REGULARIZATION FOR THE NON CUTOFF 2D RADIALLY SYMMETRIC BOLTZMANN EQUATION WITH A VELOCITY DEPENDENT CROSS SECTION

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Abstract

We extend in this work previous results of regularization for the spatially homogeneous non cutoff 2D radially symmetric Boltzmann equation which held only in the case of Maxwellian molecules. Velocity dependent cross sections can now be taken in account.

1 Introduction

The homogeneous Boltzmann equation for rarefied gases writes

$$\frac{\partial f}{\partial t} = Q(f), \tag{1.1}$$

where Q is a quadratic collision kernel acting only on the variable v and taking in account any collisions preserving momentum and kinetic energy (Cf. [Ce], [Ch, Co], [Tr, Mu]).

In the particular case when $v \in I\!R^2$, one can write

$$Q(f)(v) = \int_{v_* \in I\!R^2} \int_{\theta=-\pi}^{\pi} \Big\{ f(\frac{v+v_*}{2} + R_\theta(\frac{v-v_*}{2}))\, f(\frac{v+v_*}{2} - R_\theta(\frac{v-v_*}{2}))$$

$$- f(v)\, f(v_*) \Big\}\, B(v-v_*, \theta)\, d\theta dv_*, \tag{1.2}$$

where B is a nonnegative cross section depending only on $|v-v_*|$ and θ, and R_θ is the rotation of angle θ in $I\!R^2$.

When the collisions in the gas come out of an inverse power law interaction in $\frac{1}{r^s}$ (with $s \geq 2$), the variables $v - v_*$ and θ are separate, and the cross section B writes

$$B(v-v_*, \theta) = D(v-v_*)\, \beta(|\theta|). \tag{1.3}$$

Moreover, $\beta \in L^\infty_{\text{loc}}(]0,\pi])$ and has a non integrable singularity in 0.

We shall from now on consider cross sections B of the form (1.3) and satisfying the following assumption:

Assumption 1: *The function D is continuous, radially symmetric, and such that*

$$\exists D_0 > 0, \qquad \forall x \in I\!R^2, \qquad D(x) \geq D_0. \tag{1.4}$$

Moreover, $\hat{D}$ is a bounded measure such that

$$\int_{\eta \in I\!R^2} (1+|\eta|^2)\, |\hat{D}(\eta)| d\eta < +\infty. \tag{1.5}$$

Finally, we suppose that

$$\beta(\theta) = |\sin(\frac{\theta}{2})|^{-\gamma} \cos(\frac{\theta}{2}), \tag{1.6}$$

where $\gamma \in]1,3[$.

The type of singularities of β studied here covers exactly the range of singularities observed in dimension 3 for inverse power forces in $\frac{1}{r^s}$ when $s > 2$, in other words for soft and hard potentials (the case of Coulombian interaction being excluded). However, the hypotheses on D are unrealistic. This work is only aimed at showing that regularization properties hold for the non cutoff Boltzmann equation even with a velocity dependent cross section. The study of a realistic D needs more careful estimates and will be discussed in other works.

We recall that when the weak angular cutoff assumption of Grad (Cf. [Gr]) is made, which means that $\beta \in L^1([0,\pi])$, no regularizing effect for the homogeneous Boltzmann equation is expected. On the contrary, one can prove (under suitable assumptions) (Cf. [L], [We]) that the solution of the equation keeps exactly the same regularity (in the space of velocities) as the initial datum. No improvement of regularity can occur in this case because the solution retains a lot of the properties of the initial datum.

However, we proved in an earlier work (Cf. [De 1]) that when D is a constant function (that is in the case of Maxwellian molecules) and β is as in (1.6), the solution f of eq. (1.1) lies in $L^\infty(]0,+\infty[_t; H^{1-0}(\mathbb{R}^2_v))$ as soon as the initial datum satisfies the following assumption:

Assumption 2: *The initial datum* $f_0 > 0$ *is radially symmetric and such that*

$$\int_{v\in\mathbb{R}^2} f_0(v)\,(1+|v|^2+|\log f_0(v)|)\,dv < +\infty. \tag{1.7}$$

We extend here this result in the case when D is not necessarily constant, but satisfies nevertheless assumption 1. Note first that the following theorem of existence can easily be deduced from the proofs of [A] (we shall also discuss those matters in a forthcoming paper (Cf. [De 2])):

Theorem 1: *Under assumption 2 on the initial datum* f_0 *and assumption 1 on the cross section* B*, there exists a nonnegative weak solution* $f \in L^\infty([0,+\infty[_t; L^1(\mathbb{R}^2_v))$ *to eq. (1.1) satisfying for all* $T > 0$*,*

$$\sup_{t\in[0,T]} \int_{v\in\mathbb{R}} f(t,v)\,(1+|v|^2+|\log f(t,v)|)\,dv < +\infty. \tag{1.8}$$

Moreover, the total mass of f *is conserved:*

$$\forall t \in [0,+\infty[, \qquad \int_{v\in\mathbb{R}} f(t,v)\,dv \;=\; \int_{v\in\mathbb{R}} f_0(v)\,dv, \tag{1.9}$$

and the total energy does not increase:

$$\forall t \in [0,+\infty[, \qquad \int_{v\in \mathbb{R}} f(t,v)\,|v|^2\,dv \,\leq\, \int_{v\in \mathbb{R}} f_0(v)\,|v|^2\,dv. \tag{1.10}$$

As will be seen in the sequel, the analysis must be done here in an L^2 context (whereas it could be done entirely in L^∞ in the case of Maxwellian molecules). Therefore we will need an L^2 assumption on the initial datum.

The paper is structured as follows: the Fourier transform of the collision term Q is studied in section 2. Then, estimates are presented in section 3 for the non dominant parts of $< f, Q(f) >_{L^2}$, and in section 4 for the dominant part. Section 5 is devoted to a first analysis of the properties of regularization, which are more thoroughly discussed in section 6. We expose in section 7 some ideas to investigate those properties in slightly different contexts.

2 Extraction from $\widehat{Q(f)}$ of the dominant term

From now on, we will note $x^\perp = R_{\frac{\pi}{2}}(x)$ and $\tau_y f(x) = f(x+y)$. This section is devoted to the proof of the following lemma:

Lemma 2.1: *Under assumption 1 on the cross section, the Boltzmann kernel Q is such that for any radially symmetric function $f \in L^1(\mathbb{R}^2)$:*

$$(2\,\pi)^2 \int_{\xi\in\mathbb{R}^2} \overline{\hat{f}(\xi)}\,\widehat{Q(f)}(\xi)\,d\xi = -\chi_0 + \chi_1 + \chi_2 + \chi_3 + \chi_4, \tag{2.1}$$

where

$$\chi_0 = 4 \int_{\xi\in\mathbb{R}^2}\int_{\eta\in\mathbb{R}^2}\int_{u=-\infty}^{+\infty}\int_{y\in\mathbb{R}^2} |\xi|^{\frac{\gamma-1}{2}}\,\sin(\frac{u}{2}\frac{\xi}{|\xi|}\cdot y)\overline{\hat{f}(\xi)}\,|\xi+\eta^\perp|^{\frac{\gamma-1}{2}}$$

$$\sin(\frac{u}{2}\frac{\xi+\eta^\perp}{|\xi+\eta^\perp|}\cdot y)\,\hat{f}(\xi+\eta^\perp)\,f(y)\,\widehat{\tau_{-y}D}(\eta)\,dy\,|u|^{-\gamma}\,du d\eta d\xi, \tag{2.2}$$

$$\chi_1 = 4 \int_{\xi\in\mathbb{R}^2}\int_{\eta\in\mathbb{R}^2}\int_{|u|\geq|\xi|}\int_{y\in\mathbb{R}^2} |\xi|^{\gamma-1}\,\sin^2(\frac{u}{2}\frac{\xi}{|\xi|}\cdot y)\,\overline{\hat{f}(\xi)}\,\hat{f}(\xi+\eta^\perp)$$

$$f(y)\,\widehat{\tau_{-y}D}(\eta)\,dy\,|u|^{-\gamma}\,du d\eta d\xi, \tag{2.3}$$

$$\chi_2 = -4 \int_{\xi\in\mathbb{R}^2}\int_{\eta\in\mathbb{R}^2}\int_{u=-\infty}^{+\infty}\int_{y\in\mathbb{R}^2} |\xi|^{\gamma-1}\,\sin(\frac{u}{2}\frac{\xi}{|\xi|}\cdot y)\,\overline{\hat{f}(\xi)}\,\Big\{\sin(\frac{u}{2}\frac{\xi}{|\xi|}\cdot y)$$

$$- \sin(\frac{u}{2}\frac{\xi+\eta^\perp}{|\xi+\eta^\perp|}\cdot y)\Big\} \hat f(\xi+\eta^\perp)\, f(y)\, \widehat{\tau_{-y}D}(\eta)\, dy\, |u|^{-\gamma}\, dud\eta d\xi, \qquad (2.4)$$

$$\chi_3 = 4\int_{\xi\in I\!R^2}\int_{\eta\in I\!R^2}\int_{u=-\infty}^{+\infty}\int_{y\in I\!R^2} |\xi|^{\frac{\gamma-1}{2}}\ \sin(\frac{u}{2}\frac{\xi}{|\xi|}\cdot y)\,\overline{\hat f(\xi)}\,\Big\{|\xi+\eta^\perp|^{\frac{\gamma-1}{2}}$$

$$-|\xi|^{\frac{\gamma-1}{2}}\Big\}\ \sin(\frac{u}{2}\frac{\xi+\eta^\perp}{|\xi+\eta^\perp|}\cdot y)\,\hat f(\xi+\eta^\perp)\, f(y)\,\widehat{\tau_{-y}D}(\eta)\, dy\, |u|^{-\gamma}\, dud\eta d\xi, \qquad (2.5)$$

and

$$\chi_4 = \int_{\xi\in I\!R^2}\int_{\eta\in I\!R^2}\int_{\theta=-\pi}^{\pi} \overline{\hat f(\xi)}\,\{\hat f(\cos\frac{\theta}{2}\,\xi+\eta^\perp) - \hat f(\xi+\eta^\perp)\}\,\hat f(-\sin\frac{\theta}{2}\xi+\eta)$$

$$\times\, \hat D(\eta)\,|\sin\frac{\theta}{2}|^{-\gamma}\ \cos\frac{\theta}{2}\, d\theta d\eta d\xi. \qquad (2.6)$$

Proof of lemma 2.1: Note first that

$$(2\,\pi)^2\,\widehat{Q(f)}(\xi) = \int_{\eta\in I\!R^2}\int_{\theta=-\pi}^{\pi}\Big\{\hat f(\cos\frac{\theta}{2}\,\xi+\eta^\perp)\,\hat f(-\sin\frac{\theta}{2}\,\xi+\eta)$$

$$-\hat f(\xi+\eta^\perp)\,\hat f(\eta)\Big\}\,\hat D(\eta)\,|\sin\frac{\theta}{2}|^{-\gamma}\ \cos\frac{\theta}{2}\, d\theta d\eta. \qquad (2.7)$$

Then,

$$(2\,\pi)^2\,\widehat{Q(f)}(\xi) = l_1(\xi) + l_2(\xi), \qquad (2.8)$$

where

$$l_1(\xi) = \int_{\eta\in I\!R^2}\int_{\theta=-\pi}^{\pi}\Big\{\frac{1}{2}\hat f(-\sin\frac{\theta}{2}\,\xi+\eta) + \frac{1}{2}\hat f(\sin\frac{\theta}{2}\,\xi+\eta) - \hat f(\eta)\Big\}$$

$$\hat f(\xi+\eta^\perp)\,\hat D(\eta)\,|\sin\frac{\theta}{2}|^{-\gamma}\ \cos\frac{\theta}{2}\, d\theta d\eta, \qquad (2.9)$$

and

$$l_2(\xi) = \int_{\eta\in I\!R^2}\int_{\theta=-\pi}^{\pi}\Big\{\hat f(\cos\frac{\theta}{2}\,\xi+\eta^\perp) - \hat f(\xi+\eta^\perp)\Big\}$$

$$\hat f(-\sin\frac{\theta}{2}\xi+\eta)\,\hat D(\eta)\,|\sin\frac{\theta}{2}|^{-\gamma}\ \cos\frac{\theta}{2}\, d\theta d\eta. \qquad (2.10)$$

Note that this computation is a generalization of that of [De 1]. In this work, the same form was given for $\widehat{Q(f)}$ in the particular case of Maxwellian molecules (i.-e. when $\hat D = \delta_0$).

It is clear that

$$\chi_4 = \int_{\xi \in I\!R^2} \overline{\hat{f}(\xi)}\, l_2(\xi)\, d\xi. \tag{2.11}$$

Then,

$$\int_{\xi \in I\!R^2} \overline{\hat{f}(\xi)}\, l_1(\xi)\, d\xi = \chi_1 - 4 \int_{\xi \in I\!R^2} \int_{\eta \in I\!R^2} \int_{u=-\infty}^{+\infty} |\xi|^{\gamma-1}\, \overline{\hat{f}(\xi)}\, \hat{f}(\xi + \eta^\perp)$$

$$\times \int_{y \in I\!R^2} \sin^2(\frac{u}{2}\frac{\xi}{|\xi|} \cdot y)\, f(y)\, \widehat{\tau_{-y} D}(\eta)\, dy\, |u|^{-\gamma}\, du d\eta d\xi$$

$$= \chi_1 + \chi_2 - 4 \int_{\xi \in I\!R^2} \int_{\eta \in I\!R^2} \int_{u=-\infty}^{+\infty} |\xi|^{\gamma-1}\, \overline{\hat{f}(\xi)}\, \hat{f}(\xi + \eta^\perp)$$

$$\int_{y \in I\!R^2} \sin(\frac{u}{2}\frac{\xi}{|\xi|} \cdot y)\, \sin(\frac{u}{2}\frac{\xi + \eta^\perp}{|\xi + \eta^\perp|} \cdot y)\, f(y)\, \tau_{-y}\hat{D}(\eta)\, dy\, |u|^{-\gamma}\, du d\eta d\xi$$

$$= \chi_1 + \chi_2 + \chi_3 - 4 \int_{\xi \in I\!R^2} \int_{\eta \in I\!R^2} \int_{u=-\infty}^{+\infty} \int_{y \in I\!R^2} |\xi|^{\frac{\gamma-1}{2}}\, \overline{\hat{f}(\xi)}\, \sin(\frac{u}{2}\frac{\xi}{|\xi|} \cdot y)$$

$$|\xi + \eta^\perp|^{\frac{\gamma-1}{2}}\, \hat{f}(\xi + \eta^\perp)\, \sin(\frac{u}{2}\frac{\xi + \eta^\perp}{|\xi + \eta^\perp|} \cdot y)\, f(y)\, \tau_{-y}\hat{D}(\eta)\, dy\, |u|^{-\gamma}\, du d\eta d\xi$$

$$= \chi_1 + \chi_2 + \chi_3 - \chi_0. \tag{2.12}$$

3 Estimates for the non dominant terms

From now on, various constants will be denoted by C, or by $C(\gamma)$ when they depend on γ. Moreover, we shall denote by $||f||_{L^1_q}$ the L^1 norm of $(1 + |x|^q) f(x)$. We examine successively each of the terms χ_i, $(i = 1, 2, 3, 4)$, but give a proof only for the estimate of χ_1. To get the proof of the other lemmas, we refer to [De 3].

Lemma 3.1: *The first term χ_1 is such that:*

$$|\chi_1| \leq C(\gamma) \int_{\eta \in I\!R^2} |\hat{D}(\eta)| d\eta \quad ||f||_{L^1} \int_{\xi \in I\!R^2} |\hat{f}(\xi)|^2\, d\xi. \tag{3.1}$$

Proof of lemma 3.1: We compute

$$|\chi_1| \leq 4 \int_{\xi \in I\!R^2} \int_{\eta \in I\!R^2} \int_{|u| \geq |\xi|} \int_{y \in I\!R^2} |\xi|^{\gamma-1}\, \sin^2(\frac{u}{2}\frac{\xi}{|\xi|} \cdot y)\, |\hat{f}(\xi)|\, |\hat{f}(\xi + \eta^\perp)|$$

$$f(y)\,|\hat{D}(\eta)|\,dy\,|u|^{-\gamma}\,dud\eta d\xi$$

$$\leq 2\int_{\xi\in I\!R^2}\int_{\eta\in I\!R^2}\int_{|w|\geq 1}\int_{y\in I\!R^2}\sin^2(\frac{w}{2}\xi\cdot y)\,(|\hat{f}(\xi)|^2+|\hat{f}(\xi+\eta^{\perp})|^2)$$

$$\times\,|w|^{-\gamma}\,f(y)\,|\hat{D}(\eta)|\,dydwd\eta d\xi$$

$$\leq 4\,(\gamma-1)^{-1}\int_{\eta\in I\!R^2}|\hat{D}(\eta)|d\eta\quad ||f||_{L^1}\int_{\xi\in I\!R^2}|\hat{f}(\xi)|^2\,d\xi, \tag{3.2}$$

and lemma 3.1. is proved.

Lemma 3.2: *The second term* χ_2 *is such that*

$$|\chi_2|\leq C(\gamma)\int_{\eta\in I\!R^2}(|\eta|^{\frac{\gamma-1}{2}}+|\eta|^{\gamma-1})\,|\hat{D}(\eta)|\,d\eta\quad ||f||_{L^1_{\frac{\gamma+1}{2}}}$$

$$\times\int_{\xi\in I\!R^2}(1+|\xi|^{\frac{\gamma-1}{2}})\,|\hat{f}(\xi)|^2\,d\xi. \tag{3.2}$$

Lemma 3.3: *The third term* χ_3 *is such that*

$$|\chi_3|\leq C(\gamma)\int_{\eta\in I\!R^2}(|\eta|^{\frac{\gamma-1}{2}}+|\eta|^{\gamma-1})\,|\hat{D}(\eta)|\,d\eta\quad ||f||_{L^1_2}$$

$$\times\int_{\xi\in I\!R^2}(1+|\xi|^{\frac{\gamma-1}{2}})\,|\hat{f}(\xi)|^2\,d\xi. \tag{3.3}$$

Lemma 3.4: *One can find* $\zeta>0$ *such that*

$$|\chi_4|\leq C(\gamma)\,||f||_{L^1_2}\int_{\eta\in I\!R^2}(1+|\eta|^{\sup{(\gamma-1,3-\gamma)}})\,|\hat{D}|(\eta)\,d\eta$$

$$\times\Big(||f||_{L^1_2}+\int_{\xi\in I\!R^2}(1+|\xi|^2)^{\frac{\gamma-1}{2}-\zeta}\,|\hat{f}(\xi)|^2\,d\xi\Big). \tag{3.4}$$

4 Treatment of the dominant term of $\widehat{Q(f)}$

Lemma 4.1: *The dominant term χ_0 is real and satisfies the following estimate:*

$$\chi_0 \geq C(\gamma)\, D_0^{\frac{5-\gamma}{2}}\, ||D||_{L^\infty(\mathbb{R}^2)}^{\frac{\gamma-3}{2}}$$

$$\times \int_{y\in\mathbb{R}^2} |y|^{\gamma-1}\, f(y)\, dy \int_{\xi\in\mathbb{R}^2} |\xi|^{\gamma-1}\, |\hat{f}(\xi)|^2 d\xi. \tag{4.1}$$

Proof of lemma 4.1: Using Plancherel's formula, one gets

$$\chi_0 = 2\int_{x\in\mathbb{R}^2}\int_{y\in\mathbb{R}^2}\int_{u=-\infty}^{+\infty} |L_{u\,y}f|^2(x)\, D(x^\perp - y)\, f(y)\, |u|^{-\gamma}\, dudydx, \tag{4.2}$$

where

$$\widehat{L_{u\,y}}\, f(\xi) = |\xi|^{\frac{\gamma-1}{2}}\, \sin(\frac{u\,y}{2}\cdot\frac{\xi}{|\xi|})\, \hat{f}(\xi). \tag{4.3}$$

We now introduce

$$\widehat{T_{u\,y}}\, f(\xi) = |\xi|^{\frac{\gamma-1}{2}}\, \frac{u\,y}{2}\cdot\frac{\xi}{|\xi|}\, \hat{f}(\xi). \tag{4.4}$$

Then, for all $\epsilon_0 > 0$,

$$\chi_0 \geq C\, D_0\, \epsilon_0^{3-\gamma} \left(\inf_{|p|=1} \int_{y\in\mathbb{R}^2} |y|^{\gamma-1}\, f(y)\, \frac{y}{|y|} \otimes \frac{y}{|y|}\, dy \,:\, p\otimes p \right)$$

$$\times \int_{\xi\in\mathbb{R}^2} |\xi|^{\gamma-1}\, |\hat{f}(\xi)|^2\, d\xi$$

$$- C\, ||D||_{L^\infty(\mathbb{R}^2)}\, \epsilon_0^{5-\gamma} \int_{y\in\mathbb{R}^2} f(y)\, |y|^{\gamma-1}\, dy \int_{\xi\in\mathbb{R}^2} |\xi|^{\gamma-1}\, |\hat{f}(\xi)|^2\, d\xi. \tag{4.5}$$

But since f is radially symmetric,

$$\inf_{|p|=1} \int_{y\in\mathbb{R}^2} |y|^{\gamma-1}\, f(y)\, \frac{y}{|y|} \otimes \frac{y}{|y|}\, dy \,:\, p\otimes p \,=\, \frac{1}{2}\int_{y\in\mathbb{R}^2} |y|^{\gamma-1}\, f(y)\, dy, \tag{4.6}$$

and lemma 4.1 is proved.

5 Regularization, part I

We now give an intermediate result of regularization:

Theorem 2: *Let f_0 be an initial datum satisfying assumption 2 and such that $||f_0||_{L^2(I\!R^2_v)} < +\infty$. Then, if f is a solution given by theorem 1 of the Boltzmann equation (1.1) with this initial datum and a cross–section satisfying assumption 1, f lies in fact in $L^1_{loc}([0,+\infty[_t; H^{\frac{\gamma-1}{2}}(I\!R^2_v))$.*

Proof of theorem 2: Note first that according to theorem 1,

$$\sup_{t\in[0,+\infty[} ||f(t,\cdot)||_{L^1_2(I\!R^2_v)} < +\infty. \tag{5.1}$$

Therefore,

$$\sup_{t\in[0,+\infty[} ||\hat{f}(t,\cdot)||_{L^\infty(I\!R^2_\xi)} + ||\nabla\hat{f}(t,\cdot)||_{L^\infty(I\!R^2_\xi)}$$
$$+ ||\Delta\hat{f}(t,\cdot)||_{L^\infty(I\!R^2_\xi)} < +\infty. \tag{5.2}$$

Moreover,

$$\inf_{t\in[0,+\infty[} \int_{v\in I\!R^2} f(t,v)\,|v|^{\gamma-1}\,dv > 0, \tag{5.3}$$

since on one hand the total mass is conserved, and on the other hand the family $(f(t,\cdot))_{t\in[0,+\infty[}$ lies in a weakly compact set of $L^1(I\!R^2_v)$ (because of theorem 1).

We shall now use various strictly positive constants $C_1, C_2, etc..$, which may depend on γ and f_0.

According to the lemmas already stated, one can find $\zeta > 0$ (depending on γ) such that when f is as in theorem 1,

$$\mathcal{R}e\left(\int_{\xi\in I\!R^2} \overline{\hat{f}(\xi)}\,\widehat{Q(f)}(\xi)\,d\xi \right)$$

$$\leq C_1 + C_5 \int_{\xi\in I\!R^2} |\hat{f}(\xi)|^2\,d\xi - C_6 \int_{\xi\in I\!R^2} |\xi|^{\gamma-1}\,|\hat{f}(\xi)|^2\,d\xi. \tag{5.4}$$

Using now the fact that f is solution of the Boltzmann equation (1.1), we get the following estimate for the L^2 norm of f:

$$\frac{d}{dt}\int_{\xi\in I\!R^2} |\hat{f}(t,\xi)|^2\,d\xi \leq C_1 + C_5 \int_{\xi\in I\!R^2} |\hat{f}(t,\xi)|^2\,d\xi$$

$$- C_6 \int_{\xi \in I\!R^2} |\xi|^{\gamma-1} \, |\hat f(t,\xi)|^2 \, d\xi. \tag{5.5}$$

Therefore, there exists for all $T>0$ a constant $C_T > 0$ such that

$$\sup_{t\in[0,T]} \int_{\xi \in I\!R^2} |\hat f(t,\xi)|^2 \, d\xi \le C_T, \tag{5.6}$$

and

$$\int_0^T \int_{\xi \in I\!R^2} |\xi|^{\gamma-1} \, |\hat f(s,\xi)|^2 \, d\xi ds \le C_T, \tag{5.7}$$

which concludes the proof of theorem 2.

6 Regularization, part II

We now try to give an optimal version of the previous results of regularization. We only give the main steps of the proof.

Lemma 6.1: *For all $p>0$, one has*

$$(2\pi)^2 \int_{\xi \in I\!R^2} \overline{\hat f(\xi)} \, \widehat{Q(f)}(\xi) \, |\xi|^p \, d\xi = -\chi_{0,p} + \chi_{1,p} + \chi_{2,p} + \chi_{3,p} + \chi_{4,p}, \tag{6.1}$$

where

$$\chi_{0,p} = 4 \int_{\xi \in I\!R^2} \int_{\eta \in I\!R^2} \int_{u=-\infty}^{+\infty} \int_{y \in I\!R^2} |\xi|^{\frac{\gamma-1}{2}+\frac{p}{2}} \, \sin(\frac{u}{2}\frac{\xi}{|\xi|}\cdot y) \, \overline{\hat f(\xi)}$$

$$|\xi+\eta^\perp|^{\frac{\gamma-1}{2}+\frac{p}{2}} \sin(\frac{u}{2}\frac{\xi+\eta^\perp}{|\xi+\eta^\perp|}\cdot y) \, \hat f(\xi+\eta^\perp) \, f(y) \, \widehat{\tau_{-y}D}(\eta) \, dy \, |u|^{-\gamma} \, dud\eta d\xi, \tag{6.2}$$

$$\chi_{1,p} = 4 \int_{\xi \in I\!R^2} \int_{\eta \in I\!R^2} \int_{|u|\ge|\xi|} \int_{y \in I\!R^2} |\xi|^{\gamma-1+p} \, \sin^2(\frac{u}{2}\frac{\xi}{|\xi|}\cdot y) \overline{\hat f(\xi)} \, \hat f(\xi+\eta^\perp)$$

$$f(y) \, \widehat{\tau_{-y}D}(\eta) \, dy \, |u|^{-\gamma} \, dud\eta d\xi, \tag{6.3}$$

$$\chi_{2,p} = -4 \int_{\xi \in I\!R^2} \int_{\eta \in I\!R^2} \int_{u=-\infty}^{+\infty} \int_{y \in I\!R^2} |\xi|^{\gamma-1+p} \, \sin(\frac{u}{2}\frac{\xi}{|\xi|}\cdot y) \, \overline{\hat f(\xi)}$$

$$\Big\{ \sin(\frac{u}{2}\frac{\xi}{|\xi|}\cdot y) - \sin(\frac{u}{2}\frac{\xi+\eta^\perp}{|\xi+\eta^\perp|}\cdot y) \Big\} \, \hat f(\xi+\eta^\perp) \, f(y) \, \widehat{\tau_{-y}D}(\eta) \, dy \, |u|^{-\gamma} \, dud\eta d\xi, \tag{6.4}$$

$$\chi_{3,p} = 4 \int_{\xi \in I\!R^2} \int_{\eta \in I\!R^2} \int_{u=-\infty}^{+\infty} \int_{y \in I\!R^2} |\xi|^{\frac{\gamma-1}{2}+\frac{p}{2}} \, \sin(\frac{u}{2}\frac{\xi}{|\xi|}\cdot y) \, \overline{\hat f(\xi)}$$

$$\Big\{|\xi+\eta^{\perp}|^{\frac{\gamma-1}{2}+\frac{p}{2}}-|\xi|^{\frac{\gamma-1}{2}+\frac{p}{2}}\Big\}\,\sin(\frac{u}{2}\frac{\xi+\eta^{\perp}}{|\xi+\eta^{\perp}|}\cdot y)\,\hat{f}(\xi+\eta^{\perp})$$

$$f(y)\,\widehat{\tau_{-y}D}(\eta)\,dy\,|u|^{-\gamma}\,dud\eta d\xi, \tag{6.5}$$

and

$$\chi_{4,p}=\int_{\xi\in\mathbb{R}^2}\int_{\eta\in\mathbb{R}^2}\int_{\theta=-\pi}^{\pi}|\xi|^p\,\overline{\hat{f}(\xi)}\,\{\hat{f}(\cos\frac{\theta}{2}\,\xi+\eta^{\perp})-\hat{f}(\xi+\eta^{\perp})\}\,\hat{f}(-\sin\frac{\theta}{2}\xi+\eta)$$

$$\hat{D}(\eta)\,|\sin\frac{\theta}{2}|^{-\gamma}\,\cos\frac{\theta}{2}\,d\theta d\eta d\xi. \tag{6.6}$$

Lemma 6.2: *$\chi_{0,p}$ is real and*

$$\chi_{0,p}\geq C(\gamma,p)\,D_0^{\frac{5-\gamma}{2}}\,||D||_{L^\infty(\mathbb{R}^2)}^{\frac{\gamma-3}{2}}$$

$$\times\int_{y\in\mathbb{R}^2}|y|^{\gamma-1}\,f(y)\,dy\int_{\xi\in\mathbb{R}^2}|\xi|^{\gamma-1+p}\,|\hat{f}(\xi)|^2d\xi. \tag{6.7}$$

Moreover,

$$|\chi_{1,p}|\leq C(\gamma,p)\int_{\eta\in\mathbb{R}^2}(1+|\eta|^p)\,|\hat{D}(\eta)|\,d\eta\quad ||f||_{L^1}\int_{\xi\in\mathbb{R}^2}|\xi|^p\,|\hat{f}(\xi)|^2\,d\xi. \tag{6.8}$$

$$|\chi_{2,p}|\leq C(\gamma,p)\int_{\eta\in\mathbb{R}^2}(|\eta|^{\frac{\gamma-1}{2}}+|\eta|^{\gamma-1+p})\,|\hat{D}(\eta)|\,d\eta\quad ||f||_{L^1_{\frac{\gamma+1}{2}}}$$

$$\times\int_{\xi\in\mathbb{R}^2}(1+|\xi|^{\frac{\gamma-1}{2}+p})\,|\hat{f}(\xi)|^2\,d\xi. \tag{6.9}$$

Finally, for all $\epsilon>0$, one can find $\zeta>0$ such that

$$|\chi_{3,p}|\leq C(\gamma,p)\int_{\eta\in\mathbb{R}^2}(1+|\eta|^{\gamma-1+p})\,|\hat{D}(\eta)|\,d\eta\quad ||f||_{L^1_2}$$

$$\times\int_{\xi\in\mathbb{R}^2}(1+|\xi|^{\gamma-1+p-\zeta})\,|\hat{f}(\xi)|^2\,d\xi, \tag{6.10}$$

and

$$|\chi_{4,p}|\leq C(\gamma,p)\,||f||_{L^1_2(\mathbb{R}^2)}\int_{\eta\in\mathbb{R}^2}(1+|\eta|^{\sup(\gamma-1+p,3-\gamma)})\,|\hat{D}|(\eta)\,d\eta$$

$$\times\Big(||f||_{L^1_2}+\int_{\xi\in\mathbb{R}^2}(1+|\xi|^2)^{\sup(\frac{\gamma-1}{2}+\frac{p}{2}-\zeta,\gamma-2+p+4\,\epsilon)}\,|\hat{f}(\xi)|^2\,d\xi\Big). \tag{6.11}$$

We now give the main theorem of this work:

Theorem 3: *Let f_0 be an initial datum satisfying assumption 2 and such that $||f_0||_{L^2(I\!R^2_v)} < +\infty$. Then, if f is a solution given by theorem 1 of the Boltzmann equation (1.1) with this initial datum and a cross section satisfying assumption 1, f lies in fact in $L^\infty_{loc}([\,\overline{t}, +\infty[_t; H^{1-\epsilon}(I\!R^2_v))$ for all $\overline{t}, \epsilon > 0$. In abridged notation, f lies in $L^\infty_{loc}(]0, +\infty[_t; H^{1-0}(I\!R^2_v))$.*

Proof of theorem 3: According to lemmas 6.1 and 6.2, one can find $\zeta > 0$ (depending on γ and p) such that

$$\mathcal{R}e\left(\int_{\xi\in I\!R^2} |\xi|^p\, \overline{\hat f(\xi)}\, \widehat{Q(f)}(\xi)\, d\xi\right) \le C_1 + C_2 \int_{\xi\in I\!R^2} |\hat f(\xi)|^2\, d\xi$$

$$+ C_3 \int_{\xi\in I\!R^2} |\xi|^{\gamma-1+p-\zeta}\, |\hat f(\xi)|^2\, d\xi - C_4 \int_{\xi\in I\!R^2} |\xi|^{\gamma-1+p}\, |\hat f(\xi)|^2\, d\xi, \tag{6.12}$$

as long as

$$0 < p < 3 - \gamma. \tag{6.13}$$

Then, under this hypothesis, one can prove that

$$\frac{d}{dt}\int_{\xi\in I\!R^2} |\xi|^p\, |\hat f(t,\xi)|^2\, d\xi \le C_1 + C_5 \int_{\xi\in I\!R^2} |\hat f(t,\xi)|^2\, d\xi$$

$$- C_6 \int_{\xi\in I\!R^2} |\xi|^{\gamma-1+p}\, |\hat f(t,\xi)|^2\, d\xi. \tag{6.14}$$

Theorem 3 is then proved by induction.

Namely, if for some $\tilde t > 0$,

$$\int_{\xi\in I\!R^2} |\xi|^p\, |\hat f(\tilde t,\xi)|^2\, d\xi < +\infty, \tag{6.15}$$

then for all $T > \tilde t$, there exists $C_T > 0$ such that

$$\sup_{t\in[\,\tilde t, T]} \int_{\xi\in I\!R^2} |\xi|^p\, |\hat f(t,\xi)|^2\, d\xi < C_T, \tag{6.16}$$

and

$$\int_0^T \int_{\xi\in I\!R^2} |\xi|^{\gamma-1+p}\, |\hat f(t,\xi)|^2\, d\xi dt < C_T. \tag{6.17}$$

Finally, for all $\epsilon > 0$, one can find $t' \in]\tilde{t}, \tilde{t} + \epsilon[$ such that

$$\int_{\xi \in \mathbb{R}^2} |\xi|^{\gamma-1+p} \, |\hat{f}(t', \xi)|^2 \, d\xi < +\infty. \tag{6.18}$$

The induction ends when $p \geq 3 - \gamma$, which means that $\gamma - 1 + p \geq 2$, whence theorem 3.

7 How to go further

Note first that the limitation to the regularity of the solution of the Boltzmann equation comes out of an inequality of interpolation between derivatives. Therefore if derivatives of higher orders are known to be bounded for $\hat{f}$, the computation can be improved, and f will be more regular at the end. But derivatives of $\hat{f}$ are related to polynomial moments of f, and one can prove that under suitable assumptions such moments remain bounded when they are initially finite. Finally, the regularity of f will depend on the speed of decreasing at infinity of f_0. Note that this phenomenon was already observed in the case of Maxwellian molecules (Cf. [De 1]).

One can also wonder if it is possible to relax some of the assumptions on the cross section D. Note for example that if $\hat{D}$ does not decrease fast enough at infinity, less regularity will be observed for f at the end. The case when D is not bounded below seems even more difficult to handle.

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