# Boltzmann's Kernel and the Spatially Homogeneous Boltzmann Equation

L. Desvillettes,

March 20, 2001

Centre de Mathématiques et leurs Applications
Unité de Recherche Associée 1611 (CNRS), Ecole Normale Supérieure de Cachan
61 avenue du Président Wilson, 94235 Cachan Cedex
e-mail: desville@cmla.ens-cachan.fr

**Abstract**

In this work, we recall many results by various authors about Boltzmann's kernel of monoatomic gases. Applications of those results in the context of the spatially homogeneous Boltzmann equation are then presented.

## 1 Introduction

Since the pioneering works of Carleman (Cf. [18]), many results concerning the Boltzmann equation of monoatomic rarefied gases (Cf. [22], [24], [74]) or its variants (for example the Fokker–Planck–Landau equation, Cf. [49]) have been proven. A large number of these results can in fact be viewed as applications of functional properties of Boltzmann's kernel, that is, of estimates in which the Boltzmann kernel $Q$ and a given function $f = f(v)$ are involved, but in which no reference is made to "the" solution $f(t, v)$ of the (spatially homogeneous) Boltzmann equation.

This point of view will systematically be adopted in the sequel. For each subject (e.g. uniqueness, large time behavior, etc..), we try to extract from the existing proofs the functional estimates which seem relevant to us, and to show how they are used to get a given result for the spatially homogeneous Boltzmann equation.

We intend in this way to try to focus on properties of Boltzmann's kernel which are succeptible to yield applications in many different situations (spatially inhomogeneous Boltzmann equation, Vlasov–Boltzmann equation, Boltzmann equation coupled with a fluid or another kinetic equation, etc..).

## 2 Various kinds of cross sections

The spatially homogeneous Boltzmann equation of rarefied gases writes

$$\partial_t f(t,v) = Q(f,f)(t,v), \tag{1}$$

(or to be more coherent with our point of view, $\partial_t f(t,v) = [Q(f(t,\cdot), f(t,\cdot))](v)$), where $Q$ is a quadratic operator acting only on the $v$ variable and describing the effect of the binary collisions on the density $f(t,v)$ of particles which at time $t \in I\!R_+$ have velocity $v \in I\!R^3$.

The bilinear form associated with $Q$ (and also denoted by $Q$, or $Q_B$ when the dependance with respect to $B$ is stressed) writes

$$Q(g,f)(v) = \int_{v_* \in I\!R^3} \int_{\sigma \in I\!R^2} \Big\{ f(v')\, g(v'_*) - f(v)\, g(v_*) \Big\}$$
$$\times B\left( |v - v_*|, \frac{v - v_*}{|v - v_*|} \cdot \sigma \right) d\sigma dv_*, \tag{2}$$

where $v', v'_*$ are the pre-collisional velocities defined by

$$\begin{cases} v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \\ v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \end{cases}$$

and $B$ is a nonnegative cross section whose form depends on the interaction between particles.

For interaction forces in $r^{-s}$ (where $r$ is the distance between particles and $s > 2$), $B$ takes the form

$$B(|u|, \cos\theta) = |u|^{\alpha} \beta_s(\theta). \tag{3}$$

In the sequel, we shall only consider that kind of cross sections, that is, cross sections which are a tensorial product between a kinetic cross section which is some power of $|u|$ and an angular cross section (depending only on $\theta$).

In (3), $\alpha$ is given by the formula $\alpha = \frac{s-5}{s-1}$ and $\beta_s$ is a continuous function on $]0, \pi]$ such that

$$\beta_s(\theta) \sim_{\theta \to 0} Cte(s)\, |\theta|^{-\frac{s+1}{s-1}},$$

that is, very many grazing collisions (those collisions for which $v'$ is close to $v$ and $v'_*$ is close to $v_*$, or in an equivalent way, $\theta$ is close to 0) occur. Here and in the sequel, $Cte$ will denote any constant, sometimes depending on parameters (like $s$ here).

Because of the strong singularity of $\beta_s$ at 0, it is not possible to give a sense to $Q(f,g)(v)$ for a given $v$ when $f,g \in C_c(I\!\!R^3)$. It is however possible to define $Q(f,g)$ in the following weak sense when $s > 7/3$ for all $\phi \in \mathcal{S}(I\!\!R^3)$, and $f,g \in L^1_2(I\!\!R^3)$,

$$\int_{v\in I\!\!R^3} Q(f,g)(v)\,\phi(v)\,dv = \int_{v\in I\!\!R^3}\int_{v_*\in I\!\!R^3}\int_{\sigma\in S^2} f(v)\,g(v_*)$$
$$\times\{\phi(v') - \phi(v)\}\,|v-v_*|^\alpha\,\beta_s(\theta)\,d\sigma dv_*. \tag{4}$$

For this reason, the so–called angular cutoff of Grad (Cf. [42]) is often introduced. It consists in replacing $\beta_s(\theta)$ by $\breve{\beta}_s(\theta) = \beta_s(\theta)\wedge n$ for some large $n \geq 0$ (or equivalently, to replace $\beta_s(\theta)$ by $\beta_s(\theta)$ if $|\theta| \geq \theta_0$ and 0 if $|\theta| \leq \theta_0$, whence the name of "angular" cutoff). In this situation, $Q(f,g)(v)$ is well–defined for a given $v$ as soon as (for example) $f,g \in C_c(I\!\!R^3)$, and we decompose $Q$ in its positive and negative parts:

$$Q(f,g)(v) = Q^+(f,g)(v) - f(v)\,Lg(v),$$

where

$$Q^+(f,g)(v) = \int_{v_*\in I\!\!R^3}\int_{\sigma\in S^2} f(v')\,g(v'_*)\,|v-v_*|^\alpha\,\breve{\beta}_s(\theta)\,d\sigma dv_*$$

and

$$Lg(v) = \int_{v_*\in I\!\!R^3} g(v_*)\,|v-v_*|^\alpha\,\breve{\beta}_s(\theta)\,d\sigma dv_*.$$

Note that no such decomposition is available in the non cutoff case.

Note that an interesting variant of the Boltzmann equation is obtained when one considers

$$L(f,f)(v) = \lim_{\varepsilon\to 0} Q_{B_\varepsilon}(f,f)(v), \tag{5}$$

where

$$B_\varepsilon(|u|,\cos\theta) = \frac{|u|^\alpha}{\varepsilon^3}\,\beta_s(\frac{\theta}{\varepsilon}),$$

that is, the angular cross section is concentrating on grazing collisions (for an early mention of the relationship between grazing collosions and Landau's operator, Cf. [22]). Note that slightly different asymptotics, closer to the real physics, also lead to $L$, Cf. [80] and the references therein.

It is then possible to prove that (at least formally, Cf. [10], [25], [26], [32])

$$L(f,f)(v) = Cte \times div_v \int_{v_* \in \mathbb{R}^3} |v-v_*|^\alpha \left\{ |v-v_*|^2 \, Id - (v-v_*) \otimes (v-v_*) \right\}$$

$$\left\{ f(v_*) \nabla_v f(v) - f(v) \nabla_{v_*} f(v_*) \right\} dv_*.$$

This formula defines the Landau (or Fokker–Planck–Landau) kernel. The relationship between the cutoff Boltzmann kernel, the non cutoff Boltzmann kernel and the Landau kernel is the following: in the first kernel, most of the collisions are non grazing, in the second, most of the collisions are grazing, and in the last, all collisions are grazing.

Traditionally the kinetic parts of the cross sections are classified with respect to $s$. When $s > 5$ ($\alpha \in ]0,1[$), we speak of hard potentials; for $s = 5$ ($\alpha = 0$), of Maxwellian molecules; in the case when $s \in ]7/3, 5[$ ($\alpha \in ]-2,0[$), of soft potentials; and finally, for $s \in ]2, 7/3[$ ($\alpha \in ]-3, 2[$), of very soft potentials (Cf. [80]). The case when $s = 2$ (that is, Coulomb potential, and $\alpha = -3$) has very particular features: it doesn't seem possible to give a reasonable sense to the associated non cutoff Boltzmann kernel, so that in the sequel, we shall only consider the cutoff Boltzmann kernel and the Landau kernel in this case.

The cross sections which are of interest to us are then summarized in the following table, where X means that the kernel cannot be defined, CB means cutoff Boltzmann's kernel, NCB non cutoff Boltzmann's kernel and L Landau's kernel. Such a table will systematically be used in the sequel.

| | CB | NCB | L |
|---|---|---|---|
| Hard Potentials ($\alpha \in ]0,1[$) | | | |
| Maxwellian Molecules ($\alpha = 0$) | | | |
| Soft Potentials ($\alpha \in ]-2,0[$) | | | |
| Very Soft Potentials ($\alpha \in ]-3,-2]$) | | | |
| Coulomb Potential ($\alpha = -3$) | | X | |

The two upper left–hand–side parts of this table (that is, cutoff hard potentials or cutoff Maxwellian molecules) are sometimes refered as "regular cross sections" while the other cases will be called "singular", since at least one of the two parts of the cross section is not continuous in this case.

We end this section by making some comments on the case when the cross section is not of the form (3). It is often possible to extend the proofs written for a cross section of the form (3) in this case, provided that $B(|u|, \cos\theta)$ has a polynomial behavior (in the variable $|u|$) when $|u| \to +\infty$. Note that the cross section $B(|u|, \cos\theta) = |u| \cos\theta$ corresponds to hard–spheres collisions. Most of the results of hard potentials with cutoff also hold for this cross section.

## 3 Notations and formal results

We use in the sequel the notation $L^p_s$ for the weighted $L^p(\mathbb{R}^3)$ space defined by the norm:

$$||f||^p_{L^p_s} = \int_{v\in\mathbb{R}^3} |f(v)|^p \, (1+|v|^2)^{ps/2} \, dv,$$

and $H^1_r$ for the weighted $H^1(\mathbb{R}^3)$ space defined by the norm:

$$||f||^2_{H^1_r} = \int_{v\in\mathbb{R}^3} (|f(v)|^2 + |\nabla f(v)|^2) \, (1+|v|^2)^r \, dv.$$

Then, we define the respective mass, momentum, energy and entropy of a nonnegative function $f$ by

$$\begin{pmatrix} \rho_f \\ \rho_f \, u_f \\ \rho_f \, \frac{|u_f|^2}{2} + \frac{3}{2}\, \rho_f \, T_f \\ H(f) \end{pmatrix} = \int_{v\in\mathbb{R}^3} f(v) \begin{pmatrix} 1 \\ v \\ \frac{|v|^2}{2} \\ \log f(v) \end{pmatrix} dv.$$

At the formal level, it is easy to see that a solution of (1) whose initial datum $f(0,\cdot) = f_{in}$ is nonnegative remains so in the evolution (when $t > 0$). In the sequel, we shall only consider such solutions.

Then, using the identity

$$\int_{v\in\mathbb{R}^3} Q(f,f)(v) \begin{pmatrix} 1 \\ v \\ \frac{|v|^2}{2} \end{pmatrix} dv = 0,$$

we see that (still at the formal level), a solution of (1) satisfies the conservation of mass, momentum and energy:

$$\begin{pmatrix} \rho_{f(t,\cdot)} \\ \rho_{f(t,\cdot)} \, u_{f(t,\cdot)} \\ \rho_{f(t,\cdot)} \, \frac{|u_{f(t,\cdot)}|^2}{2} + \frac{3}{2}\, \rho_{f(t,\cdot)} \, T_{f(t,\cdot)} \end{pmatrix} = \begin{pmatrix} \rho_{f_{in}} \\ \rho_{f_{in}} \, u_{f_{in}} \\ \rho_{f_{in}} \, \frac{|u_{f_{in}}|^2}{2} + \frac{3}{2}\, \rho_{f_{in}} \, T_{f_{in}} \end{pmatrix}. \qquad (6)$$

Then, the nonpositivity of the dissipation of entropy (sometimes called first part of Boltzmann's H–theorem)

$$D_Q(f) = \int_{v\in \mathbb{R}^3} Q(f,f)(v)\, \log f(v)\, dv \le 0 \tag{7}$$

entails the decay of the entropy (at the formal level) for the solutions of eq. (1):

$$\forall 0 \le s \le t, \qquad H(f(t,\cdot)) \le H(f(s,\cdot)). \tag{8}$$

As a consequence of (6)–(8), we get the following (formal) a priori estimates on the solution of eq. (1) (Cf. for example [34] in the inhomogeneous setting):

$$\forall T>0, \qquad \sup_{t\in[0,T]} \int_{v\in \mathbb{R}^3} (1+|v|^2+|\log f(t,v)|)\, f(t,v)\, dv$$

$$\le Cte(T, \rho_{f_{in}}, u_{f_{in}}, T_{f_{in}}, H(f_{in})), \tag{9}$$

$$\int_0^{+\infty} D_Q(f(t,\cdot))\, dt \le Cte(\rho_{f_{in}}, u_{f_{in}}, T_{f_{in}}, H(f_{in})). \tag{10}$$

The case of equality in (7) is the second part of Boltzmann's H–theorem:

$$\forall v \in \mathbb{R}^3, \qquad Q(f,f)(v) = 0 \qquad \Longleftrightarrow \qquad D_Q(f) = 0$$

$$\Longleftrightarrow \qquad f(v) = M_f(v), \tag{11}$$

where $M_f$ is the Maxwellian function of $v$ having the same mass, momentum and energy as $f$, namely

$$M_f(v) = \frac{\rho_f}{(2\pi T_f)^{3/2}}\, e^{-\frac{|v-u_f|^2}{2\,T_f}}.$$

This is the key to the long time behavior of the solutions of (1). Formally, we expect that the entropy decreases to its minimum (among functions having the same mass, momentum and energy as $f$),

$$\lim_{t\to+\infty} H(f(t,\cdot)) = \inf\{H(f), / \quad \rho_f = \rho_{f_{in}}, u_f = u_{f_{in}}, T_f = T_{f_{in}}\},$$

and that

$$\lim_{t\to+\infty} f(t,v) = M_{f_{in}}.$$

All of the previous results (conservation of energy, decay of entropy, long time behavior, etc..) can be proven only once existence (and uniqueness) is established for (1) (under a given assumption on the cross section). The study of the smoothness of solutions of (1) will enable to get strong solutions. Then, a rigorous proof of (6) will require estimates on the behavior when $|v| \to +\infty$ of the solution of (1), while a rigorous proof of (8) will require some knowledge about the lower bounds on these solutions.

All those issues (existence, uniqueness, behavior when $|v| \to +\infty$, smoothness, lower bounds, behavior when $t \to +\infty$) will successively be treated in sections 4 to 9. Then, in section 10, we try to give a synthetic result in the most standard case (cutoff hard potentials). At this point will be given the only precise theorem (all the other results of this paper are detailed in the references). Finally, various results on other issues concerning the solutions of (1) are reviewed in section 11.

# 4 Existence

When the cross section is regular (that is, for cutoff hard potentials or Maxwellian molecules), existence can be obtained through an inductive procedure, using for example monotonicity (Cf. [6], [59] and [60]). One has to cope with the following difficulties:

1. The nonnegativity of the solution must be preserved in the inductive procedure;

2. The conservation of mass (or energy) must be used to prevent blow–ups due to the quadratic character of the kernel.

At the end, one gets "strong" solutions, in the sense that if $f_{in} \in L^1_2$, then there exists a solution $f$ to (1) in $C_t(L^1_{2,v})$ such that $Q(f,f) \in L^1_{loc,t,v}$. In order to get equality in (1) for all $v$ (and not for a.e. $v$), one can use the study of smoothness presented in section 7.

For (not too) singular cross sections (that is, for cutoff or non cutoff, hard or soft (but not very soft) potentials), solutions are obtained by weak $L^1$ compactness without using estimate (10) (Cf. [7], [41]). If $f_{in} \in L^1_2$ and $f_{in} \log f_{in} \in L^1$, estimate (9) ensures that a sequence $f_n$ of solutions to (1) with a cross section $B_n$ obtained by smoothing the singular cross section $B$ will be compact in $L^1_{t,v}$, thanks to Dunford–Pettis theorem (Cf. [16] for example). Then, one passes to the limit ($f_n \to f$) in the weak form (4) of the kernel. No problems occur because of the variable $v$ since the kernel (in its weak form (4)) is close to a tensor product with respect to this variable.

Strong compactness (in time) of the velocity averages of $f_n$ are then easily obtained thanks (for example) to Aubin's lemma (Cf. [69]) and ensure that $f$ satisfies the limit equation.

Of course at the end, we only get weak solutions of the equation. Nevertheless, in some cases, results of smoothness are known which ensure that the solution is in fact strong (Cf. section 7).

Finally, for very singular cross sections (that is, for cutoff or non cutoff very soft potentials, or cutoff Coulombian potential), solutions are also obtained by (weak $L^1$) compactness. However, one now needs to use the entropy dissipation estimate (10) to give a sense to the kernel. Those solutions are called entropy solutions or H–solutions (Cf. [75]). An alternative way of obtaining solutions in this case (but only under the cutoff assumption) is to use the renormalization techniques of [34] and [50]. Finally, in the non cutoff case, the singularity of the angular cross section is sometimes strong enough to produce a regularising effect allowing to recover "usual" weak solutions (Cf. section 7 and [4]).

Note that solutions to the Landau equation can also be obtained by a weak $L^1$ compactness argument (Cf. [32]), using the limiting process of (5). It is however possible to directly use techniques coming from the theory of parabolic equations (Cf. [10] and [32]) to prove existence in this case.

We summarize the results about existence in a table, with the following abbreviations:

1. The sign IS means that existence is obtained by an inductive scheme.
2. The sign comp means that existence is obtained by a weak compactness argument.
3. The sign H means that existence of entropy solutions is proven.
4. The sign renorm means that existence of renormalized solutions is proven.

| | CB | NCB | L |
|---|---|---|---|
| Hard Potentials ($\alpha \in ]0,1[$) | IS | comp | comp |
| Maxwellian Molecules ($\alpha = 0$) | IS | comp | comp |
| Soft Potentials ($\alpha \in ]-2,0[$) | comp | comp | comp |
| Very Soft Potentials ($\alpha \in ]-3,-2]$) | H or renorm | H or comp in some cases | H |
| Coulombian Potential ($\alpha = -3$) | H or renorm | X | H |

# 5 Uniqueness

Uniqueness is an open question for soft (and of course very soft, or Coulombian) potentials.

For the cutoff Boltzmann equation with hard potentials, it is a consequence of a Gronwall type lemma, which takes into account the gain of moments (Cf. section 6). For a precise statement in a weighted $L^1$ setting, Cf. [6] (Cf. also [59] and [60] in the case of Maxellian molecules).

For Landau's kernel with hard potentials, one can also use a Gronwall type lemma, but this time it takes into account not only the gain of moments but also the gain of smoothness (Cf. sections 6 and 7). This lemma is a consequence of the following type of functional estimates on Landau's kernel (Cf. [32]):

$$\int (L(f,f) - L(g,g))\,(f-g)\,(1+|v|^2)^q\,dv$$
$$\leq Cte(||f||_{H^1_r}, ||g||_{H^1_r})\,||f-g||^2_{L^2_q},$$

for well chosen $q, r > 0$. At the end, uniqueness holds in a weighted $L^2$ space (where existence is also known to hold).

In the particular case of Maxwellian molecules, it is possible to use a Gronwall lemma in a weak topology, which enables to get a result even in the non cutoff situation (Cf. [71]). Note finally that (still in this case) uniqueness for a martingale problem related to the equation can also be proven (Cf. [67], [68], [31]).

Finally, one must keep in mind that some assumption on the energy of solutions must be made in the uniqueness theorem (for example, at least that the energy does not increase), since strange solutions with a growing energy are known to exist, even for regular cross sections (Cf. [84]).

We end up this section with a table explaining whether uniqueness is proven or not for each type of cross sections.

| | CB | NCB | L |
|---|---|---|---|
| Hard Potentials ($\alpha \in ]0,1[$) | yes | | yes |
| Maxwellian Molecules ($\alpha = 0$) | yes | yes | yes |
| Soft Potentials ($\alpha \in ]-2,0[$) | | | |
| Very Soft Potentials ($\alpha \in ]-3,-2]$) | | | |
| Coulombian Potential ($\alpha = -3$) | | X | |

# 6 Behavior for large velocities

Most of the results on the behavior of the solution of eq. (1) when $|v| \to +\infty$ are in fact written in terms of the moments of the solution, that is, of its $L^1_s$ norm for $s > 0$. The main feature of $Q$ with respect to these moments is that as soon as one looks to the superquadratic case (that is, $s > 2$), the loss term of $Q$ is dominant. For $\delta > 0$ (not too large), this can be seen on the following functional estimate, valid in most of the situations studied here (cutoff or noncutoff kernel, hard or soft potentials (for very soft potentials, the constant are slightly different), Landau kernel, etc..):

$$\int Q(f,f)(v)\,|v|^{2+\delta}dv \le -Cte(\rho_f,u_f,T_f) \int f(v)\,|v|^{2+\delta+\alpha}\,dv + Cte(\rho_f,u_f,T_f). \tag{12}$$

This estimate can be seen as an integrated version of the Povzner inequality for a given collision (Cf. [61]).

An application of this inequality is the following: all superquadratic moments are immediately created (and then preserved uniformly in time) for hard potentials, if one of them initially exists (Cf. [27], [35], [32]).

Moreover, this last condition can be relaxed for the (cutoff or non cutoff) Boltzmann (but not the Landau !) equation, thanks to a reverse Povzner inequality (Cf. [58]).

For Maxwellian molecules, polynomial moments are never created, but propagated (and bounded when $t \to +\infty$). They are given by an explicit formula (Cf. [47]) . "Maxwellian moments" like $\int f(v)\, \exp(\lambda\, |v|^2)\, dv$ can also be studied. This is the interesting theory of Maxwellian tails (Cf. [12]). It also works for "exponential" moments.

Finally, for soft potentials, moments are propagated (this is still a consequence of (12)) but may blow up when $t \to +\infty$ (Cf. [27] and [73]).

Thanks to this study, it is possible to prove that in most situations, the conservation of energy (6) rigorously holds.

We summarize in the table below the results of this section, with the following convention:

1. The sign P means that (polynomial superquadratic) moments are propagated .
2. The sign $\infty$ means that these moments remain bounded when $t \to +\infty$.
3. The sign C means that (polynomial superquadratic) moments are immediately created .
4. The sign ? means that the result is presumably true, but not explicitly proven in an article.

| | CB | NCB | L |
|---|---|---|---|
| Hard Potentials ($\alpha \in ]0,1[$) | CP$\infty$ | CP$\infty$ | CP$\infty$ |
| Maxwellian Molecules ($\alpha = 0$) | P$\infty$ | P$\infty$ | P$\infty$ |
| Soft Potentials ($\alpha \in ]-2,0[$) | P | P ? | P |
| Very Soft Potentials ($\alpha \in ]-3,-2]$) | P ? | P ? | P |
| Coulombian Potential ($\alpha = -3$) | P ? | X | P ? |

## 7 Smoothness

The results on smoothness for the solutions of (1) can be summarized in the following way: smoothness (including weighted $L^p$ regularity) is propagated (but also singularities !) when the cross section is cutoff. It is created as soon as $t > 0$ when the cross section is non cutoff (or for Landau's kernel).

In the cutoff case (more precisely, for hard potentials, Maxwellian molecules and reasonably soft potentials), the following functional estimate can be obtained thanks to Fourier integral operators (Cf. [50]), Radon transform (Cf. [82]) or Fourier transform theory (Cf. [15] and [52]):

$$f \in L^2 \qquad \Rightarrow \qquad Q^+(f,f), Lf \in H^q_{loc},$$

where $q = 1$ for hard potentials and $q \in ]0,1[$ for reasonably soft potentials. The propagation of smoothness (and singularities) is then a consequence of Duhamel's formula

$$f(t) = f(0)\, e^{-\int_0^t L(f)(\sigma)d\sigma} + \int_0^t Q^+(f,f)(s)\, e^{-\int_s^t L(f)(\sigma)d\sigma}\, ds.$$

In particular, one can see that the $L^2$ singularities of the initial datum never disappear, but are exponentially damped. Note also that the propagation of (weighted) $L^\infty$ norms (Cf. [8], [18], [55]) or weighted $L^p$ (for $p \in ]1,+\infty[$) norms (Cf. [45] and [46]) has been proven.

In the non cutoff equation, it is possible to get the following functional estimate thanks to a Fourier analysis (Cf. [4]):

$$D_Q(f) = -\int Q(f,f)(v)\, \log f(v)\, dv$$

$$\geq Cte(R, \rho_f, u_f, T_f, H(f))\, ||\sqrt{f}||^2_{H^q(B_R)} - Cte(\rho_f, u_f, T_f, H(f))\, ||f||_{L^1_2},$$

where $q > 0$ depends on the angular cross section $\beta$ and $B_R$ is the ball of center 0 and radius $R$ in $\mathbb{R}^3$. Smoothness (in $L^1_t(H^1_{s,v})$) for $\sqrt{f}$ is then obtained thanks to the entropy dissipation estimate (10). Higher derivatives are known to be created and to propagate (sometimes up to infinty, Cf. [21]) in many particular cases (Cf. [28], [29], [30], [62]). Note also the approach to this question using the Malliavin calculus (Cf. [43], [39]).

Finally, for Landau's equation, it is possible to apply techniques designed for parabolic equations (Cf. [10], [32]). Then, for hard potentials, its solution lies in $C^\infty(\mathbb{R}^*_+; \mathcal{S}(\mathbb{R}^3))$ as soon as mass, entropy and a superquadratic moment initially exist.

We now summarize the results about smoothness in a table, with the following convention:

1. The sign P means that smoothness (and singularities) is propagated .
2. The sign $\infty$ means that some (weighted $L^2$) norm of a derivative is bounded when $t \to +\infty$ (in the case when it initially exists for cutoff cross sections).
3. The sign C means that smoothness is immediately created .
4. The sign [ ] means that the result is known to hold only for a mollified version of the (soft potential) cross section.
5. The sign ? means that the result is presumably true, but not explicitly proven in an article.

| | CB | NCB | L |
|---|---|---|---|
| Hard Potentials ($\alpha \in ]0,1[$) | P | CP | CP |
| Maxwellian Molecules ($\alpha = 0$) | P$\infty$ | CP$\infty$ | P$\infty$ ? |
| Soft Potentials ($\alpha \in ]-2,0[$) | [P] | C | [C] ? [P] |
| Very Soft Potentials ($\alpha \in ]-3,-2]$) | | C | |
| Coulombian Potential ($\alpha = -3$) | | X | |

# 8 Lower bounds

In this section, we use the following idea: the support of $Q^+(f,f)$ is bigger than that of $f$: because of the collisions, large velocities appear even if they were not present at the beginning. A quantitative version of that remark leads to Maxwellian lower bounds for the cutoff hard potentials (Cf. [18], [57], [64], [65]).

For the Landau equation, the same kind of estimates is a consequence of maximum principle techniques (Cf. [32]).

Finally, in the non cutoff case, no Maxwellian lower bound is known to hold. In the case of Maxwellian molecules, strict positivity when $t > 0$ of $f$ is obtained thanks to Malliavin calculus techniques (Cf. [37], [38]).

Note that the study of lower bounds (and smoothness) enables to rigorously prove the decay of entropy (8).

The following table summarizes what is known on the existence of lower bounds for the solution of (1): The sign [ ] means that the result is known to hold only for a mollified version of the (soft potential) cross section, the sign () means that only the strict positivity of the solution is known.

| | CB | NCB | L |
|---|---|---|---|
| Hard Potentials ($\alpha \in ]0, 1[$) | yes | | yes |
| Maxwellian Molecules ($\alpha = 0$) | yes | (yes) | yes |
| Soft Potentials ($\alpha \in ]-2, 0[$) | [yes] | | |
| Very Soft Potentials ($\alpha \in ]-3, -2]$) | | | |
| Coulombian Potential ($\alpha = -3$) | | X | |

# 9 Large time behavior

The decay of $f(t, \cdot)$ towards $M_{f_{in}}$, has been known for a long time in many situations (Cf. [56] for example).

In order to get estimates on the speed of this decay, one can use spectral theory on the linearized equation (since after some time, $f(t, \cdot)$ will be close to $M_{f_{in}}$) (Cf. [9] and [81]). In this way it is possible to prove that the convergence is exponential in weighted $L^1$ and $L^p$ (for $p \in ]1, +\infty[$) spaces for cutoff hard potentials. Note however that the constants involved in these estimates are not explicit.

In order to get explicit constants, one can try another approach, which consists in comparing the entropy dissipation $D_Q(f)$ and the relative entropy

$$H(f|M_f) \equiv \int f(v) \, \log(f(v)/M_f(v)) \, dv.$$

It means that one tries to prove weak versions of Cercignani's conjecture (Cf. [23]):

$$D_Q(f) \geq Cte(f) \, \Phi(H(f|M_f)),$$

for some function $\Phi$ which increases not too slowly at point 0, and some $Cte(f)$ depending on various norms of $f$. The conjecture itself (i.-e. with $\Phi(x) = x$, and $C(f)$ depending only on mass, energy and entropy of $f$ ) is true in the case of the Landau equation (with Maxwellian cross section), but not in the case of Boltzmann's equation (Cf. [13] and [33]).

Then, one uses the H-theorem in the form

$$\frac{d}{dt} H(f|M_f) = -D_Q(f),$$

and some variant of Gronwall's lemma.

Such weak versions of the Cercignani conjecture have been introduced first in [19] and [20], and then in [33] for the Landau equation, in [72] for hard potentials and Maxwellian molecules, and in [73] for cutoff soft potentials. They rely on the logarithmic Sobolev inequality of Gross (Cf. [44]), or on ideas used in (some of the) proofs of this inequality..

At the end, one gets a polynomial convergence in the case of cutoff hard or soft potentials, and an exponential convergence for the Landau equation (and for the Boltzmann equation with Maxwellian molecules, Cf. [40] and [21]), all constants being explicit.

Note also that the convergence to equilibrium is sometimes true in the case when the entropy of the initial datum is infinite (Cf. [1]).

We summarize below the results of this section with the following conventions:

1. The sign pol means that the convergence has at least an algebraic rate.
2. The sign exp means that the convergence has an exponential rate.
3. The sign $E$ means that all constants can be explicitly bounded.
4. The sign [ ] means that the result is known to hold only for a mollified version of the (soft potential) cross section.

| | CB | NCB | L |
|---|---|---|---|
| Hard Potentials ($\alpha \in ]0,1[$) | exp, E pol | | E pol |
| Maxwellian Molecules ($\alpha = 0$) | E exp | E exp | E exp |
| Soft Potentials ($\alpha \in ]-2,0[$) | [E pol] | | [E pol] |
| Very Soft Potentials ($\alpha \in ]-3,-2]$) | | | [E pol] |
| Coulombian Potential ($\alpha = -3$) | | X | |

## 10 Synthetic result for cutoff hard potentials

In this section, we detail the hypothesis of a theorem on the solutions of (1) in the most standard case, namely, that of cutoff hard potentials. The proof of the various statements included in this theorem can be found in the references described in the sections above.

**Theorem**: *Let $f_{in}$ be an initial datum with finite mass, energy and entropy (that is, $\int_{\mathbb{R}^3} f_{in}\,(1+|v|^2+|\log f_{in}|)\,dv < +\infty$), and $B$ be defined by $B(|u|,\cos\theta) = |u|^\alpha\,\breve{\beta}_s(\theta)$, for $\alpha \in ]0,1[$ (and $\breve{\beta}_s \in L^\infty(]0,\pi[)$). Then there exists a solution to the Boltzmann equation with cross section $B$ in $C^1([0,+\infty[; L^1_{loc}(\mathbb{R}^3))$ for which mass, momentum and energy are conserved (that is, (6) holds).*

*Any other solution (in the same space) such that the energy is conserved (or at least decreases) is equal to this solution.*

*For any time $t > 0$, this solution is bigger than a given Maxwellian and has all its (polynomial) moments bounded. Moreover those estimates are uniform on $[T,+\infty[$ for all $T > 0$.*

*Then, $f(t,\cdot)$ lies in $H^q_{loc}(\mathbb{R}^3)$ (for a given $q \in \mathbb{N}$ and a given $t > 0$) if and only if $f_{in}$ also lies in $H^q_{loc}(\mathbb{R}^3)$.*

*Finally, $f$ satisfies the estimate of decay of entropy (8) rigorously and converges exponentially fast in $L^1(\mathbb{R}^3)$ (and algebraically fast with computable constants) towards $M_{f_{in}}$.*

## 11 Other issues

In this section, we try to review some of the issues about the solutions of (1) which have not been discussed previously.

1. Explicit solutions: For the Boltzmann equation, only one family of (non steady) solutions is explicitly known: the so–called BKW mode, in the case of Maxwellian molecules (Cf. [11], [48]). Note that still for Maxwellian molecules, all the polynomial moments of any solution can be computed explicitly (Cf. [47]), and "semi-explicit" expressions can be given (Wild sums, etc..) (Cf. [85]).

2. Special ways of writing the kernel: Different formulas for the kernel are useful, among which one can quote: the Fourier transform formulation (in particular in the case of Maxwellian molecules) (Cf. [12], [63]), the Carleman representation (with the generalized Radon transform) (Cf. [18] and [83]), the divergence form of the kernel (Cf. [78]), the martingale problem related to the equation (Cf. [66], [67], [68], [31]), and the pseudodifferential approach (Cf. [3]).

3. Eternal solutions: For the Landau equation with Maxwellian molecules, no non–trivial eternal solutions exist (Cf. [80]). The question is open for the Boltzmann operator, but solved for some related equations (Cf. [17]).

4. Behavior of functionals with higher derivatives: The Fisher information is decreasing along solutions of the Boltzmann and Landau equation with Maxwellian molecules (Cf. [77] and [76]). Note that those results were previously proven in 1D (Kac's model) and 2D for the Boltzmann equation with Maxwellian molecules (Cf. [53] and [70]). Finally, a study of the functionals which decrease along the solutions of the Boltzmann equation with Maxwellian molecules can be found in [14].

5. Stability with respect to initial data or cross sections: Results linked to the uniqueness are proven for the Landau equation with hard potentials (Cf. [32]).

6. Complex kernels: When the cross section is not a tensor product, many of the previous results remain true. The situation becomes more intricate for polyatomic gases (Cf. [54]), or inelastic collisions, or kernels with quantum mechanics or relativistic effects. To get an example of the difficulties inherent to such complex kernels, Cf. [5], [36].

Finally, the numerical discretization of Boltzmann's kernel is an important subject that we do not try to tackle here.

**Acknowledgments**: I wish to thank Leif Arkeryd, Carlo Cercignani, Sylvie Méléard, Cédric Villani and Bernt Wennberg for their most valuable comments during the preparation of this work, and for all the references they have pointed out to me.

The support of the TMR contract "Asymptotic Methods in Kinetic Theory", ERB FMBX CT97 0157 is also acknowledged.

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