^{(i),P}\left(\frac{|V|}{\sqrt{m_i}}, J\right) \frac{2}{3}\, V_1^4\, \frac{\varphi_i(JT)}{q_i(T)}\, dV dJ \Bigg)$$

$$\times \left[\frac{\nabla_x u + \nabla_x u^T}{2} - \frac{1}{3}\, \nabla_x \cdot u\, Id \right]_{kl}$$

$$+ T\, \nabla_x \cdot u\, \delta_{kl} \Bigg(\sum_{i=1}^{A} n^{(i)} \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2}}}{(2\pi)^{3/2}}\, \tilde{h}^{(i),D}\left(\frac{|V|}{\sqrt{m_i}}\right) V_1^2\, dV$$

$$+ \sum_{i=A+1}^{A+B} n^{(i)} \int_0^\infty \int_{\mathbb{R}^3} \frac{e^{\frac{|V|^2}{2} - J}}{(2\pi)^{3/2}}\, \tilde{h}^{(i),D}\left(\frac{|V|}{\sqrt{m_i}}, J\right) V_1^2\, \frac{\varphi_i(JT)}{q_i(T)}\, dV dJ \Bigg), \tag{57}$$

so that viscosity terms are recovered.

We finally compute (for $k = 1,\ldots,3$)

$$G_k = \sum_{i=1}^{A} \int_{\mathbb{R}^3} M^{(i)}\, g^{(i)}\, m_i\, \frac{|v|^2}{2}\, v_k\, dv + \sum_{i=A+1}^{A+B} \int_0^\infty \int_{\mathbb{R}^3} M^{(i)}\, g^{(i)} \left(m_i\, \frac{|v|^2}{2} + I\right) v_k\, \varphi_i(I)\, dvdI.$$

Hence by using a change of variable, we obtain for $k = 1, \dots, 3$

$$G_k = T\sqrt{T} \sum_{i=1}^{A} n^{(i)} \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}} \, g^{(i)}(V\sqrt{T} + u) \, m_i \, \frac{|V|^2}{2} \, V_k \, dV$$

$$+ T\sqrt{T} \sum_{i=A+1}^{A+B} n^{(i)} \int_0^\infty \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2} - J}}{(2\pi/m_i)^{3/2}} \, g^{(i)}(V\sqrt{T} + u, J\,T) \left[m_i \, \frac{|V|^2}{2} + J \right]$$

$$\times V_k \, \frac{\varphi_i(JT)}{q_i(T)} \, dV \, dJ.$$

Therefore, by using the expression of $g^{(i)}$ (cf. (52) and (53)) and evenness properties, we get

$$G_k = \sum_l F_{kl} \, u_l + T \left\{ \sum_{i=1}^{A} \frac{n^{(i)}}{m_i} \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2}}}{(2\pi)^{3/2}} \, \tilde{h}^{(i),Q} \left(\frac{|V|}{\sqrt{m_i}} \right) \frac{|V|^2}{2} \, V_1^2 \, dV \right\} \partial_{x_k} T$$

$$+ T \left\{ \sum_{i=A+1}^{A+B} \frac{n^{(i)}}{m_i} \int_0^\infty \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2} - J}}{(2\pi)^{3/2} q_i(T)} \, \tilde{h}^{(i),Q} \left(\frac{|V|}{\sqrt{m_i}}, J \right) \left(\frac{|V|^2}{2} + J \right) V_1^2 \, \varphi_i(JT) \, dV \, dJ \right\} \partial_{x_k} T$$

$$+ T^2 \left\{ \sum_{i=1}^{A} n^{(i)} \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}} \, h^{(i),W}_{s^{(1)},\dots,s^{(A+B)}}(V) \, m_i \, \frac{|V|^2}{2} \, V_k \, dV \right\}$$

$$+ T^2 \left\{ \sum_{i=A+1}^{A+B} n^{(i)} \int_0^\infty \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2} - J}}{(2\pi/m_i)^{3/2} q_i(T)} \, h^{(i),W}_{s^{(1)},\dots,s^{(A+B)}}(V, J) \left[m_i \, \frac{|V|^2}{2} + J \right] \right.$$

$$\left. \times V_k \, \varphi_i(JT) \, dV \, dJ \right\}. \tag{58}$$

In the previous relation, the terms $h^{(i),W}_{s^{(1)},\dots,s^{(A+B)}}$ contain a linear combination of the gradients of the concentrations. Hence, this final computation shows the dissipative terms corresponding to the (Fourier) diffusion of temperature, and those related to the Dufour effect.

We finally write down the system (29) – (32) in the following semi-explicit form (neglecting the $O(\varepsilon^2)$ terms):

$$\partial_t (m_i \, n^{(i)}) + \nabla_x \cdot (m_i \, n^{(i)} u) = -\varepsilon \nabla_x \cdot D^{(i)}, \tag{59}$$

$$\partial_t \left(\sum_{i=1}^{A+B} m_i \, n^{(i)} \, u_k \right) + \sum_l \partial_{x_l} \left(\sum_{i=1}^{A+B} [m_i \, n^{(i)} \, u_k \, u_l + n^{(i)} \, T \, \delta_{kl}] \right) = -\varepsilon \sum_l \partial_{x_l} F_{kl}, \tag{60}$$

$$\partial_t \left(\sum_{i=1}^{A} [m_i \, n^{(i)} \, \frac{|u|^2}{2} + \frac{3}{2} \, n^{(i)} \, T] + \sum_{i=A+1}^{A+B} [m_i \, n^{(i)} \, \frac{|u|^2}{2} + \frac{5}{2} \, n^{(i)} \, T] \right) \tag{61}$$

$$+ \sum_l \partial_{x_l} \left(\sum_{i=1}^{A} [m_i \, n^{(i)} \, \frac{|u|^2}{2} \, u_l + \frac{5}{2} \, n^{(i)} \, T \, u_l] + \sum_{i=A+1}^{A+B} [m_i \, n^{(i)} \, \frac{|u|^2}{2} \, u_l + \frac{7}{2} \, n^{(i)} \, T \, u_l] \right)$$

$$= -\varepsilon \nabla_x \cdot G.$$

In those equations, the terms $D^{(i)}$, F_{kl} and G are given by formulas (55), (56), (57) and (58) in terms of the functions $\tilde{h}^{(i),Q}$, $h^{(i),W}$, $\tilde{h}^{(i),P}$ and $\tilde{h}^{(i),D}$.

4. Explicit computations in the case of constant cross sections

The quantities $h^{(i)}$ which appear in the definition of $g^{(i)}$ and therefore in the dissipative quantities $D_k^{(i)}$, F_{kl} and G_k (which are part of the Navier-Stokes system of compressible monoatomic and polyatomic gas mixtures) cannot in general be explicitly computed.

As in the case of a single monotaomic gas, it is however possible to compute them when the cross sections (here denoted by B_{ij}) appearing in the collision operators Q_{ij} are very simple. Therefore, in this section, we shall systematically use the assumption that B_{ij} is constant (and $B_{ij} = B_{ji}$). Moreover, in order to be coherent with the fact that in the air, the main polyatomic species (that is, O_2 and N_2) are in fact diatomic, we also shall assume that for all $i = A+1, \dots, A+B$, one has $\varphi_i(I) = 1$, so that one also has $q_i(T) = T$.

The next four subsections are respectively devoted to the computation of h^W, h^P, h^D and h^Q. Then, subsection 4.5 contains the computation of $D_k^{(i)}$, F_{kl} and G_k, starting from the values obtained for h^W, h^P, h^D and h^Q. In the procedure, use will be made of integrals reported in the Appendix. These computations will be really long but, even if some points could be easily recovered from the results for general kernels presented in the previous section, we prefer to derive all coefficients till the very end. Indeed, any numerical code for a mixture of monatomic and polyatomic gases needs to be tested on cases in which all is explicit, before being used in physics or engineering applications mentioned in the Introduction, and a complete Navier–Stokes hydrodynamics from kinetic models in the case of monoatomic and polyatomic mixtures is still lacking in the literature.

4.1. **Computation of h^W.**

We begin by computing for all $i, j = 1, \dots, A+B$ the quantity
$$K_{ij}(v \mapsto W_1^{(i)}\, m_i\, v_1, v \mapsto W_1^{(j)}\, m_j\, v_1),$$
where $W_1^{(i)}, W_1^{(j)} \in \mathbb{R}$ are constants.

For $i = 1, \dots, A$, $j = 1, \dots, A$,
$$\begin{aligned}
K_{ij}(v \mapsto W_1^{(i)}\, m_i\, v_1, v \mapsto W_1^{(j)}\, m_j\, v_1)(v) &= \int_{\mathbb{R}^3} \int_{S^2} \frac{e^{-\frac{m_j}{2}|v_*|^2}}{(2\pi/m_j)^{3/2}} \\
\times \Bigg[W_1^{(j)}\, m_j \left(\frac{m_i\, v_1 + m_j\, v_{1*}}{m_i + m_j} - \frac{m_i}{m_i + m_j} |v - v_*|\, \sigma_1 \right) \\
+ W_1^{(i)}\, m_i \left(\frac{m_i\, v_1 + m_j\, v_{1*}}{m_i + m_j} + \frac{m_j}{m_i + m_j} |v - v_*|\, \sigma_1 \right) \\
- m_j\, W_1^{(j)}\, v_{1*} - m_i\, W_1^{(i)}\, v_1 \Bigg] B_{ij}\, d\sigma\, dv_* \\
= B_{ij}\, \mu_{ij}\, (W_1^{(j)} - W_1^{(i)})\, v_1.
\end{aligned}$$

For $i = 1, \dots, A$, $j = A+1, \dots, A+B$,

$$K_{ij}(v \mapsto W_1^{(i)}\, m_i\, v_1, v \mapsto W_1^{(j)}\, m_j\, v_1)(v) = \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} \int_0^1 \frac{e^{-\frac{m_j}{2}\,|v_*|^2 - I_*}}{(2\pi/m_j)^{3/2}}$$

$$\times \Bigg[m_j W_1^{(j)} \left(\frac{m_i\, v_1 + m_j\, v_{1*}}{m_i + m_j} - \frac{m_i}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_1 \right)$$

$$+ m_i\, W_1^{(i)} \left(\frac{m_i\, v_1 + m_j\, v_{1*}}{m_i + m_j} + \frac{m_j}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_1 \right)$$

$$- m_j\, W_1^{(j)}\, v_{1*} - m_i\, W_1^{(i)}\, v_1 \Bigg]\, R^{1/2}\, dR\, d\sigma\, dI_*\, dv_*$$

$$= \frac{2}{3}\, B_{ij}\, \mu_{ij}\, (W_1^{(j)} - W_1^{(i)})\, v_1.$$

For $i = A+1, \dots, A+B$, $j = 1, \dots, A$,

$$K_{ij}(v \mapsto W_1^{(i)}\, m_i\, v_1, v \mapsto W_1^{(j)}\, m_j\, v_1)(v, I) = \int_{\mathbb{R}^3} \int_{S^2} \int_0^1 \frac{e^{-\frac{m_j}{2}\,|v_*|^2}}{(2\pi/m_j)^{3/2}}$$

$$\times \Bigg[m_j\, W_1^{(j)} \left(\frac{m_i\, v_1 + m_j\, v_{1*}}{m_i + m_j} - \frac{m_i}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_1 \right)$$

$$+ m_i\, W_1^{(i)} \left(\frac{m_i\, v_1 + m_j\, v_{1*}}{m_i + m_j} + \frac{m_j}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_1 \right)$$

$$- m_j\, W_1^{(j)}\, v_{1*} - m_i\, W_1^{(i)}\, v_1 \Bigg]\, B_{ij}\, R^{1/2}\, dR\, d\sigma\, dv_*$$

$$= \frac{2}{3}\, B_{ij}\, \mu_{ij}\, (W_1^{(j)} - W_1^{(i)})\, v_1.$$

For $i = A+1, \dots, A+B$, $j = A+1, \dots, A+B$,

$$K_{ij}(v \mapsto W_1^{(i)}\, m_i\, v_1, v \mapsto W_1^{(j)}\, m_j\, v_1)(v, I) = \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} \int_0^1 \int_0^1 \frac{e^{-\frac{m_j}{2}\,|v_*|^2 - I_*}}{(2\pi/m_j)^{3/2}}$$

$$\times \Bigg[m_j\, W_1^{(j)} \left(\frac{m_i\, v_1 + m_j\, v_{1*}}{m_i + m_j} - \frac{m_i}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_1 \right)$$

$$+ m_i\, W_1^{(i)} \left(\frac{m_i\, v_1 + m_j\, v_{1*}}{m_i + m_j} + \frac{m_j}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_1 \right)$$

$$- m_j\, W_1^{(j)}\, v_{1*} - m_i\, W_1^{(i)}\, v_1 \Bigg]\, B_{ij}\, (1-R)\, R^{1/2}\, dR\, dr\, d\sigma\, dI_*\, dv_*$$

$$= \frac{4}{15}\, B_{ij}\, \mu_{ij}\, (W_1^{(j)} - W_1^{(i)})\, v_1.$$

We define then

$$\tilde{B}_{ij} = \begin{cases} B_{ij} & \text{if} \quad i,j = 1,\dots,A, \\ \frac{2}{3} B_{ij} & \text{if} \quad i = 1,\dots,A,\ j = A+1,\dots,A+B, \\ \frac{2}{3} B_{ij} & \text{if} \quad i = A+1,\dots,A+B,\ j = 1,\dots,A, \\ \frac{4}{15} B_{ij} & \text{if} \quad i = A+1,\dots,A+B,\ j = A+1,\dots,A+B. \end{cases}$$

The problem

$$\mathcal{K}(h^{(i),W}_{s^{(1)},\dots,s^{(A+B)};i=1,\dots,A+B}) = k^{W}_{i;s^{(1)},\dots,s^{(A+B)};\, i=1,\dots,A+B}$$

with

$$j = 1,\dots,A, \qquad \int_{\mathbb{R}^3} \frac{e^{-m_j \frac{|V|^2}{2}}}{(2\pi/m_j)^{3/2}}\, h^{(j),W}_{s^{(1)},\dots,s^{(A+B)}}(V)\, dV = 0,$$

$$j = A+1,\dots,A+B, \qquad \int_{\mathbb{R}^3}\int_0^\infty \frac{e^{-m_j \frac{|V|^2}{2} - J}}{(2\pi/m_j)^{3/2}}\, h^{(j),W}_{s^{(1)},\dots,s^{(A+B)}}(V,J)\, dV dJ = 0,$$

$$\sum_{i=1}^{A} n^{(i)} \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}}\, h^{(i),W}_{s^{(1)},\dots,s^{(A+B)}}(V) \begin{pmatrix} m_i\, V_k \\ m_i\, \frac{|V|^2}{2} \end{pmatrix} dV$$

$$+ \sum_{i=A+1}^{A+B} n^{(i)} \int_{\mathbb{R}^3}\int_0^\infty \frac{e^{-m_i \frac{|V|^2}{2} - J}}{(2\pi/m_i)^{3/2}}\, h^{(i)W}_{s^{(1)},..,s^{(A+B)}}(V,J) \begin{pmatrix} m_i\, V_k \\ m_i\, \frac{|V|^2}{2} + J \end{pmatrix} dV dJ = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

becomes, after exchanging the variable 1 with any of the variables p,

$$h^{(i),W}_{s^{(1)},\dots,s^{(A+B)}}(V) = m_i\, W^{(i)} \cdot V, \qquad i = 1,\dots,A,$$

$$h^{(i),W}_{s^{(1)},\dots,s^{(A+B)}}(V,J) = m_i\, W^{(i)} \cdot V, \qquad i = A+1,\dots,A+B,$$

where the tridimensional constants $W^{(i)}$ must satisfy the system

$$\sum_{j=1}^{A+B} n^{(j)}\, \tilde{B}_{ij}\, \mu_{ij}\, (W^{(j)} - W^{(i)}) = s^{(i)},$$

$$\sum_{i=1}^{A+B} m_i\, n^{(i)}\, W^{(i)} = 0.$$

Note that the first part of the system only contains $A+B-1$ independant equations. It can be solved only under the constraint $\sum_{i=1}^{A+B} n^{(i)}\, s^{(i)} = 0$.

We finish the computation by noticing that in the special case of a mixture of two gases, the system above can be solved very easily (remember that $\tilde{B}_{12} = \tilde{B}_{21}$):

$$W^{(1)} = \frac{m_2\, s^{(1)}}{(m_1\, n^{(1)} + m_2\, n^{(2)})\, \tilde{B}_{12}\, \mu_{12}},$$

$$W^{(2)} = \frac{-m_1\, s^{(2)}}{(m_1\, n^{(1)} + m_2\, n^{(2)})\, \tilde{B}_{12}\, \mu_{12}},$$

with

$$s^{(1)} = \frac{\nabla_x n^{(1)}}{n^{(1)}} - \frac{m_1(\nabla_x n^{(1)} + \nabla_x n^{(2)})}{m_1 n^{(1)} + m_2 n^{(2)}} + \left(1 - \frac{m_1(n^{(1)} + n^{(2)})}{m_1 n^{(1)} + m_2 n^{(2)}}\right) \frac{\nabla_x T}{T},$$

$$s^{(2)} = \frac{\nabla_x n^{(2)}}{n^{(2)}} - \frac{m_2(\nabla_x n^{(1)} + \nabla_x n^{(2)})}{m_1 n^{(1)} + m_2 n^{(2)}} + \left(1 - \frac{m_2(n^{(1)} + n^{(2)})}{m_1 n^{(1)} + m_2 n^{(2)}}\right) \frac{\nabla_x T}{T}.$$

4.2. **Computation of h^P.** The computation of h^P follows the same lines as the computation of h^W.

We start with the nondiagonal part of the tensor. For the sake of simplicity, we consider the case with components $p = 1, q = 2$, the other ones being obtained by an immediate change of indices.

For $i = A+1,, \dots, A+B$, $j = A+1, \dots, A+B$, and $\Pi_{12}^{(i)}$, $\Pi_{12}^{(j)}$ real constants,

$$K_{ij}(v \mapsto m_i\, \Pi_{12}^{(i)}\, v_1\, v_2,\ v \mapsto m_j\, \Pi_{12}^{(j)}\, v_1\, v_2)(v, I) = \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} \int_0^1 \int_0^1 \frac{e^{-\frac{m_j}{2}\, |v_*|^2 - I_*}}{(2\pi/m_j)^{3/2}}$$

$$\times \Bigg[m_j\, \Pi_{12}^{(j)} \left(\frac{m_i\, v_1 + m_j\, v_{1*}}{m_i + m_j} - \frac{m_i}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_1 \right)$$

$$\times \left(\frac{m_i\, v_2 + m_j\, v_{2*}}{m_i + m_j} - \frac{m_i}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_2 \right)$$

$$+ m_i\, \Pi_{12}^{(i)} \left(\frac{m_i\, v_1 + m_j\, v_{1*}}{m_i + m_j} + \frac{m_j}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_1 \right)$$

$$\times \left(\frac{m_i\, v_2 + m_j\, v_{2*}}{m_i + m_j} + \frac{m_j}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_2 \right)$$

$$- m_j\, \Pi_{12}^{(j)}\, v_{1*}\, v_{2*} - m_i\, \Pi_{12}^{(i)}\, v_1\, v_2 \Bigg] B_{ij}\, (1-R)\, R^{1/2}\, dR\, dr\, d\sigma\, dI_*\, dv_*$$

$$= \frac{4}{15}\, B_{ij}\, \mu_{ij}^2 \left(\frac{\Pi_{12}^{(j)} - 2\,\Pi_{12}^{(i)}}{m_j} - \frac{\Pi_{12}^{(i)}}{m_i} \right) v_1\, v_2.$$

In the same way, for $i = 1, \dots, A$, $j = 1, \dots, A$,

$$K_{ij}(v \mapsto m_i\, \Pi_{12}^{(i)}\, v_1\, v_2, v \mapsto m_j\, \Pi_{12}^{(j)}\, v_1\, v_2)(v) = B_{ij}\, \mu_{ij}^2 \left(\frac{\Pi_{12}^{(j)} - 2\,\Pi_{12}^{(i)}}{m_j} - \frac{\Pi_{12}^{(i)}}{m_i} \right) v_1\, v_2,$$

for $i = 1, \dots, A$, $j = A+1, \dots, A+B$,

$$K_{ij}(v \mapsto m_i\, \Pi_{12}^{(i)}\, v_1\, v_2, v \mapsto m_j\, \Pi_{12}^{(j)}\, v_1\, v_2)(v) = \frac{2}{3}\, B_{ij}\, \mu_{ij}^2 \left(\frac{\Pi_{12}^{(j)} - 2\,\Pi_{12}^{(i)}}{m_j} - \frac{\Pi_{12}^{(i)}}{m_i} \right) v_1\, v_2,$$

for $i = A+1, \dots, A+B$, $j = 1, \dots, A$,

$$K_{ij}(v \mapsto m_i\, \Pi_{12}^{(i)}\, v_1\, v_2, v \mapsto m_j\, \Pi_{12}^{(j)}\, v_1\, v_2)(v, I) = \frac{2}{3}\, B_{ij}\, \mu_{ij}^2 \left(\frac{\Pi_{12}^{(j)} - 2\,\Pi_{12}^{(i)}}{m_j} - \frac{\Pi_{12}^{(i)}}{m_i} \right) v_1\, v_2.$$

Finally, with the notations of the previous paragraph, for $i = 1, \dots, A+B$, $j = 1, \dots, A+B$,

$$K_{ij}(v \mapsto m_i\, \Pi_{12}^{(i)}\, v_1\, v_2, v \mapsto m_j\, \Pi_{12}^{(j)}\, v_1\, v_2) = \tilde{B}_{ij}\, \mu_{ij}^2 \left(\frac{\Pi_{12}^{(j)} - 2\,\Pi_{12}^{(i)}}{m_j} - \frac{\Pi_{12}^{(i)}}{m_i} \right) v_1\, v_2.$$

We now turn to diagonal coefficients.

For $i = A+1, \dots, A+B$, $j = A+1, \dots, A+B$, and $\Pi_{11}^{(i)}$, $\Pi_{11}^{(j)}$ real constants,

$$K_{ij}\Big(v \mapsto m_i\, \Pi_{11}^{(i)}\, [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2], v \mapsto m_j\, \Pi_{11}^{(j)}\, [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2] \Big)(v, I)$$

$$= \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} \int_0^1 \int_0^1 \frac{e^{-\frac{m_j}{2}\, |v_*|^2 - I_*}}{(2\pi/m_j)^{3/2}}$$

$$\times \Big[m_j\, \Pi_{11}^{(j)} \Big(\frac{2}{3} \Big| \frac{m_i\, v_1 + m_j\, v_{1*}}{m_i + m_j} - \frac{m_i}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_1 \Big|^2$$
$$- \frac{1}{3} \Big| \frac{m_i\, v_2 + m_j\, v_{2*}}{m_i + m_j} - \frac{m_i}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_2 \Big|^2$$
$$- \frac{1}{3} \Big| \frac{m_i\, v_3 + m_j\, v_{3*}}{m_i + m_j} - \frac{m_i}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_3 \Big|^2 \Big)$$
$$+ m_i\, \Pi_{11}^{(i)} \Big(\frac{2}{3} \Big| \frac{m_i\, v_1 + m_j\, v_{1*}}{m_i + m_j} + \frac{m_j}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_1 \Big|^2$$
$$- \frac{1}{3} \Big| \frac{m_i\, v_2 + m_j\, v_{2*}}{m_i + m_j} + \frac{m_j}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_2 \Big|^2$$
$$- \frac{1}{3} \Big| \frac{m_i\, v_3 + m_j\, v_{3*}}{m_i + m_j} + \frac{m_j}{m_i + m_j} \sqrt{\frac{2R\,E}{\mu_{ij}}}\, \sigma_3 \Big|^2 \Big)$$
$$- m_j\, \Pi_{11}^{(j)} \Big(\frac{2}{3}\, |v_{1*}|^2 - \frac{1}{3}\, |v_{2*}|^2 - \frac{1}{3}\, |v_{3*}|^2 \Big) - m_i\, \Pi_{11}^{(i)} \Big(\frac{2}{3}\, |v_1|^2 - \frac{1}{3}\, |v_2|^2 - \frac{1}{3}\, |v_3|^2 \Big) \Big]$$
$$\times B_{ij}\, (1 - R)\, R^{1/2}\, dR\, dr\, d\sigma\, dI_*\, dv_*$$
$$= \frac{4}{15}\, B_{ij}\, \mu_{ij}^2 \left(\frac{\Pi_{12}^{(j)} - 2\,\Pi_{12}^{(i)}}{m_j} - \frac{\Pi_{12}^{(i)}}{m_i} \right) [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2].$$

In the same way, for $i = 1, \dots, A$, $j = 1, \dots, A$,

$$K_{ij} \Big(v \mapsto m_i\, \Pi_{11}^{(i)}\, [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2], v \mapsto m_j\, \Pi_{11}^{(j)}\, [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2] \Big)(v)$$
$$= B_{ij}\, \mu_{ij}^2 \left(\frac{\Pi_{12}^{(j)} - 2\,\Pi_{12}^{(i)}}{m_j} - \frac{\Pi_{12}^{(i)}}{m_i} \right) [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2],$$

for $i = 1, \dots, A$, $j = A + 1, \dots, A + B$,

$$K_{ij} \Big(v \mapsto m_i\, \Pi_{11}^{(i)}\, [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2], v \mapsto m_j\, \Pi_{11}^{(j)}\, [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2] \Big)(v)$$
$$= \frac{2}{3}\, B_{ij}\, \mu_{ij}^2 \left(\frac{\Pi_{12}^{(j)} - 2\,\Pi_{12}^{(i)}}{m_j} - \frac{\Pi_{12}^{(i)}}{m_i} \right) [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2],$$

for $i = A + 1, \dots, A + B$, $j = 1, \dots, A$,

$$K_{ij} \Big(v \mapsto m_i\, \Pi_{11}^{(i)}\, [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2], v \mapsto m_j\, \Pi_{11}^{(j)}\, [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2] \Big)(v, I)$$
$$= \frac{2}{3}\, B_{ij}\, \mu_{ij}^2 \left(\frac{\Pi_{12}^{(j)} - 2\,\Pi_{12}^{(i)}}{m_j} - \frac{\Pi_{12}^{(i)}}{m_i} \right) [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2].$$

Finally, for $i = 1, \dots, A + B$, $j = 1, \dots, A + B$,

$$K_{ij} \Big(v \mapsto m_i\, \Pi_{11}^{(i)}\, [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2], v \mapsto m_j\, \Pi_{11}^{(j)}\, [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2] \Big)(v, I)$$
$$= \tilde{B}_{ij}\, \mu_{ij}^2 \left(\frac{\Pi_{12}^{(j)} - 2\,\Pi_{12}^{(i)}}{m_j} - \frac{\Pi_{12}^{(i)}}{m_i} \right) [\frac{2}{3}\, v_1^2 - \frac{1}{3}\, v_2^2 - \frac{1}{3}\, v_3^2].$$

The problem

$$\mathcal{K}(h^{(i),P,p,q}_{i=1,\dots,A+B}) = k^{P,p,q}_{i;\,i=1,\dots,A+B}$$

with

$$j = 1,\dots,A, \qquad \int_{\mathbb{R}^3} \frac{e^{-m_j\,\frac{|V|^2}{2}}}{(2\pi/m_j)^{3/2}}\, h^{(j),P,p,q}(V)\,dV = 0,$$

$$j = A+1,\dots,A+B, \qquad \int_{\mathbb{R}^3}\int_0^\infty \frac{e^{-m_j\,\frac{|V|^2}{2}-J}}{(2\pi/m_j)^{3/2}}\, h^{(j),P,p,q}(V,J)\,dV\,dJ = 0,$$

$$\sum_{i=1}^{A} n^{(i)} \int_{\mathbb{R}^3} \frac{e^{-m_i\,\frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}}\, h^{(i),P,p,q}(V) \begin{pmatrix} m_i\,V_k \\ m_i\,\frac{|V|^2}{2} \end{pmatrix} dV$$

$$+ \sum_{i=A+1}^{A+B} n^{(i)} \int_{\mathbb{R}^3}\int_0^\infty \frac{e^{-m_i\,\frac{|V|^2}{2}-J}}{(2\pi/m_i)^{3/2}}\, h^{(i)P,p,q}(V,J) \begin{pmatrix} m_i\,V_k \\ m_i\,\frac{|V|^2}{2}+J \end{pmatrix} dV\,dJ = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

becomes

$$h^{(i),P,p,q}(V) = m_i\,\Pi^{(i)}_{pq}\,P^{(i)}_{p,q}(V), \qquad i = 1,\dots,A,$$
$$h^{(i),P,p,q}(V,J) = m_i\,\Pi^{(i)}_{pq}\,P^{(i)}_{p,q}(V), \qquad i = A+1,\dots,A+B,$$

where the constants $\Pi^{(i)}_{pq}$ have to satisfy the system

$$\sum_{j=1}^{A+B} n^{(j)}\,\tilde{B}_{ij}\,\mu_{ij}^2 \left(\frac{\Pi^{(j)}_{pq} - 2\,\Pi^{(i)}_{pq}}{m_j} - \frac{\Pi^{(i)}_{pq}}{m_i} \right) = m_i.$$

We obtain as announced that the coefficients $\Pi^{(i)}_{pq}$ depend on i but not on p,q (we denote them by $\Pi^{(i)}$). Using the Galilean invariance, we get $h^{(i),P}(V) = \tilde{h}^{(i),P}(|V|)\,P(|V|)$, with

$$\tilde{h}^{(i),P}(|V|) = m_i\,\Pi^{(i)}, \qquad i = 1,\dots,A,$$
$$\tilde{h}^{(i),P}(|V|,J) = m_i\,\Pi^{(i)}, \qquad i = A+1,\dots,A+B.$$

4.3. **Computation of h^D.** We recall that we consider here a mixture of monoatomic and diatomic gases, so that $\varphi_i(I) = 1$ for $i = A+1,\dots,A+B$. Then, the quantity $\Lambda(T)$ defined in (49) simplifies very much, and turns out to be independent of T:

$$\Lambda = \frac{\sum_{j=1}^{A+B} n^{(j)}}{3\,\sum_{j=1}^{A+B} n^{(j)} + 2\,\sum_{j=A+1}^{A+B} n^{(j)}}. \tag{62}$$

Note that for a completely monoatomic mixture, $\Lambda = \frac{1}{3}$, while for a completely diatomic mixture, $\Lambda = \frac{1}{5}$.

We have to solve the problem

$$\mathcal{K}(h^{(i),D}_{i=1,\dots,A+B}) = \sum_j n^{(j)}\,K_{ij}(h^{(i),D}, h^{(j),D}) = k^D_{i;\,i=1,\dots,A+B}, \tag{63}$$

where

$$k^D_{i;\,i=1,\dots,A+B} = (m_i|V|^2 - 3)\left(\frac{1}{3} - \Lambda\right) - 2\,r_i\,\Lambda(J-1),$$

with

$$j = 1,\dots,A, \qquad \int_{\mathbb{R}^3} \frac{e^{-m_j\,\frac{|V|^2}{2}}}{(2\pi/m_j)^{3/2}}\, h^{(j),D}(V)\,dV = 0,$$

$$j = A+1,\dots,A+B, \qquad \int_{\mathbb{R}^3}\int_0^\infty \frac{e^{-m_j\,\frac{|V|^2}{2}-J}}{(2\pi/m_j)^{3/2}}\,h^{(j),D}(V,J)\,dV\,dJ = 0,$$

$$\sum_{i=1}^{A} n^{(i)} \int_{\mathbb{R}^3} \frac{e^{-m_i\,\frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}}\,h^{(i),D}(V)\begin{pmatrix} m_i\,V_k \\ m_i\,\frac{|V|^2}{2}\end{pmatrix} dV$$

$$+\sum_{i=A+1}^{A+B} n^{(i)} \int_{\mathbb{R}^3}\int_0^\infty \frac{e^{-m_i\,\frac{|V|^2}{2}-J}}{(2\pi/m_i)^{3/2}}\,h^{(i),D}(V,J)\begin{pmatrix} m_i\,V_k \\ m_i\,\frac{|V|^2}{2}+J\end{pmatrix} dV\,dJ = \begin{pmatrix} 0 \\ 0\end{pmatrix}.$$

Note that among the orthogonality relations, the one related to mass and momentum can be checked immediately. Only the orthogonality related to total energy is not immediately seen, we therefore explicitly check it:

$$\sum_{i=1}^{A} n^{(i)} \int_{\mathbb{R}^3} \frac{e^{-m_i\,\frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}}\,(m_i|V|^2-3)\left(\frac{1}{3}-\Lambda\right) m_i\,\frac{|V|^2}{2}\,dV$$

$$+\sum_{i=A+1}^{A+B} n^{(i)} \int_{\mathbb{R}^3}\int_0^\infty \frac{e^{-m_i\,\frac{|V|^2}{2}-J}}{(2\pi/m_i)^{3/2}}\,(m_i|V|^2-3)\Big(\left(\frac{1}{3}-\Lambda\right)-2\,\Lambda(J-1)\Big)\,(m_i\,\frac{|V|^2}{2}+J)\,dV\,dJ = 0.$$

A direct computation of the left-hand side gives

$$\sum_{i=1}^{A} n^{(i)}\left(\frac{1}{3}-\Lambda\right)(\frac{15}{2}-\frac{9}{2}) + \sum_{i=A+1}^{A+B} n^{(i)}\Big((\frac{15}{2}-\frac{9}{2})\left(\frac{1}{3}-\Lambda\right)-2\,\Lambda\Big).$$

So the left-hand side gives

$$\sum_{i=1}^{A} n^{(i)}\,3\left(\frac{1}{3}-\Lambda\right) + \sum_{i=A+1}^{A+B} n^{(i)}\left(3\left(\frac{1}{3}-\Lambda\right)-2\,\Lambda\right)$$

$$= \sum_{i=1}^{A+B} n^{(i)} - 3\Lambda\sum_{i=1}^{A+B} n^{(i)} - 2\Lambda\sum_{i=A+1}^{A+B} n^{(i)} = 0\,. \tag{64}$$

Here the last term is equal to 0 because of the definition of Λ provided in (62).

In each computation of this subsection, the objective will be to try to cast the final results as a proper combination of $m_i|v|^2-3$ and $I-1$. We will skip a lot of intermediate steps, which repeat the line of computing adopted in the previous paragraphs, and that may be recovered using the integrals reported in the Appendix.

- For $i=1,\dots,A$ and $j=1,\dots,A$ we get

$$B_{ij}^{-1}K_{ij}\Big(v\mapsto \Delta^{(i)}(m_i|v|^2-3), \quad v\mapsto \Delta^{(j)}(m_j|v|^2-3)\Big)(v) =$$

$$= \int_{\mathbb{R}^3}\int_{S^2}\left(\frac{m_j}{2\pi}\right)^{3/2} e^{-\frac{1}{2}m_j|v_*|^2}\left[\Delta^{(j)}\left\{m_j\left|\frac{m_iv+m_jv_*}{m_i+m_j}-\frac{m_i}{m_i+m_j}|v-v_*|\sigma\right|^2-3\right\}\right.$$

$$+\Delta^{(i)}\left\{m_i\left|\frac{m_iv+m_jv_*}{m_i+m_j}+\frac{m_j}{m_i+m_j}|v-v_*|\sigma\right|^2-3\right\}-\Delta^{(j)}(m_j|v_*|^2-3)$$

$$\left.-\Delta^{(i)}(m_i|v|^2-3)\right]d\sigma\,dv_* = \frac{2\,\mu_{ij}}{m_i+m_j}(\Delta^{(j)}-\Delta^{(i)})(m_i|v|^2-3).$$

- For $i = 1, \dots, A$ and $j = A+1, \dots, A+B$ we get

$$B_{ij}^{-1} K_{ij} \left(v \mapsto \Delta^{(i)}(m_i|v|^2 - 3), \quad (v, I) \mapsto \Delta^{(j)}(m_j|v|^2 - 3) + \widetilde{\Delta}^{(j)}(I-1) \right)(v) =$$

$$= \int_{\mathbb{R}^3} \int_{S^2} \int_0^\infty \int_0^1 \left(\frac{m_j}{2\pi}\right)^{3/2} e^{-\frac{1}{2}m_j|v_*|^2 - I_*} \left[\Delta^{(j)} \left\{ m_j \left| \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{m_i + m_j} \sqrt{\frac{2RE}{\mu_{ij}}}\, \sigma \right|^2 - 3 \right\} \right.$$

$$+ \widetilde{\Delta}^{(j)}((1-R)E - 1) + \Delta^{(i)} \left\{ m_i \left| \frac{m_i v + m_j v_*}{m_i + m_j} + \frac{m_j}{m_i + m_j} \sqrt{\frac{2RE}{\mu_{ij}}}\, \sigma \right|^2 - 3 \right\}$$

$$\left. - \Delta^{(j)}(m_j|v_*|^2 - 3) - \widetilde{\Delta}^{(j)}(I_* - 1) - \Delta^{(i)}(m_i|v|^2 - 3) \right] \sqrt{R}\, dR\, dI_*\, d\sigma\, dv_*$$

$$= \frac{2}{15} \frac{m_j}{m_i + m_j} \left(\Delta^{(j)} \frac{8\, m_i}{m_i + m_j} + \widetilde{\Delta}^{(j)} \right)(m_i|v|^2 - 3)$$

$$+ \Delta^{(i)} \left[-\frac{4}{3} \frac{m_j}{(m_i + m_j)^2} \left(m_i + \frac{1}{5} m_j \right) \right](m_i|v|^2 - 3).$$

- For $i = A+1, \dots, A+B$ and $j = 1, \dots, A$ we get

$$B_{ij}^{-1} K_{ij} \left((v, I) \mapsto \Delta^{(i)}(m_i|v|^2 - 3) + \widetilde{\Delta}^{(i)}(I-1), \quad v \mapsto \Delta^{(j)}(m_j|v|^2 - 3) \right)(v, I) =$$

$$= \int_{\mathbb{R}^3} \int_{S^2} \int_0^1 \left(\frac{m_j}{2\pi}\right)^{3/2} e^{-\frac{1}{2}m_j|v_*|^2} \left[\Delta^{(j)} \left\{ m_j \left| \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{m_i + m_j} \sqrt{\frac{2RE}{\mu_{ij}}}\, \sigma \right|^2 - 3 \right\} \right.$$

$$+ \Delta^{(i)} \left\{ m_i \left| \frac{m_i v + m_j v_*}{m_i + m_j} + \frac{m_j}{m_i + m_j} \sqrt{\frac{2RE}{\mu_{ij}}}\, \sigma \right|^2 - 3 \right\} + \widetilde{\Delta}^{(i)}[(1-R)E - 1]$$

$$\left. - \Delta^{(j)}(m_j|v_*|^2 - 3) - \Delta^{(i)} m_i(|v|^2 - 3) - \widetilde{\Delta}^{(i)}(I-1) \right] \sqrt{R}\, dR\, d\sigma\, dv_*$$

$$= \Delta^{(j)} \left[\frac{16}{15} \frac{\mu_{ij}}{m_i + m_j}(m_i|v|^2 - 3) + \frac{4}{5} \frac{m_i}{m_i + m_j}(I-1) \right]$$

$$- \frac{2}{15} \frac{m_j}{m_i + m_j} \left(2\, \Delta^{(i)} \frac{5\, m_i + m_j}{m_i + m_j} - \widetilde{\Delta}^{(i)} \right)(m_i|v|^2 - 3) + \frac{2}{5} \left(\Delta^{(i)} \frac{2\, m_j}{m_i + m_j} - \widetilde{\Delta}^{(i)} \right)(I-1).$$

- For $i = A+1, \dots, A+B$ and $j = A+1, \dots, A+B$ we get

$$B_{ij}^{-1} K_{ij} \left((v, I) \mapsto \Delta^{(i)}(m_i|v|^2 - 3) + \widetilde{\Delta}^{(i)}(I-1), \quad (v, I) \mapsto \Delta^{(j)}(m_j|v|^2 - 3) + \widetilde{\Delta}^{(j)}(I-1) \right)(v, I) =$$

$$= \int_{\mathbb{R}^3} \int_{S^2} \int_0^\infty \int_0^1 \int_0^1 \left(\frac{m_j}{2\pi}\right)^{3/2} e^{-\frac{1}{2}m_j|v_*|^2 - I_*} \left[\Delta^{(j)} \left\{ m_j \left| \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{m_i + m_j} \sqrt{\frac{2RE}{\mu_{ij}}}\, \sigma \right|^2 \right. \right.$$

$$\left. - 3 \right\} + \widetilde{\Delta}^{(j)}[(1-r)(1-R)E - 1] + \Delta^{(i)} \left\{ m_i \left| \frac{m_i v + m_j v_*}{m_i + m_j} + \frac{m_j}{m_i + m_j} \sqrt{\frac{2RE}{\mu_{ij}}}\, \sigma \right|^2 - 3 \right\}$$

$$\left. + \widetilde{\Delta}^{(i)}[r(1-R)E - 1] - \Delta^{(j)}(m_j|v_*|^2 - 3) - \widetilde{\Delta}^{(j)}(I_* - 1) - \Delta^{(i)} m_i(|v|^2 - 3) - \widetilde{\Delta}^{(i)}(I-1) \right]$$

$$\times (1-R)\sqrt{R}\, dR\, dr\, dI_*\, d\sigma\, dv_*$$

$$
\begin{aligned}
&= \frac{4}{21}\,\frac{m_j}{(m_i+m_j)^2}\left(2\,\Delta^{(j)}\,m_i + \frac{1}{5}\,\widetilde{\Delta}^{(j)}(m_i+m_j)\right)(m_i|v|^2-3)\\
&\qquad + \frac{8}{35}\left(\Delta^{(j)}\,\frac{m_i}{m_i+m_j} + \frac{1}{3}\,\widetilde{\Delta}^{(j)}\right)(I-1)\\
&- \frac{4}{15}\,\frac{m_j}{(m_i+m_j)^2}\left[\Delta^{(i)}\left(2\,m_i + \frac{4}{7}\,m_j\right) - \frac{1}{7}\,\widetilde{\Delta}^{(i)}(m_i+m_j)\right](m_i|v|^2-3)\\
&\qquad + \frac{4}{7}\left(\frac{2}{5}\,\Delta^{(i)}\,\frac{m_j}{m_i+m_j} - \frac{1}{3}\,\widetilde{\Delta}^{(i)}\right)(I-1).
\end{aligned}
$$

In conclusion, system (63) may be rewritten as :

- for $i = 1,\dots,A$,

$$
\begin{aligned}
&\sum_{j=1}^{A} 2n^{(j)}\,B_{ij}\,\frac{\mu_{ij}}{m_i+m_j}\,(\Delta^{(j)}-\Delta^{(i)})\\
&+ \sum_{j=A+1}^{A+B} n^{(j)}\,B_{ij}\left\{\frac{16}{15}\frac{m_j}{m_i+m_j}\,\frac{m_i}{m_i+m_j}\Delta^{(j)} + \frac{2}{15}\frac{m_j}{m_i+m_j}\,\widetilde{\Delta}^{(j)}\right.\\
&\qquad\left. - \frac{4}{3}\frac{m_j(m_i+\frac{1}{5}m_j)}{(m_i+m_j)^2}\Delta^{(i)}\right\} = \frac{1}{3} - \Lambda,
\end{aligned}
\tag{65}
$$

- for $i = A+1,\dots,A+B$,

$$
\begin{aligned}
&\sum_{j=1}^{A} n^{(j)}\,B_{ij}\left\{\frac{16}{15}\frac{\mu_{ij}}{m_i+m_j}\,\Delta^{(j)} - \frac{4}{15}\frac{m_j}{m_i+m_j}\,\frac{5m_i+m_j}{m_i+m_j}\,\Delta^{(i)} + \frac{2}{15}\frac{m_j}{m_i+m_j}\,\widetilde{\Delta}^{(i)}\right\}\\
&+ \sum_{j=A+1}^{A+B} n^{(j)}\,B_{ij}\left\{\frac{8}{21}\frac{m_i m_j}{(m_i+m_j)^2}\,\Delta^{(j)} + \frac{4}{105}\frac{m_j}{m_i+m_j}\,\widetilde{\Delta}^{(j)} - \frac{4}{15}\frac{m_j(2m_i+\frac{4}{7}m_j)}{(m_i+m_j)^2}\Delta^{(i)}\right.\\
&\qquad\left. + \frac{4}{105}\frac{m_j}{m_i+m_j}\,\widetilde{\Delta}^{(i)}\right\} = \frac{1}{3} - \Lambda,
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
&\sum_{j=1}^{A} n^{(j)}\,B_{ij}\left\{\frac{4}{5}\frac{m_i}{m_i+m_j}\Delta^{(j)} + \frac{4}{5}\frac{m_j}{m_i+m_j}\Delta^{(i)} - \frac{2}{5}\widetilde{\Delta}^{(i)}\right\}\\
&+ \sum_{j=A+1}^{A+B} n^{(j)}\,B_{ij}\left\{\frac{8}{35}\frac{m_i}{m_i+m_j}\Delta^{(j)} + \frac{8}{105}\widetilde{\Delta}^{(j)} + \frac{8}{35}\frac{m_j}{m_i+m_j}\Delta^{(i)} - \frac{4}{21}\widetilde{\Delta}^{(i)}\right\} = -2\Lambda.
\end{aligned}
\tag{67}
$$

We can note that a suitable combination of the right hand sides of the system (65)–(67) vanishes, as proved in (64), which shows the orthogonality condition with respect to the energy. We can check that the same linear combination vanishes even when we consider the left hand sides of the equations (65)–(67), yielding that one of these $A+2B$ equations is redundant.

Indeed, if we denote by Υ_i^1, Υ_i^2, Υ_i^3 the left hand sides of the i–th equation in (65), (66), (67), respectively, we have

$$
\sum_{i=1}^{A} 3\,n^{(i)}\Upsilon_i^1 + \sum_{i=A+1}^{A+B} 3\,n^{(i)}\Upsilon_i^2 + \sum_{i=A+1}^{A+B} n^{(i)}\Upsilon_i^3 =
$$

$$= \sum_{i=1}^{A}\sum_{j=1}^{A} 3\, n^{(i)} n^{(j)}\, B_{ij}\, \frac{2\,\mu_{ij}}{m_i + m_j}\, (\Delta^{(j)} - \Delta^{(i)})$$

$$+ \sum_{i=1}^{A}\sum_{j=A+1}^{A+B} 3\, n^{(i)} n^{(j)}\, B_{ij}\, \left\{ \frac{16}{15} \frac{m_j}{m_i+m_j} \frac{m_i}{m_i+m_j} \Delta^{(j)} + \frac{2}{15} \frac{m_j}{m_i+m_j} \widetilde{\Delta}^{(j)} - \frac{4}{3} \frac{m_j (m_i + \frac{1}{5} m_j)}{(m_i+m_j)^2} \Delta^{(i)} \right\}$$

$$+ \sum_{i=A+1}^{A+B}\sum_{j=1}^{A} 3\, n^{(i)} n^{(j)}\, B_{ij}\, \left\{ \frac{16}{15} \frac{\mu_{ij}}{m_i+m_j} \Delta^{(j)} - \frac{4}{15} \frac{m_j}{m_i+m_j} \frac{5m_i + m_j}{m_i+m_j} \Delta^{(i)} + \frac{2}{15} \frac{m_j}{m_i+m_j} \widetilde{\Delta}^{(i)} \right\}$$

$$+ \sum_{i=A+1}^{A+B}\sum_{j=A+1}^{A+B} 3\, n^{(i)} n^{(j)}\, B_{ij}\, \left\{ \frac{8}{21} \frac{m_i m_j}{(m_i+m_j)^2} \Delta^{(j)} + \frac{4}{105} \frac{m_j}{m_i+m_j} \widetilde{\Delta}^{(j)} \right.$$

$$\left. - \frac{4}{15} \frac{m_j (2m_i + \frac{4}{7} m_j)}{(m_i+m_j)^2} \Delta^{(i)} + \frac{4}{105} \frac{m_j}{m_i+m_j} \widetilde{\Delta}^{(i)} \right\}$$

$$+ \sum_{i=A+1}^{A+B}\sum_{j=1}^{A} n^{(i)} n^{(j)}\, B_{ij}\, \left\{ \frac{4}{5} \frac{m_i}{m_i+m_j} \Delta^{(j)} + \frac{4}{5} \frac{m_j}{m_i+m_j} \Delta^{(i)} - \frac{2}{5} \widetilde{\Delta}^{(i)} \right\}$$

$$+ \sum_{i=A+1}^{A+B}\sum_{j=A+1}^{A+B} n^{(i)} n^{(j)}\, B_{ij}\, \left\{ \frac{8}{35} \frac{m_i}{m_i+m_j} \Delta^{(j)} + \frac{8}{105} \widetilde{\Delta}^{(j)} + \frac{8}{35} \frac{m_j}{m_i+m_j} \Delta^{(i)} - \frac{4}{21} \widetilde{\Delta}^{(i)} \right\}.$$

Let us study these sums separately. As concerns the term relevant to $i = 1, \dots, A$, $j = 1, \dots, A$ we have

$$\sum_{i=1}^{A}\sum_{j=1}^{A} 3\, n^{(i)} n^{(j)}\, B_{ij}\, \frac{2\,\mu_{ij}}{m_i + m_j}\, (\Delta^{(j)} - \Delta^{(i)}) = 0$$

simply because it changes sign if we exchange the two indices $i \leftrightarrow j$. Analogous arguments allow to prove that also the sums relevant to $i = A+1, \dots, A+B$, $j = A+1, \dots, A+B$ vanish:

$$\sum_{i=A+1}^{A+B}\sum_{j=A+1}^{A+B} 3\, n^{(i)} n^{(j)}\, B_{ij}\, \left\{ \frac{8}{21} \frac{m_i m_j}{(m_i+m_j)^2} \Delta^{(j)} + \frac{4}{105} \frac{m_j}{m_i+m_j} \widetilde{\Delta}^{(j)} \right.$$

$$\left. - \frac{4}{15} \frac{m_j (2m_i + \frac{4}{7} m_j)}{(m_i+m_j)^2} \Delta^{(i)} + \frac{4}{105} \frac{m_j}{m_i+m_j} \widetilde{\Delta}^{(i)} \right\}$$

$$+ \sum_{i=A+1}^{A+B}\sum_{j=A+1}^{A+B} n^{(i)} n^{(j)}\, B_{ij}\, \left\{ \frac{8}{35} \frac{m_i}{m_i+m_j} \Delta^{(j)} + \frac{8}{105} \widetilde{\Delta}^{(j)} + \frac{8}{35} \frac{m_j}{m_i+m_j} \Delta^{(i)} - \frac{4}{21} \widetilde{\Delta}^{(i)} \right\}$$

$$= \sum_{i=A+1}^{A+B}\sum_{j=A+1}^{A+B} 3\, n^{(i)} n^{(j)}\, B_{ij}\, \left\{ \frac{8}{21} \frac{m_i m_j}{(m_i+m_j)^2} \Delta^{(i)} + \frac{4}{105} \frac{m_i}{m_i+m_j} \widetilde{\Delta}^{(i)} \right.$$

$$\left. - \frac{4}{15} \frac{m_i (2m_j + \frac{4}{7} m_i)}{(m_i+m_j)^2} \Delta^{(j)} + \frac{4}{105} \frac{m_i}{m_i+m_j} \widetilde{\Delta}^{(j)} \right\}$$

$$+ \sum_{i=A+1}^{A+B}\sum_{j=A+1}^{A+B} n^{(i)} n^{(j)}\, B_{ij}\, \left\{ \frac{8}{35} \frac{m_j}{m_i+m_j} \Delta^{(i)} + \frac{8}{105} \widetilde{\Delta}^{(i)} + \frac{8}{35} \frac{m_i}{m_i+m_j} \Delta^{(j)} - \frac{4}{21} \widetilde{\Delta}^{(j)} \right\}$$

$$= \frac{1}{2} \sum_{i=A+1}^{A+B}\sum_{j=A+1}^{A+B} n^{(i)} n^{(j)}\, B_{ij}\, \left\{ \frac{\Delta^{(i)}}{(m_i+m_j)^2} \left[-\frac{8}{5} m_i m_j - \frac{16}{35} m_j^2 + \frac{8}{35} m_j (m_i + m_j) + \frac{8}{7} m_i m_j \right. \right.$$

$$+\frac{8}{35}m_j(m_i+m_j)\Big] + \frac{\Delta^{(j)}}{(m_i+m_j)^2}\left[\frac{8}{7}m_im_j + \frac{8}{35}m_i(m_i+m_j) - \frac{8}{5}m_im_j - \frac{16}{35}m_i^2\right.$$
$$\left.+\frac{8}{35}m_i(m_i+m_j)\right] + \frac{\widetilde{\Delta}^{(i)}+\widetilde{\Delta}^{(j)}}{m_i+m_j}\left[\frac{4}{35}m_j + \left(\frac{8}{105}-\frac{4}{21}\right)(m_i+m_j) + \frac{4}{35}m_i\right]\Bigg\} = 0.$$

Finally, as concerns the remaining sums involving monatomic and diatomic species, we exchange indices only in the sums relevant to $i = A+1,\dots,A+B$, $j = 1,\dots,A$:

$$\sum_{i=A+1}^{A+B}\sum_{j=1}^{A} 3\,n^{(i)}n^{(j)}\,B_{ij}\left\{\frac{16}{15}\frac{\mu_{ij}}{m_i+m_j}\,\Delta^{(j)} - \frac{4}{15}\frac{m_j}{m_i+m_j}\,\frac{5m_i+m_j}{m_i+m_j}\,\Delta^{(i)} + \frac{2}{15}\frac{m_j}{m_i+m_j}\,\widetilde{\Delta}^{(i)}\right\} =$$
$$=\sum_{i=1}^{A}\sum_{j=A+1}^{A+B} 3\,n^{(i)}n^{(j)}\,B_{ij}\left\{\frac{16}{15}\frac{\mu_{ij}}{m_i+m_j}\,\Delta^{(i)} - \frac{4}{15}\frac{m_i}{m_i+m_j}\,\frac{5m_j+m_i}{m_i+m_j}\,\Delta^{(j)} + \frac{2}{15}\frac{m_i}{m_i+m_j}\,\widetilde{\Delta}^{(j)}\right\},$$

and

$$\sum_{i=A+1}^{A+B}\sum_{j=1}^{A} n^{(i)}n^{(j)}\,B_{ij}\left\{\frac{4}{5}\frac{m_i}{m_i+m_j}\Delta^{(j)} + \frac{4}{5}\frac{m_j}{m_i+m_j}\Delta^{(i)} - \frac{2}{5}\widetilde{\Delta}^{(i)}\right\} =$$
$$=\sum_{i=1}^{A}\sum_{j=A+1}^{A+B} n^{(i)}n^{(j)}\,B_{ij}\left\{\frac{4}{5}\frac{m_j}{m_i+m_j}\Delta^{(i)} + \frac{4}{5}\frac{m_i}{m_i+m_j}\Delta^{(j)} - \frac{2}{5}\widetilde{\Delta}^{(j)}\right\}.$$

Taking into account these results we get

$$\sum_{i=1}^{A}\sum_{j=A+1}^{A+B} 3\,n^{(i)}n^{(j)}\,B_{ij}\left\{\frac{16}{15}\frac{m_j}{m_i+m_j}\,\frac{m_i}{m_i+m_j}\Delta^{(j)} + \frac{2}{15}\frac{m_j}{m_i+m_j}\,\widetilde{\Delta}^{(j)} - \frac{4}{3}\frac{m_j(m_i+\frac{1}{5}m_j)}{(m_i+m_j)^2}\Delta^{(i)}\right\}$$
$$+\sum_{i=A+1}^{A+B}\sum_{j=1}^{A} 3\,n^{(i)}n^{(j)}\,B_{ij}\left\{\frac{16}{15}\frac{\mu_{ij}}{m_i+m_j}\,\Delta^{(j)} - \frac{4}{15}\frac{m_j}{m_i+m_j}\,\frac{5m_i+m_j}{m_i+m_j}\,\Delta^{(i)} + \frac{2}{15}\frac{m_j}{m_i+m_j}\,\widetilde{\Delta}^{(i)}\right\}$$
$$+\sum_{i=A+1}^{A+B}\sum_{j=1}^{A} n^{(i)}n^{(j)}\,B_{ij}\left\{\frac{4}{5}\frac{m_i}{m_i+m_j}\Delta^{(j)} + \frac{4}{5}\frac{m_j}{m_i+m_j}\Delta^{(i)} - \frac{2}{5}\widetilde{\Delta}^{(i)}\right\} =$$
$$=\sum_{i=1}^{A}\sum_{j=A+1}^{A+B} n^{(i)}n^{(j)}\,B_{ij}\left\{\frac{\Delta^{(j)}}{(m_i+m_j)^2}\left[\frac{16}{5}m_im_j - 4m_im_j - \frac{4}{5}m_i^2 + \frac{4}{5}m_i(m_i+m_j)\right]\right.$$
$$+\frac{\Delta^{(i)}}{(m_i+m_j)^2}\left[-4m_im_j - \frac{4}{5}m_j^2 + \frac{16}{5}m_im_j + \frac{4}{5}m_j(m_i+m_j)\right]$$
$$\left.+\frac{\widetilde{\Delta}^{(j)}}{m_i+m_j}\left[\frac{2}{5}m_j + \frac{2}{5}m_i - \frac{2}{5}(m_i+m_j)\right]\right\} = 0$$

and this concludes our proof.

The solution of the problem (63) is therefore

$$h^{(i),D}_{i=1,\dots,A+B}(V) = \Delta^{(i)}(m_i|V|^2-3), \qquad i=1,\dots,A,$$
$$h^{(i),D}_{i=1,\dots,A+B}(V,J) = \Delta^{(i)}(m_i|V|^2-3) + \widetilde{\Delta}^{(i)}(J-1), \qquad i=A+1,\dots,A+B,$$

that, as expected from the Galilean invariance, depends on the vector V only through its modulus $|V|$; coefficients $\Delta^{(i)}$, $\widetilde{\Delta}^{(i)}$ have to satisfy $A+2B-1$ among the equations (65)–(67).

4.4. **Computation of h^Q.** We recall that we wish to solve the problem

$$\mathcal{K}(h^{(i),Q,p}_{i=1,\dots,A+B}) = k^{Q,p}_{i;\,i=1,\dots,A+B}, \tag{68}$$

where

$$k^{Q,p}_{i;\,i=1,\dots,A+B} = \frac{1}{2}(m_i|V|^2-5)V_p + r_i(J-1)V_p\,.$$

We test for that the effect of K_{ij} on combinations of $|v|^2v_1$ and v_1.

- For $i=1,\dots,A$ and $j=1,\dots,A$ we get

$$B_{ij}^{-1}K_{ij}\left(v\mapsto Q^{(i)}m_i|v|^2v_1,\quad v\mapsto Q^{(j)}m_j|v|^2v_1\right)(v)=$$

$$= \int_{\mathbb{R}^3}\int_{S^2}\left(\frac{m_j}{2\pi}\right)^{3/2} e^{-\frac{1}{2}m_j|v_*|^2}\left[Q^{(j)}m_j\left|\frac{m_iv+m_jv_*}{m_i+m_j}-\frac{m_i}{m_i+m_j}|v-v_*|\sigma\right|^2\right.$$
$$\times\left(\frac{m_iv_1+m_jv_{1*}}{m_i+m_j}-\frac{m_i}{m_i+m_j}|v-v_*|\sigma_1\right)+Q^{(i)}m_i\left|\frac{m_iv+m_jv_*}{m_i+m_j}+\frac{m_j}{m_i+m_j}|v-v_*|\sigma\right|^2$$
$$\left.\times\left(\frac{m_iv_1+m_jv_{1*}}{m_i+m_j}+\frac{m_j}{m_i+m_j}|v-v_*|\sigma_1\right)-Q^{(j)}m_j|v_*|^2v_{1*}-Q^{(i)}m_i|v|^2v_1\right]d\sigma\,dv_*$$
$$= Q^{(j)}\frac{m_i}{(m_i+m_j)^3}\left[\frac{8}{3}\,m_i^2m_j|v|^2v_1+5\left(m_i^2+m_j^2-\frac{2}{3}\,m_im_j\right)v_1\right]$$
$$+Q^{(i)}\frac{m_i}{(m_i+m_j)^3}\left[-m_j\left(3\,m_i^2+m_j^2+\frac{4}{3}\,m_im_j\right)|v|^2v_1+10\,m_j\left(m_i-\frac{1}{3}\,m_j\right)v_1\right].$$

From the computations relevant to h^W (see Subsection 4.1) we easily get:

$$B_{ij}^{-1}K_{ij}\left(v\mapsto Q^{(i)}v_1,\quad v\mapsto Q^{(j)}v_1\right)(v)=\frac{1}{m_i+m_j}\left(m_iQ^{(j)}-m_jQ^{(i)}\right)v_1.$$

Thus, combining the two previous results we finally get

$$B_{ij}^{-1}K_{ij}\left(v\mapsto Q^{(i)}\big(m_i|v|^2-5\big)v_1,\quad v\mapsto Q^{(j)}\big(m_j|v|^2-5\big)v_1\right)(v)=$$
$$=\left\{Q^{(j)}\,\frac{8}{3}\,m_i^2-Q^{(i)}\left(3\,m_i^2+\frac{4}{3}m_im_j+m_j^2\right)\right\}\frac{m_j}{(m_i+m_j)^3}(m_i|v|^2-5)v_1.$$

- For $i=1,\dots,A$ and $j=A+1,\dots,A+B$ we get

$$B_{ij}^{-1}K_{ij}\left(v\mapsto Q^{(i)}m_i|v|^2v_1,\quad (v,I)\mapsto Q^{(j)}m_j|v|^2v_1+\widetilde{Q}^{(j)}Iv_1\right)(v)=$$

$$=\int_{\mathbb{R}^3}\int_{S^2}\int_0^\infty\int_0^1\left(\frac{m_j}{2\pi}\right)^{3/2}e^{-\frac{1}{2}m_j|v_*|^2-I_*}\left[Q^{(j)}m_j\left|\frac{m_iv+m_jv_*}{m_i+m_j}-\frac{m_i}{m_i+m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma\right|^2\right.$$
$$\times\left(\frac{m_iv_1+m_jv_{1*}}{m_i+m_j}-\frac{m_i}{m_i+m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma_1\right)+\widetilde{Q}^{(j)}(1-R)E\left(\frac{m_iv_1+m_jv_{1*}}{m_i+m_j}-\frac{m_i}{m_i+m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma_1\right)$$
$$+Q^{(i)}m_i\left|\frac{m_iv+m_jv_*}{m_i+m_j}+\frac{m_j}{m_i+m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma\right|^2\left(\frac{m_iv_1+m_jv_{1*}}{m_i+m_j}+\frac{m_j}{m_i+m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma_1\right)$$
$$\left.-Q^{(j)}m_j|v_*|^2v_{1*}-\widetilde{Q}^{(j)}I_*v_{1*}-Q^{(i)}m_i|v|^2v_1\right]\sqrt{R}\,dR\,dI_*\,d\sigma\,dv_*$$

$$= Q^{(j)}\,\frac{2}{3}\,\frac{m_i}{(m_i+m_j)^3}\Big[2\,m_i^2 m_j|v|^2 v_1 + 5\big(m_i^2+m_j^2\big)v_1\Big] + \widetilde{Q}^{(j)}\,\frac{2}{3}\,\frac{m_i^2}{(m_i+m_j)^2}\left[\frac{1}{5}\,m_j|v|^2 v_1 + v_1\right]$$

$$+ Q^{(i)}\,\frac{2}{3}\,\frac{m_i m_j}{(m_i+m_j)^3}\Big[-\big(3\,m_i^2 + 2\,m_i m_j + m_j^2\big)|v|^2 v_1 + 10\,m_i\,v_1\Big].$$

From Subsection 4.1, we easily get

$$B_{ij}^{-1}K_{ij}\left(v\mapsto Q^{(i)}v_1,\quad v\mapsto Q^{(j)}v_1\right)(v) = \frac{2}{3}\,\frac{1}{m_i+m_j}\Big(m_i Q^{(j)} - m_j Q^{(i)}\Big)v_1$$

(and this allows to compute also $K_{ij}\left(v\mapsto 0,\quad v\mapsto -\widetilde{Q}^{(j)}v_1\right)(v)$). Thus, combining the results, we obtain

$$B_{ij}^{-1}K_{ij}\left(v\mapsto Q^{(i)}\big(m_i|v|^2-5\big)v_1,\quad v\mapsto Q^{(j)}\big(m_j|v|^2-5\big)v_1 + \widetilde{Q}^{(j)}(I-1)v_1\right)(v) =$$

$$= \left\{2\,Q^{(j)}\,m_i - Q^{(i)}\left(3\,m_i + 2\,m_j + \frac{m_j^2}{m_i}\right) + \frac{1}{5}\,\widetilde{Q}^{(j)}(m_i+m_j)\right\}\frac{2}{3}\,\frac{m_i m_j}{(m_i+m_j)^3}(m_i|v|^2-5)v_1.$$

- For $i = A+1,\dots,A+B$ and $j = 1,\dots,A$, we get

$$B_{ij}^{-1}K_{ij}\left((v,I)\mapsto Q^{(i)}m_i|v|^2 v_1 + \widetilde{Q}^{(i)}Iv_1,\quad v\mapsto Q^{(j)}m_j|v|^2 v_1\right)(v,I) =$$

$$= \int_{\mathbb{R}^3}\int_{S^2}\int_0^1\left(\frac{m_j}{2\pi}\right)^{3/2} e^{-\frac{1}{2}m_j|v_*|^2}\left[Q^{(j)}m_j\left|\frac{m_i v + m_j v_*}{m_i+m_j} - \frac{m_i}{m_i+m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma\right|^2\right.$$

$$\times\left(\frac{m_i v_1 + m_j v_{1*}}{m_i+m_j} - \frac{m_i}{m_i+m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma_1\right) + Q^{(i)}m_i\left|\frac{m_i v + m_j v_*}{m_i+m_j} + \frac{m_j}{m_i+m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma\right|^2$$

$$\times\left(\frac{m_i v_1 + m_j v_{1*}}{m_i+m_j} + \frac{m_j}{m_i+m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma_1\right) + \widetilde{Q}^{(i)}(1-R)E\left(\frac{m_i v_1 + m_j v_{1*}}{m_i+m_j}\right.$$

$$\left.\left. + \frac{m_j}{m_i+m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma_1\right) - Q^{(j)}m_j|v_*|^2 v_{1*} - Q^{(i)}m_i|v|^2 v_1 - \widetilde{Q}^{(i)}Iv_1\right]\sqrt{R}\,dR\,d\sigma\,dv_*$$

$$= Q^{(j)}\,\frac{m_i}{(m_i+m_j)^3}\left[\frac{4}{3}\,m_i^2 m_j|v|^2 v_1 + \frac{4}{3}\,m_i(m_i+m_j)Iv_1 + 2\left(m_i^2 - \frac{2}{3}\,m_i m_j + \frac{5}{3}\,m_j^2\right)v_1\right]$$

$$+ Q^{(i)}\,\frac{m_i m_j}{(m_i+m_j)^3}\left[-\left(2\,m_i^2 + \frac{4}{3}\,m_i m_j + \frac{2}{3}\,m_j^2\right)|v|^2 v_1 + \frac{4}{3}(m_i+m_j)Iv_1 + \frac{4}{3}(4\,m_i - m_j)v_1\right]$$

$$+ \widetilde{Q}^{(i)}\,\frac{2}{15}\,\frac{1}{(m_i+m_j)^2}\Big[m_i^2 m_j|v|^2 v_1 - \big(3\,m_i^2 + 8\,m_i m_j + 5\,m_j^2\big)Iv_1 + \big(3\,m_i - 2\,m_j\big)m_i v_1\Big].$$

From Subsection 4.1, we have

$$B_{ij}^{-1}K_{ij}\left(v\mapsto Q^{(i)}v_1,\quad v\mapsto Q^{(j)}v_1\right)(v) = \frac{2}{3}\,\frac{1}{m_i+m_j}\Big(m_i Q^{(j)} - m_j Q^{(i)}\Big)v_1,$$

therefore

$$B_{ij}^{-1}K_{ij}\left(v\mapsto Q^{(i)}\big(m_i|v|^2-5\big)v_1 + \widetilde{Q}^{(i)}(I-1)v_1,\quad v\mapsto Q^{(j)}\big(m_j|v|^2-5\big)v_1\right)(v,I) =$$

$$= \left\{ \frac{2}{3} \, Q^{(j)} \, m_i^2 - Q^{(i)} \left(m_i^2 + \frac{2}{3} \, m_i m_j + \frac{1}{3} \, m_j^2 \right) + \frac{1}{15} \, \widetilde{Q}^{(i)} \, m_i (m_i + m_j) \right\}$$

$$\times 2 \, \frac{m_j}{(m_i + m_j)^3} (m_i |v|^2 - 5) v_1$$

$$+ \left\{ 2 \, Q^{(j)} \, m_i^2 + 2 \, Q^{(i)} \, m_i \, m_j - \frac{1}{5} \, \widetilde{Q}^{(i)} \big(3 \, m_i^2 + 8 \, m_i m_j + 5 m_j^2 \big) \right\} \frac{2}{3} \, \frac{1}{(m_i + m_j)^2} (I - 1) v_1.$$

- For $i = A+1, \ldots, A+B$ and $j = A+1, \ldots, A+B$, we get

$$B_{ij}^{-1} K_{ij} \Big((v, I) \mapsto Q^{(i)} m_i |v|^2 v_1 + \widetilde{Q}^{(i)} I v_1, \quad (v, I) \mapsto Q^{(j)} m_j |v|^2 v_1 + \widetilde{Q}^{(j)} I v_1 \Big) (v, I) =$$

$$= \int_{\mathbb{R}^3} \int_{S^2} \int_0^\infty \int_0^1 \int_0^1 \left(\frac{m_j}{2\pi} \right)^{3/2} e^{-\frac{1}{2} m_j |v_*|^2 - I_*} \left[\left(Q^{(j)} m_j \left| \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{m_i + m_j} \sqrt{\frac{2RE}{\mu_{ij}}} \, \sigma \right|^2 \right. \right.$$

$$\left. + \widetilde{Q}^{(j)} (1 - r)(1 - R) E \right) \left(\frac{m_i v_1 + m_j v_{1*}}{m_i + m_j} - \frac{m_i}{m_i + m_j} \sqrt{\frac{2RE}{\mu_{ij}}} \, \sigma_1 \right)$$

$$+ \left(Q^{(i)} m_i \left| \frac{m_i v + m_j v_*}{m_i + m_j} + \frac{m_j}{m_i + m_j} \sqrt{\frac{2RE}{\mu_{ij}}} \, \sigma \right|^2 + \widetilde{Q}^{(i)} r (1 - R) E \right)$$

$$\times \left(\frac{m_i v_1 + m_j v_{1*}}{m_i + m_j} + \frac{m_j}{m_i + m_j} \sqrt{\frac{2RE}{\mu_{ij}}} \, \sigma_1 \right)$$

$$\left. - Q^{(j)} m_j |v_*|^2 v_{1*} - \widetilde{Q}^{(j)} I_* v_{1*} - Q^{(i)} m_i |v|^2 v_1 - \widetilde{Q}^{(i)} I v_1 \right] (1 - R) \sqrt{R} \, dR \, dr \, dI_* \, d\sigma \, dv_*$$

$$= Q^{(j)} \, \frac{m_i}{(m_i + m_j)^3} \left[\frac{16}{35} \, m_i^2 m_j |v|^2 v_1 + \frac{8}{21} \, m_i (m_i + m_j) I v_1 + \left(\frac{20}{21} \, m_i^2 + \frac{4}{3} \, m_j^2 \right) v_1 \right]$$

$$+ \widetilde{Q}^{(j)} \, \frac{8}{105} \, \frac{m_i}{(m_i + m_j)^2} \left[\frac{1}{2} \, m_i m_j |v|^2 v_1 + (m_i + m_j) I v_1 + \frac{5}{2} \, m_i v_1 \right]$$

$$+ Q^{(i)} \, \frac{m_i}{(m_i + m_j)^3} \left[-\frac{4}{5} \, m_j \left(m_i^2 + \frac{16}{21} \, m_i m_j + \frac{1}{3} \, m_j^2 \right) |v|^2 v_1 + \frac{8}{21} \, m_j (m_i + m_j) I v_1 + \frac{16}{7} \, m_i m_j v_1 \right]$$

$$+ \widetilde{Q}^{(i)} \, \frac{8}{105} \, \frac{1}{(m_i + m_j)^2} \left[\frac{1}{2} \, m_i^2 m_j |v|^2 v_1 - \frac{1}{2} (5 m_i + 7 m_j)(m_i + m_j) I v_1 + \frac{5}{2} \, m_i^2 v_1 \right].$$

From Subsection 4.1, we have

$$B_{ij}^{-1} K_{ij} \Big(v \mapsto Q^{(i)} v_1, \quad v \mapsto Q^{(j)} v_1 \Big) (v) = \frac{4}{15} \, \frac{1}{m_i + m_j} \Big(m_i Q^{(j)} - m_j Q^{(i)} \Big) v_1,$$

therefore

$$B_{ij}^{-1} K_{ij} \Big(v \mapsto Q^{(i)} \big(m_i |v|^2 - 5 \big) v_1 + \widetilde{Q}^{(i)} (I - 1) v_1, \quad v \mapsto Q^{(j)} \big(m_j |v|^2 - 5 \big) v_1 + \widetilde{Q}^{(j)} (I - 1) v_1 \Big) (v, I) =$$

$$= \left\{ \frac{4}{7} \, Q^{(j)} \, m_i^2 - Q^{(i)} \left(m_i^2 + \frac{16}{21} \, m_i m_j + \frac{1}{3} \, m_j^2 \right) + \frac{1}{21} \, (\widetilde{Q}^{(j)} + \widetilde{Q}^{(i)}) m_i (m_i + m_j) \right\}$$

$$\times \frac{4}{5} \, \frac{m_j}{(m_i + m_j)^3} (m_i |v|^2 - 5) v_1$$

$$+ \left\{ Q^{(j)} \, m_i^2 + Q^{(i)} \, m_i \, m_j + \frac{1}{5} \, (m_i + m_j) \left(\widetilde{Q}^{(j)} \, m_i - \frac{1}{2} \, \widetilde{Q}^{(i)} (5 m_i + 7 m_j) \right) \right\} \frac{8}{21} \, \frac{1}{(m_i + m_j)^2} (I - 1) v_1.$$

In conclusion, the solution of problem (68) is thus

$$h^{(i),Q,p}_{i=1,\dots,A+B}(V) = Q^{(i)}(m_i|V|^2-5)V, \qquad i=1,\dots,A,$$
$$h^{(i),Q,p}_{i=1,\dots,A+B}(V,J) = Q^{(i)}(m_i|V|^2-5)V + \widetilde{Q}^{(i)}(J-1)V, \qquad i=A+1,\dots,A+B,$$

and we note that, as already anticipated owing to the Galilean invariance, $h^{(i),Q}(V,J) = \tilde{h}^{(i),Q}(|V|,J)\,V$ where

$$\tilde{h}^{(i),Q,p}_{i=1,\dots,A+B}(|V|) = Q^{(i)}(m_i|V|^2-5), \qquad i=1,\dots,A,$$
$$\tilde{h}^{(i),Q,p}_{i=1,\dots,A+B}(|V|,J) = Q^{(i)}(m_i|V|^2-5) + \widetilde{Q}^{(i)}(J-1), \qquad i=A+1,\dots,A+B.$$

Coefficients $Q^{(i)}$ and $\widetilde{Q}^{(i)}$ must satisfy the system

- for $i=1,\dots,A$,

$$\sum_{j=1}^{A} n^{(j)}\,B_{ij}\left\{Q^{(j)}\,\frac{8}{3}\,m_i^2 - Q^{(i)}\left(3\,m_i^2+\frac{4}{3}m_i m_j + m_j^2\right)\right\}\frac{m_j}{(m_i+m_j)^3}$$
$$+\sum_{j=A+1}^{A+B} n^{(j)}\,B_{ij}\left\{2\,Q^{(j)}\,m_i - Q^{(i)}\left(3\,m_i+2\,m_j+\frac{m_j^2}{m_i}\right)+\frac{1}{5}\,\widetilde{Q}^{(j)}(m_i+m_j)\right\}$$
$$\times\frac{2}{3}\,\frac{m_i m_j}{(m_i+m_j)^3} = \frac{1}{2},$$

- for $i=A+1,\dots,A+B$,

$$\sum_{j=1}^{A} n^{(j)}\,B_{ij}\left\{\frac{2}{3}\,Q^{(j)}\,m_i^2 - Q^{(i)}\left(m_i^2+\frac{2}{3}\,m_i m_j+\frac{1}{3}\,m_j^2\right)+\frac{1}{15}\,\widetilde{Q}^{(i)}\,m_i(m_i+m_j)\right\}2\,\frac{m_j}{(m_i+m_j)^3}$$
$$+\sum_{j=A+1}^{A+B} n^{(j)}\,B_{ij}\left\{\frac{4}{7}\,Q^{(j)}\,m_i^2 - Q^{(i)}\left(m_i^2+\frac{16}{21}\,m_i m_j+\frac{1}{3}\,m_j^2\right)+\frac{1}{21}\,(\widetilde{Q}^{(j)}+\widetilde{Q}^{(i)})m_i(m_i+m_j)\right\}$$
$$\times\frac{4}{5}\,\frac{m_j}{(m_i+m_j)^3} = \frac{1}{2},$$

$$\sum_{j=1}^{A} n^{(j)}\,B_{ij}\left\{2\,Q^{(j)}\,m_i^2 + 2\,Q^{(i)}\,m_i\,m_j - \frac{1}{5}\,\widetilde{Q}^{(i)}\big(3\,m_i^2+8\,m_i m_j+5m_j^2\big)\right\}\frac{2}{3}\,\frac{1}{(m_i+m_j)^2}$$
$$+\sum_{j=A+1}^{A+B} n^{(j)}\,B_{ij}\left\{Q^{(j)}\,m_i^2 + Q^{(i)}\,m_i\,m_j + \frac{1}{5}\,(m_i+m_j)\left(\widetilde{Q}^{(j)}\,m_i - \frac{1}{2}\,\widetilde{Q}^{(i)}(5m_i+7m_j)\right)\right\}$$
$$\times\frac{8}{21}\,\frac{1}{(m_i+m_j)^2} = 1.$$

4.5. **Obtention of the viscosity coefficients in the case of constant cross sections.** We recall here the Navier-Stokes system obtained at the end of subsection 3.3.4: now write down eq. (29) - (32) in the special case which is considered here:

$$\partial_t(m_i\, n^{(i)}) + \nabla_x \cdot (m_i\, n^{(i)} u) = -\varepsilon \nabla_x \cdot D^{(i)}, \tag{69}$$

$$\partial_t\Big(\sum_{i=1}^{A+B} m_i\, n^{(i)}\, u_k\Big) + \sum_l \partial_{x_l}\Big(\sum_{i=1}^{A+B}[m_i\, n^{(i)}\, u_k\, u_l + n^{(i)}\, T\, \delta_{kl}]\Big) = -\varepsilon \sum_l \partial_{x_l} F_{kl}, \tag{70}$$

$$\partial_t\Big(\sum_{i=1}^{A}[m_i\, n^{(i)}\,\frac{|u|^2}{2} + \frac{3}{2}\, n^{(i)}\, T] + \sum_{i=A+1}^{A+B}[m_i\, n^{(i)}\,\frac{|u|^2}{2} + \frac{5}{2}\, n^{(i)}\, T]\Big) \tag{71}$$

$$+\sum_l \partial_{x_l}\Big(\sum_{i=1}^{A}[m_i\, n^{(i)}\,\frac{|u|^2}{2}\, u_l + \frac{5}{2}\, n^{(i)}\, T\, u_l] + \sum_{i=A+1}^{A+B}[m_i\, n^{(i)}\,\frac{|u|^2}{2}\, u_l + \frac{7}{2}\, n^{(i)}\, T\, u_l]\Big)$$
$$= -\varepsilon \nabla_x \cdot G.$$

The viscosity terms $D_k^{(i)}$, F_{kl} and G_k, also computed in Subsection 3.3.4, are given by the following formulas:

$$i = 1, \dots, A, \qquad D_k^{(i)} = n^{(i)}\, \partial_{x_k} T\, Z_1^{(i)} + n^{(i)}\, T\, Z_{2k}^{(i)}, \tag{72}$$

$$i = A+1, \dots, A+B, \qquad D_k^{(i)} = n^{(i)}\, \partial_{x_k} T\, Z_3^{(i)} + n^{(i)}\, T\; Z_{4k}^{(i)}, \tag{73}$$

$$F_{kl} = \Big(\sum_{i=1}^{A} T\,\frac{n^{(i)}}{m_i} Z_5^{(i)} + \sum_{i=A+1}^{A+B} T\,\frac{n^{(i)}}{m_i} Z_6^{(i)}\Big)\Big[\frac{\nabla_x u + \nabla_x u^T}{2} - \frac{1}{3}\,\nabla_x \cdot u\, Id\Big]_{kl} \tag{74}$$
$$+T\,\nabla_x \cdot u\,\delta_{kl}\Big(\sum_{i=1}^{A} n^{(i)} Z_7^{(i)} + \sum_{i=A+1}^{A+B} n^{(i)} Z_8^{(i)}\Big),$$

$$G_k = \sum_l F_{kl}\, u_l + T\,\partial_{x_k} T\left\{\sum_{i=1}^{A}\frac{n^{(i)}}{m_i}\, Z_9^{(i)} + \sum_{i=A+1}^{A+B}\frac{n^{(i)}}{m_i}\, Z_{10}^{(i)}\right\} \tag{75}$$
$$+T^2\left\{\sum_{i=1}^{A} n^{(i)}\, Z_{11k}^{(i)} + \sum_{i=A+1}^{A+B} n^{(i)}\, Z_{12k}^{(i)}\right\}.$$

In the previous equations, we recall that

$$s^{(i)} = \frac{\nabla_x n^{(i)}}{n^{(i)}} - \frac{m_i \sum_{j=1}^{A+B} \nabla_x n^{(j)}}{\sum_{j=1}^{A+B} m_j\, n^{(j)}} + \Big(1 - \frac{m_i \sum_{j=1}^{A+B} n^{(j)}}{\sum_{j=1}^{A+B} m_j\, n^{(j)}}\Big)\frac{\nabla_x T}{T},$$

and the terms $Z_1, \dots, Z_{12k}$ are defined by

$$Z_1^{(i)} = \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2}}}{(2\pi)^{3/2}}\, \tilde{h}^{(i),Q}\left(\frac{|V|}{\sqrt{m_i}}\right)\, V_1^2\, dV\,, \tag{76}$$

$$Z_{2k}^{(i)} = \int_{\mathbb{R}^3} \frac{e^{-\frac{m_i\,|V|^2}{2}}}{(2\pi/m_i)^{3/2}}\, h^{(i),W}(V)\, m_i V_k\, dV\,, \tag{77}$$

$$Z_3^{(i)} = \int_{\mathbb{R}^3} \int_0^\infty \frac{e^{-\frac{|V|^2}{2} - J}}{(2\pi)^{3/2}} \, \tilde{h}^{(i),Q} \left(\frac{|V|}{\sqrt{m_i}}, J \right) V_1^2 \, dV \, dJ \,, \tag{78}$$

$$Z_{4k}^{(i)} = \int_{\mathbb{R}^3} \int_0^\infty \frac{e^{-\frac{m_i \, |V|^2}{2} - J}}{(2\pi/m_i)^{3/2}} \, h^{(i),W}(V,J) \, m_i \, V_k \, dV \, dJ \,, \tag{79}$$

$$Z_5^{(i)} = \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2}}}{(2\pi)^{3/2}} \, \tilde{h}^{(i),P} \left(\frac{|V|}{\sqrt{m_i}} \right) \frac{2}{3} \, V_1^4 \, dV \,, \tag{80}$$

$$Z_6^{(i)} = \int_{\mathbb{R}^3} \int_0^\infty \frac{e^{-\frac{|V|^2}{2} - J}}{(2\pi)^{3/2}} \, \tilde{h}^{(i),P} \left(\frac{|V|}{\sqrt{m_i}}, J \right) \frac{2}{3} \, V_1^4 \, dV \, dJ \,, \tag{81}$$

$$Z_7^{(i)} = \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2}}}{(2\pi)^{3/2}} \, \tilde{h}^{(i),D} \left(\frac{|V|}{\sqrt{m_i}} \right) V_1^2 \, dV \,, \tag{82}$$

$$Z_8^{(i)} = \int_{\mathbb{R}^3} \int_0^\infty \frac{e^{-\frac{|V|^2}{2} - J}}{(2\pi)^{3/2}} \, \tilde{h}^{(i),D} \left(\frac{|V|}{\sqrt{m_i}}, J \right) V_1^2 \, dV dJ \,, \tag{83}$$

$$Z_9^{(i)} = \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2}}}{(2\pi)^{3/2}} \, \tilde{h}^{(i),Q} \left(\frac{|V|}{\sqrt{m_i}} \right) \frac{|V|^2}{2} \, V_1^2 \, dV \,, \tag{84}$$

$$Z_{10}^{(i)} = \int_{\mathbb{R}^3} \int_0^\infty \frac{e^{-\frac{|V|^2}{2} - J}}{(2\pi)^{3/2}} \, \tilde{h}^{(i),Q} \left(\frac{|V|}{\sqrt{m_i}}, J \right) \left(\frac{|V|^2}{2} + J \right) V_1^2 \, dV dJ \,, \tag{85}$$

$$Z_{11k}^{(i)} = \int_{\mathbb{R}^3} \frac{e^{-m_i \frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}} \, h^{(i),W}(V) \, m_i \frac{|V|^2}{2} \, V_k \, dV \,, \tag{86}$$

$$Z_{12k}^{(i)} = \int_{\mathbb{R}^3} \int_0^\infty \frac{e^{-m_i \frac{|V|^2}{2} - J}}{(2\pi/m_i)^{3/2}} \, h^{(i),W}(V,J) \left(m_i \frac{|V|^2}{2} + J \right) V_k \, dV dJ \,. \tag{87}$$

For constant cross sections, the functions $h^{(i),W}$, $\tilde{h}^{(i),P}$, $\tilde{h}^{(i),D}$, $\tilde{h}^{(i),Q}$, have been computed in previous subsections and may be cast in compact form, for $i = 1, \dots, A+B$, as

$$\begin{aligned} h^{(i),W} &= m_i \, W^{(i)} \cdot V, \\ \tilde{h}^{(i),P} &= m_i \, \Pi^{(i)}, \\ \tilde{h}^{(i),D} &= \Delta^{(i)} \big(m_i |V|^2 - 3 \big) + r_i \, \widetilde{\Delta}^{(i)} (J-1), \\ \tilde{h}^{(i),Q} &= Q^{(i)} \big(m_i |V|^2 - 5 \big) + r_i \, \widetilde{Q}^{(i)} (J-1), \end{aligned} \tag{88}$$

with coefficients $W^{(i)}$, $\Pi^{(i)}$, $\Delta^{(i)}$, $\widetilde{\Delta}^{(i)}$, $Q^{(i)}$, $\widetilde{Q}^{(i)}$ fulfilling the suitable systems pointed out in subsections 4.1 to 4.4.

The terms $Z_1, \dots, Z_{12k}$ can be then computed owing, whenever necessary, to integrals reported in the Appendix:

$$Z_1^{(i)} = \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2}}}{(2\pi)^{3/2}} Q^{(i)}\big(|V|^2-5\big)\, V_1^2\, dV = 0, \tag{89}$$

$$Z_{2k}^{(i)} = \int_{\mathbb{R}^3} \frac{e^{-\frac{m_i\,|V|^2}{2}}}{(2\pi/m_i)^{3/2}}\, m_i\, W^{(i)}\cdot V\;\, m_i V_k\, dV \;= m_i\, W_k^{(i)}, \tag{90}$$

$$Z_3^{(i)} = \int_{\mathbb{R}^3}\int_0^\infty \frac{e^{-\frac{|V|^2}{2}-J}}{(2\pi)^{3/2}} \Big[Q^{(i)}\big(|V|^2-5\big)+\widetilde{Q}^{(i)}(J-1)\Big]\, V_1^2\, dV\, dJ = 0, \tag{91}$$

$$Z_{4k}^{(i)} = \int_{\mathbb{R}^3}\int_0^\infty \frac{e^{-\frac{m_i\,|V|^2}{2}-J}}{(2\pi/m_i)^{3/2}}\, m_i\, W^{(i)}\cdot V\, m_i\, V_k\, dV\, dJ \;= m_i\, W_k^{(i)}, \tag{92}$$

$$Z_5^{(i)} = \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2}}}{(2\pi)^{3/2}}\, m_i\, \Pi^{(i)}\, \frac{2}{3}\, V_1^4\, dV = 2\, m_i\, \Pi^{(i)}, \tag{93}$$

$$Z_6^{(i)} = \int_{\mathbb{R}^3}\int_0^\infty \frac{e^{-\frac{|V|^2}{2}-J}}{(2\pi)^{3/2}}\, m_i\, \Pi^{(i)}\, \frac{2}{3}\, V_1^4\, dV\, dJ = 2\, m_i\, \Pi^{(i)}, \tag{94}$$

$$Z_7^{(i)} = \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2}}}{(2\pi)^{3/2}}\, \Delta^{(i)}\big(|V|^2-3\big)\, V_1^2\, dV = 2\,\Delta^{(i)}, \tag{95}$$

$$Z_8^{(i)} = \int_{\mathbb{R}^3}\int_0^\infty \frac{e^{-\frac{|V|^2}{2}-J}}{(2\pi)^{3/2}} \Big[\Delta^{(i)}\big(|V|^2-3\big)+\widetilde{\Delta}^{(i)}(J-1)\Big]\, V_1^2\, dV dJ = 2\,\Delta^{(i)}, \tag{96}$$

$$Z_9^{(i)} = \int_{\mathbb{R}^3} \frac{e^{-\frac{|V|^2}{2}}}{(2\pi)^{3/2}}\, Q^{(i)}\big(m_i|V|^2-5\big)\, \frac{|V|^2}{2}\, V_1^2\, dV = 5\, Q^{(i)}, \tag{97}$$

$$Z_{10}^{(i)} = \int_{\mathbb{R}^3}\int_0^\infty \frac{e^{-\frac{|V|^2}{2}-J}}{(2\pi)^{3/2}} \Big[Q^{(i)}\big(|V|^2-5\big)+\widetilde{Q}^{(i)}(J-1)\Big] \left(\frac{|V|^2}{2}+J\right) V_1^2\, dV dJ = 5\, Q^{(i)}+\widetilde{Q}^{(i)}, \tag{98}$$

$$Z_{11k}^{(i)} = \int_{\mathbb{R}^3} \frac{e^{-m_i\frac{|V|^2}{2}}}{(2\pi/m_i)^{3/2}}\, m_i\, W^{(i)}\cdot V\, m_i\, \frac{|V|^2}{2}\, V_k\, dV = \frac{5}{2}\, W_k^{(i)}, \tag{99}$$

$$Z_{12k}^{(i)} = \int_{\mathbb{R}^3}\int_0^\infty \frac{e^{-m_i\frac{|V|^2}{2}-J}}{(2\pi/m_i)^{3/2}}\, m_i\, W^{(i)}\cdot V \left(m_i\, \frac{|V|^2}{2}+J\right) V_k\, dV dJ = \frac{7}{2}\, W_k^{(i)}. \tag{100}$$

Consequently, viscosity terms turn out to be

$$D_k^{(i)} = m_i\, W_k^{(i)}\, n^{(i)}\, T, \qquad i = 1,\dots,A+B, \tag{101}$$

where coefficients $W_k^{(i)}$ are combinations of the quantities $s^{(i)}$, (so that they contain gradients of number densities and of temperature), while

$$F_{kl} = 2\,T \sum_{i=1}^{A+B} n^{(i)}\, \Pi^{(i)} \left[\frac{\nabla_x u+\nabla_x u^T}{2} - \frac{1}{3}\,\nabla_x\cdot u\, Id\right]_{kl} + 2\,T\,\nabla_x\cdot u\,\delta_{kl} \sum_{i=1}^{A+B} n^{(i)}\,\Delta^{(i)}, \tag{102}$$

and finally
(103)

$$G_k = \sum_l F_{kl}\, u_l + T\,\partial_{x_k} T \sum_{i=1}^{A+B} \frac{n^{(i)}}{m_i} \left(5\, Q^{(i)} + r_i\, \widetilde{Q}^{(i)}\right) + T^2 \sum_{i=1}^{A+B} n^{(i)} \left(\frac{5}{2} + r_i\right) W_k^{(i)},$$

where $\Pi^{(i)}$, $\Delta^{(i)}$, $W^{(i)}$, $Q^{(i)}$, and $\widetilde{Q}^{(i)}$ are computed in subsections 4.1 to 4.4.

To conclude, the system of Navier-Stokes equations which are obtained by the Chapman-Enskog procedure from a system of Boltzmann equations corresponding to a mixture of monoatomic and polyatomic gases with constant cross sections, can be written down explicitly thanks to equations (69) – (71), formulas (101) – (103), and the linear finite-dimensional systems defined in subsections 4.1 to 4.4.

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Appendix: Some integrals

Integrals over the variable R:

$$\int_0^1 R^{1/2}\, dR = \frac{2}{3}, \qquad \int_0^1 (1-R)\, R^{1/2}\, dR = \frac{4}{15},$$

$$\int_0^1 (1-R)\, R^{3/2}\, dR = \frac{4}{35}, \qquad \int_0^1 (1-R)^2\, R^{1/2}\, dR = \frac{16}{105}.$$

Integrals over the variable r:

$$\int_0^1 r\, dr = \frac{1}{2}\,, \qquad \int_0^1 (1-r)\, dr = \frac{1}{2}\,.$$

Integrals over the energy variable:

$$\int_0^\infty e^{-\,I_*}\, dI_* = 1, \qquad \int_0^\infty I_*\, e^{-\,I_*}\, dI_* = 1,$$

$$\int_0^\infty I_*^2\, e^{-\,I_*}\, dI_* = 2.$$

Integrals over the angular variable:

$$\int_{S^2} d\sigma = 1, \qquad \int_{S^2} |\sigma|^2\, d\sigma = 1, \qquad \int_{S^2} (\sigma_1)^2 d\sigma = \frac{1}{3}.$$

Integrals over the velocity variable:

$$\int_{\mathbb{R}^3} \frac{e^{-\,\frac{1}{2}|v_*|^2}}{(2\pi)^{3/2}}\, dv_* = 1\,, \qquad \int_{\mathbb{R}^3} |v_*|^2\, \frac{e^{-\,\frac{1}{2}|v_*|^2}}{(2\pi)^{3/2}}\, dv_* = 3\,,$$

$$\int_{\mathbb{R}^3} |v_*|^4\, \frac{e^{-\,\frac{1}{2}|v_*|^2}}{(2\pi)^{3/2}}\, dv_* = 15\,, \qquad \int_{\mathbb{R}^3} |v_*|^6\, \frac{e^{-\,\frac{1}{2}|v_*|^2}}{(2\pi)^{3/2}}\, dv_* = 105\,,$$

or, in case of particle masses $m_i \neq 1$,

$$\left(\frac{m_i}{2\pi}\right)^{3/2} \int_{\mathbb{R}^3} e^{-\,\frac{1}{2} m_i |v_*|^2}\, dv_* = 1\,, \qquad \left(\frac{m_i}{2\pi}\right)^{3/2} \int_{\mathbb{R}^3} m_i |v_*|^2\, e^{-\,\frac{1}{2} m_i |v_*|^2}\, dv_* = 3\,.$$

Use will be made also of some of the following relations:

$$\int_{\mathbb{R}^3} (v_1)^2\, A(|v|)\, dv = \frac{1}{3} \int_{\mathbb{R}^3} |v|^2\, A(|v|)\, dv\,,$$

$$\int_{\mathbb{R}^3} (v_1)^4\, A(|v|)\, dv = \frac{1}{5} \int_{\mathbb{R}^3} |v|^4\, A(|v|)\, dv\,,$$

$$\int_{\mathbb{R}^3} (v_1)^2 (v_2)^2\, A(|v|)\, dv = \frac{1}{15} \int_{\mathbb{R}^3} |v|^4\, A(|v|)\, dv\,,$$

$$\int_{\mathbb{R}^3} (v_1)^6\, A(|v|)\, dv = \frac{1}{7} \int_{\mathbb{R}^3} |v|^6\, A(|v|)\, dv\,,$$

$$\int_{\mathbb{R}^3} (v_1)^4 (v_2)^2\, A(|v|)\, dv = \frac{1}{35} \int_{\mathbb{R}^3} |v|^6\, A(|v|)\, dv\,,$$

$$\int_{\mathbb{R}^3} (v_1)^2 (v_2)^2 (v_3)^2\, A(|v|)\, dv = \frac{1}{105} \int_{\mathbb{R}^3} |v|^6\, A(|v|)\, dv\,.$$

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(C.B.) CEA-CESTA, 15 avenue des sablières - CS 60001 33116 Le Barp Cedex, France
E-mail address: celine.baranger@cea.fr

(M.B.) University of Parma, Dept. of Mathematics, Physics and Computer Sciences, Parco Area delle Scienze 53/A, I-43124, Parma, Italy.
E-mail address: marzia.bisi@unipr.it

(S.B.) Univ. Bordeaux, CNRS, Bordeaux INP, IMB, UMR 5251, F-33400 Talence, France.
E-mail address: Stephane.Brull@math.u-bordeaux.fr

(L.D.) Univ. Paris Diderot, Sorbonne Paris Cité, Institut de Mathématiques de Jussieu - Paris Rive Gauche, UMR 7586, CNRS, Sorbonne Universités, UPMC Univ. Paris 06, F-75013, Paris, France.
E-mail address: desvillettes@math.univ-paris-diderot.fr